A Family of $2^{\aleph_1}$ Logarithmic Functions of Distinct Growth Rates *

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Abstract

We construct a totally ordered sets Γ of positive infinite germs (i.e. germs of positive real valued functions that tend to $+\infty$), with order type the lexicographic product $\aleph_1 \times \mathbb{Z}^2$. We show that Γ admits $2^{\aleph_1}$ order preserving automorphisms of pairwise distinct growth rates.

1 Introduction

We consider real valued functions, defined on some final segment of the real line. Say that f and g have the same *germ* at $+\infty$ if they are ultimately equal, i.e. if $f(x) = g(x)$ for all x large enough. This defines an equivalence relation on the set of real valued functions. The germ of f is defined to be the equivalence class of f and will also be denoted by f. Write $f \prec g$ if $f(x) < g(x)$ for all x large enough. Let $\mathcal{G}$ denote the ring of all germs. The relation $\prec$ defines a partial order on $\mathcal{G}$. This partial order was first introduced by Paul Du Bois-Reymond and subsequently studied by many authors. In particular, Hausdorff [H1909a], respectively Hardy [Har1954] investigated the totally ordered subsets (respectively subfields) of $\mathcal{G}$. See [Step09] and [Koj09] for comprehensive historical accounts on this subject.

In this note we consider totally ordered sets Γ of positive infinite germs (i.e. germs of positive real valued functions that tend to $+\infty$). We construct such Γ with order type the lexicographic product $\aleph_1 \times \mathbb{Z}^2$. We show that Γ admits $2^{\aleph_1}$ order preserving automorphisms of pairwise distinct growth rates.

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We were led to such investigations while constructing models of Th($\mathbb{R}$, exp) (the elementary theory of the ordered field of real numbers with the exponential function) using fields of formal power series $\mathbb{R}((\Gamma))$. The choice of Γ as above gives a natural functional interpretation to the formal constructions described in [KS05].

2 An asymptotic scale indexed by $\aleph_1 \times \mathbb{Z}^2$

Let Γ be a set of positive infinite germs. We assume that Γ is totally ordered by $\prec$. For an automorphism σ, denote by $\sigma^{(n)}$ its n-th iterate.

We recall some terminology ([Kuh00, Remark 3.20 p. 57]) for more details. Let σ be an automorphism of a totally ordered set Λ. The *rank* of (Λ, σ) is the order type of the quotient $\Lambda / \sim_\sigma$, where $a \sim_\sigma a'$ if and only if there exists $n \in \mathbb{N}$ such that $\sigma^{(n)}(a) \geq a'$ and $\sigma^{(n)}(a') \geq a$.

We now construct Γ as promised in the introduction.

For $(p,q) \in \mathbb{Z}^2$, we denote by $g_{p,q}$ the positive infinite germ

$$x \mapsto \exp\left(x^q \exp\left(x^p\right)\right).$$

If we endow $\mathbb{Z}^2$ with the lexicographic order, then $(p,q) < (p',q')$ implies $g_{p,q} \prec g_{p',q'}$.

Now let $(h_\alpha)_{\alpha \in \aleph_1}$ be a sequence of positive infinite germs h_α, in such a way that $\alpha < \beta$ implies $h_\alpha \prec h_\beta$, see the remark below.

For all $(\alpha, p, q) \in \aleph_1 \times \mathbb{Z}^2$, we denote $f_{\alpha,p,q}$ the germ at $+\infty$ of the function $\exp_3\left(h_\alpha(x)\right) g_{p,q}(x)$ (here $\exp_3$ denotes the third iterate of exp).

These germs are defined in such a way that if $(\alpha, p, q) < (\alpha', p', q')$ for the lexicographic order, then $f_{\alpha,p,q} \prec f_{\alpha',p',q'}$.

We now construct $2^{\aleph_1}$ automorphisms on Γ of pairwise distinct ranks. To this end, we fix two automorphisms on $\Gamma_1 = \left\{g_{p,q}\,,\ (p,q) \in \mathbb{Z}^2\right\}$ defined by :

$$\begin{aligned} \sigma\left(g_{p,q}\right) &= g_{p-1,q} \\ \rho\left(g_{p,q}\right) &= g_{p,q-1} \end{aligned}$$

It follows easily from the definition of $g_{p,q}$ that the rank of (Γ_1, σ) is 1 and the rank of (Γ_1, ρ) is $\mathbb{Z}$. We define now, for every $S \subset \aleph_1$, the automorphism τ_S on Γ :

$$\tau_S\left(f_{\alpha,p,q}\right) = \begin{cases} f_{\alpha,p-1,q} = \exp_3\left(h_\alpha\right) \sigma\left(g_{p,q}\right) & \text{if } \alpha \in S \\ f_{\alpha,p,q-1} = \exp_3\left(h_\alpha\right) \rho\left(g_{p,q}\right) & \text{if } \alpha \notin S \end{cases}$$

Now it is shown in [KS05, Section 7] that such pairwise distinct automorphisms σ have pairwise distinct ranks.

Remark. The existence of such a sequence $(h_\alpha)_{\alpha\in\aleph_1}$ is established e.g. in [H1909a]. One can describe for example the first ϵ_0 terms of such a sequence. Set $h_0(x) := x$. We define h_α by transfinite induction for $\alpha < \epsilon_0$. If the Cantor normal form of α is $\omega^{\beta_r} d_r + \cdots + \omega^{\beta_1} d_1 + d_0$, with $\beta_1 < \cdots < \beta_r < \alpha$ and $d_{0,} \ldots, d_r \in \mathbb{N}$, set

$$h_\alpha(x) := \exp\left(d_r h_{\beta_r}(x) + \cdots + d_1 h_{\beta_1}(x)\right) \exp(x)^{d_0} .$$

We can set $h_{\epsilon_0} := t(x)$ where $t(x)$ is a germ of transexponential growth.

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