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On the structure of nonarchimedean exponential fields II *

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Contents

1 Introduction **1**

2 The valuation theoretical interpretation of the growth axioms for exponentials **5**

2.1 Left exponentials and the value group 7
2.2 Middle exponentials and the residue field 11
2.3 Right exponentials 12
2.4 (De)composing exponentials 13

3 The structure theory for countable exponential fields **16**

4 Contraction groups **21**

5 References **24**

1 Introduction

In the paper [K], the second author has studied the structure of nonarchimedean exponential fields, i.e., nonarchimedean ordered fields admitting an order isomorphism between the additive group and the multiplicative group of positive elements. Among other results, a necessary and sufficient criterion was given for countable nonarchimedean fields to be exponential. In view of the recent development in the model theory of exponential fields (cf. [W], [D–M–M]), however, it is desirable to have criteria for the existence of exponentials satisfying well known axioms that are also satisfied by the usual exponential on $I\!R$. In particular, the usual exponential

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satisfies axioms which relate its growth with the growth of polynomials, and (an infinite scheme of) Taylor axioms which express that it is the limit of the sequence of finite partial sums

$$E_n(x) := \sum_{i=0}^{n} \frac{x^i}{i!} .$$

Our approach in this paper is to give a valuation theoretical interpretation of the growth and Taylor axioms. Such an analysis is developped in chapter 2 and will be used in chapter 3 to obtain a strengthening of the Countable Case Characterization Theorem (Theorem 3.32 of [K]).

We are mainly interested in constructing exponential fields (K, f) satisfying at least the following theory $\forall x \in K : T(f, x)$ satisfied also by $I\!R$ with $f = \exp$:

$$T(f,x) :\equiv \begin{cases} x \geq n^2 & \rightarrow \quad f(x) > x^n & (n \in I\!N) \\ x > 0 & \rightarrow \quad f(x) > E_n(x) & (n \in I\!N) \\ x < 0 & \rightarrow \quad f(x) > E_k(x) & (k \in 2I\!N - 1) \\ x < 0 & \rightarrow \quad f(x) < E_k(x) & (k \in 2I\!N) \\ |x| \leq 1 & \rightarrow \quad |f(x) - E_n(x)| < |x^{n+1}| & (n \in I\!N) \end{cases}$$

In this case, f will be called a *strong exponential*.

For the definition and basic properties of the natural valuation v on an ordered field K and the natural valuation v_G on an ordered Abelian group G, see [K]. For $a \in K$, the residue class modulo v will be denoted by $\overline{a}$. Further, $\overline{K}$ denotes the residue field, and we will write $G := v(K^\times)$ for the value group. The valuation ring R_v and the valuation ideal I_v are convex subgroups of the additive group $(K, +, 0, <)$. Since this group is divisible, there are group complements $\mathbf{A}$ to R_v in $(K, +, 0, <)$ and $\mathbf{A}'$ to I_v in R_v such that it may be represented as the (internal) lexicographic sum

$$(K, +, 0, <) = \mathbf{A} \amalg \mathbf{A}' \amalg I_v \; , \tag{1}$$

and $\mathbf{A}'$ is (canonically) isomorphic to $(\overline{K}, +, 0, <)$ through the residue map. As it was done in [K], we will only consider exponentials f which satisfy:

1) $f(I_v)$ is equal to the subgroup $1 + I_v$ of 1–units in the ordered multiplicative group $(K^{>0}, \cdot, 1, <)$ of positive elements of K,

2) $f(R_v)$ is equal to the subgroup $\mathcal{U}_v^{>0}$ of all units in $(K^{>0}, \cdot, 1, <)$.

(Note that an element $a \in K$ is called a *unit* if $v(a) = 0$.) For such an exponential f, we find that $\mathbf{B} := f(\mathbf{A})$ is a group complement to $\mathcal{U}_v^{>0}$ in $(K^{>0}, \cdot, 1, <)$ and that $\mathbf{B}' := f(\mathbf{A}')$ is a group complement to $1 + I_v$ in $\mathcal{U}_v^{>0}$, and we have a decomposition

$$(K^{>0}, \cdot, 1, <) = \mathbf{B} \amalg \mathbf{B}' \amalg (1 + I_v) \tag{2}$$

(cf. Theorem 3.8 of [K]). If such decompositions (1) and (2) are given and if f_L is an isomorphism between the ordered groups $\mathbf{A}$ and $\mathbf{B}$, then f_L will be called a *left exponential*. Similarly, an isomorphism f_M between the ordered groups $\mathbf{A}'$ and $\mathbf{B}'$ will be called a *middle exponential*, and an isomorphism f_R between the ordered groups I_v and $1 + I_v$ will be called a *right exponential*. Every exponential (which satisfies conditions 1) and 2)) induces a left, a middle and a right exponential. According to this decomposition of exponentials, we decompose also T as follows:

$$T(f,x) \equiv$$
$$(v(x) < 0 \rightarrow T_L(f,x)) \;\wedge\; (v(x) = 0 \rightarrow T(f,x)) \;\wedge\; (v(x) > 0 \rightarrow T_R(f,x)) \tag{3}$$

with

$$T_L(f,x) :\equiv \bigwedge_{n\in I\!N} T_{L,n}(f,x) \quad \text{and} \quad T_R(f,x) :\equiv \bigwedge_{n\in I\!N} T_{R,n}(f,x)$$

where

$$T_{L,n}(f,x) \;:\equiv\; \begin{cases} x>0 \;\to\; f(x)>x^m & (m\in I\!N,\ m\le n) & (\mathrm{L1}_n(f,x)) \\ x>0 \;\to\; f(x)>E_m(x) & (m\in I\!N,\ m\le n) & (\mathrm{L2}_n(f,x)) \\ x<0 \;\to\; f(x)>E_k(x) & (k\in 2I\!N-1,\ k\le n) & (\mathrm{L3}_n(f,x)) \\ x<0 \;\to\; f(x)<E_k(x) & (k\in 2I\!N,\ k\le n) & (\mathrm{L4}_n(f,x)) \end{cases}$$

$$T_{R,n}(f,x) \;:\equiv\; \begin{cases} x>0 \;\to\; f(x)>E_m(x) & (m\in I\!N,\ m\le n) & (\mathrm{R1}_n(f,x)) \\ x<0 \;\to\; f(x)>E_k(x) & (k\in 2I\!N-1,\ k\le n) & (\mathrm{R2}_n(f,x)) \\ x<0 \;\to\; f(x)<E_k(x) & (k\in 2I\!N,\ k\le n) & (\mathrm{R3}_n(f,x)) \\ |f(x)-E_m(x)|<|x^{m+1}| & (m\in I\!N,\ m\le n) & (\mathrm{R4}_n(f,x)) \end{cases}$$

.

Note that $v(x) < 0$ implies $|x| > 1$ and $x > 0 \to x > n^2$; similarly, $v(x) > 0$ implies $|x| < 1$ and $x < n^2$. Conversely, if $x \ge n^2$ then $v(x) \le 0$ and if $|x| \le 1$ then $v(x) \ge 0$. So indeed, (K, f) satisfies $\forall x : T(f,x)$ if and only if it satisfies the three theories in (3).

Let f be an exponential on K (resp. a left exponential $f : \mathbf{A} \to \mathbf{B}$). It will be called *left strong* if it satisfies $T_L(f,x)$ for all x such that $v(x) < 0$ (resp. for all $x \in \mathbf{A}$). In section 2.1, we will study the axiom scheme $T_{L,n}(f,x)$ and show its logical relation to the assertions of the form

$$v(f(x)) < v(x^n)$$

(cf. Corollary 2.5). The crucial notion introduced in section 2.1 is that of a strong exponential group. Given an archimedean ordered Abelian group A, an *exponential group* (G,φ) *in* A is an ordered Abelian group whose archimedean components are all isomorphic to A and such that φ is an order isomorphism from $G^{<0} := \{g \in G \mid g < 0\}$ onto $\Gamma^- := v_G(G \setminus \{0\})$. The exponential group (G,φ) is said to be a *strong exponential group* if $\varphi(g) < v_G(g)$ for all $g \in G^{<0}$. With $G = v(K^\times)$, every exponential f of K induces a group exponential $\varphi = \varphi_f$ on G, as it was shown in Remark 3.21 of [K]. We will prove that (G,φ_f) is a strong exponential group if and only if for every positive x with $v(x) < 0$ (resp. $x \in \mathbf{A}$), we have $v(f(x)) < v(x^n)$ for all $n \in I\!N$. It is then deduced that f is left strong if and only if (G,φ_f) is a strong exponential group.

In section 2.2, we will then consider the relation between the exponential (or middle exponential) f on K and the exponential $\overline{f}$ that it induces on the residue field. Since this residue field with respect to the natural valuation is archimedean, we are able to show that f satisfies $T(f,x)$ for all x with $v(x) = 0$ (resp. for all $x \in \mathbf{A}'$) if and only if $\overline{f}$ coincides with the usual exponential exp on $\overline{K}$ (cf. Theorem 2.12).

In section 2.3, we will consider the right part. Let f be an exponential on K (resp. a right exponential $f : I_v \to 1 + I_v$). It will be called *right strong* if it satisfies $T_R(f,x)$ for all x such that $v(x) > 0$ (that is, for all $x \in I_v$). We will study the axiom scheme $T_{R,n}(f,x)$ and show its logical relation to assertions of the form

$$v(f(x) - E_n(x)) > v(x^n)$$

(cf. Lemma 2.13). We will then deduce that f is right strong if and only if for every x with $v(x) > 0$, we have $v(f(x) - E_n(x)) > v(x^n)$ for all $n \in I\!N$ (cf. Corollary 2.14).

The results of these three sections are put together in section 2.4 to obtain a necessary and sufficient condition for an exponential to be strong (Theorem 2.17) and to reduce the axiom system T to just two axiom schemes (Theorem 2.18). Also, exponentials may be constructed from given left, middle and right exponentials, inheriting their growth and Taylor properties (Theorem 2.20). If the value group G is a strong exponential group having an additional property, then a given exponential may be improved to an exponential which is left strong (Theorem 2.21).

In the case of countable nonarchimedean exponential fields, strong methods are available for the construction of exponentials satisfying many of the above axioms. It turns out that the following theories are the most adequate for this case, for $n \in I\!N$:

$$T_n(f,x) :\equiv$$
$$(v(x) < 0 \rightarrow T_L(f,x)) \wedge (v(x) = 0 \rightarrow T(f,x)) \wedge (v(x) > 0 \rightarrow T_{R,n}(f,x)) .$$

The ultimate result obtained in chapter 3 is Theorem 3.7. It shows that under the hypothesis that K be countable, non–archimedean and root closed for positive elements, given a strong exponential $\overline{f}$ on $\overline{K}$, it can be lifted to an exponential f on K satisfying $\forall x \in K : T_1(f,x)$ if and only if $G \simeq \coprod_{\mathbb{Q}}(\overline{K}, +, 0 <)$. If K is henselian for its natural valuation (as it is the case if K is real closed), then T_1 may be replaced by T_n, for arbitrary fixed $n \in I\!N$. This is the promised strengthening of Theorem 3.32 of [K]. It is shown by proving that a nontrivial countable divisible exponential group is always strong exponential (cf. Proposition 3.1), and that the (existing) isomorphism $f : I_v \rightarrow 1 + I_v$ can be replaced by (and improved to) an isomorphism $f_n : I_v \rightarrow 1 + I_v$ satisfying $\forall x \in I_v : T_{R,n}(f_n, x)$ (cf. Lemma 3.6). Both results are shown using back and forth constructions.

For $n = 1$, we get the theory $\forall x \in K : T_1(f,x)$, where

$$T_1(f,x) \equiv \left\{ \begin{array}{rcll} x \geq n^2 & \rightarrow & f(x) > x^n & (n \in I\!N) \\ v(x) \leq 0 \wedge x > 0 & \rightarrow & f(x) > E_n(x) & (n \in I\!N) \\ v(x) \leq 0 \wedge x < 0 & \rightarrow & f(x) > E_k(x) & (k \in 2I\!N - 1) \\ v(x) \leq 0 \wedge x < 0 & \rightarrow & f(x) < E_n(x) & (k \in 2I\!N) \\ x \neq 0 & \rightarrow & f(x) > 1 + x & \\ x < 0 & \rightarrow & f(x) < 1 & \\ |x| \leq 1 & \rightarrow & |f(x) - (1+x)| < |x|^2 . & \end{array} \right.$$

Now if an exponential f satisfies $\forall x \in K : f(x) \geq 1 + x$, then it is continuous, differentiable and equal to its own derivative (cf. Theorem 14 of [D–W]). We have thus shown that for countable real closed fields, any exponential on the archimedean residue field $\overline{K}$ which is continuous, differentiable and equal to its own derivative, may be lifted to an exponential on the field itself, satisfying the same properties, provided that $G \simeq \coprod_{\mathbb{Q}}(\overline{K}, +, 0, <)$. This gives a complete answer to problem 3) stated in the introduction of [K], for the countable case.

In the last chapter, we will give a further discussion of the structure induced by an exponential on the value group G of a nonarchimedean ordered field. For this discussion, we will introduce a new concept. It is inspired by the idea of studying a "shift" on the value group. At this point, we would like to thank A. Macintyre for bringing this idea to our attention. We are also endebted to Arne Ledet for hints concerning these notes, and to all other participants of our seminar for their interest and patience.

2 The valuation theoretical interpretation of the growth axioms for exponentials

Recall that the natural valuation on an ordered field K has the following properties, for all $a, b \in K$:

$$v(a) > v(b) \implies |a| < |b| \tag{4}$$
$$v(a-b) > v(a) \implies \mathrm{sign}(a) = \mathrm{sign}(b) \tag{5}$$

and, if $v(a) \geq 0$ and $v(b) \geq 0$,

$$\overline{a} > \overline{b} \Rightarrow a > b \quad \text{and} \quad a > b \Rightarrow \overline{a} \geq \overline{b}\,. \tag{6}$$

We will consider the following formulas:

$$\begin{aligned} P_n(x,y) &:\equiv |y - E_n(x)| < |x^n| \\ P'_n(x,y) &:\equiv |y - E_n(x)| < |x^{n+1}| \\ Q_n(x,y) &:\equiv v(y - E_n(x)) > v(x^n)\,. \end{aligned}$$

In the following, we will have a closer look at these formulas and make some basic observations. Note first that $Q_n(a,b)$ trivially holds if $a \neq 0$ and $b = E_n(a)$. From (4), we obtain: for every n, $Q_n(a,b)$ implies $P_n(a,b)$ and if $v(a) < 0$, then $P_n(a,b)$ implies $P'_n(a,b)$. Similarly, $P'_n(a,b)$ implies $P_n(a,b)$ if $v(a) > 0$.

Lemma 2.1 *i)* $Q_n(a,b)$ *implies* $b > E_{n-1}(a)$ *whenever* n *is even or* $a > 0$*, and it implies* $b < E_{n-1}(a)$ *whenever* n *is odd and* $a < 0$*.*
ii) $Q_n(a,b)$ *implies* $P'_{n-1}(a,b)$*, for every* $n \geq 2$*.*
iii) $Q_n(a,b)$ *implies* $Q_m(a,b)$*, whenever* $v(a) > 0$ *and* $m \leq n$*.*
iv) $P'_n(a,b)$ *implies* $Q_{n-1}(a,b)$*, whenever* $v(a) > 0$*.*

Proof: i): $Q_n(a,b)$ implies $a \neq 0$ and

$$v\left(b - E_{n-1}(a) - \frac{a^n}{n!}\right) > v(a^n) = v\left(\frac{a^n}{n!}\right)\ , \tag{7}$$

so by (5)

$$\mathrm{sign}(b - E_{n-1}(a)) = \mathrm{sign}\left(\frac{a^n}{n!}\right)$$

But $a^n/n! > 0$ holds if and only if n is even or a is positive. This proves i).
ii): Equation (7) implies

$$|b - E_{n-1}(a)| = \left|\frac{a^n}{n!} + a\right|$$

for some $c \in a^n I_v$. But then, $v(c) > v(a^n/n!)$ which yields

$$\left|\frac{a^n}{n!} + c\right| < 2\left|\frac{a^n}{n!}\right| \leq |a^n|$$

for $n \geq 2$. This proves assertion ii).
iii): Equation (7) implies

$$v(b - E_{n-1}(a)) = v\left(\frac{a^n}{n!}\right) = v(a^n) \; . \tag{8}$$

If $v(a) > 0$, then $v(a^n) > v(a^{n-1})$, and thus, equation (8) yields $Q_{n-1}(a,b)$. Our assertion now follows by induction.
iv): Since $|a| < |b|$ implies $v(a) \geq v(b)$, $P'_n(a,b)$ implies $v(b - E_n(a)) \geq v(a^{n+1})$ which in view of $v(a) > 0$ yields equation (7). This in turn yields

$$v(b - E_{n-1}(a)) = v\left(\frac{a^n}{n!}\right) = v(a^n) > v(a^{n-1})$$

which is $Q_{n-1}(a,b)$. □

We need a further lemma about the Q_n–predicates. Note that if K is an ordered field and root closed for positive elements, then for every $y \in K^{>0}$ and $q \in \mathbb{Q}$, the element y^q is a (uniquely defined) element in the divisible (and torsion free) group $(K^{>0}, \cdot, 1, <)$.

Lemma 2.2 *Let K be root closed for positive elements. Suppose that $a \neq 0$, $v(a) > 0$ and $v(c) > 0$. Assume that $Q_n(a,b)$ and $Q_n(c,d)$ hold. Then $Q_n(qa, b^q)$ holds for every $q \in \mathbb{Q}$, $q \neq 0$. Further, $Q_n(a+c, bd)$ holds if $v(a+c) = \min(v(a), v(c))$.*

Proof: From the theory of binomial coefficients, it is known that

$$\left(\sum_{i=0}^{n} \frac{a^i}{i!}\right) \cdot \left(\sum_{i=0}^{n} \frac{c^i}{i!}\right) = \sum_{i=0}^{n} \frac{(a+c)^i}{i!} + A$$

where A is a sum consisting only of monomials $c_{\mu,\nu} a^\mu c^\nu$ for which $\mu + \nu \geq n+1$. W.l.o.g., let $v(a) \leq v(c)$. Then $v(A) \geq v(a^{n+1})$, and we obtain

$$\left(\sum_{i=0}^{n} \frac{a^i}{i!}\right) \cdot \left(\sum_{i=0}^{n} \frac{c^i}{i!}\right) \equiv \sum_{i=0}^{n} \frac{(a+c)^i}{i!} \mod a^{n+1} R_v \; . \tag{9}$$

Suppose now that $Q_n(a,b)$ and $Q_n(c,d)$ hold. Since both sums on the left side of (9) have v–value 0 and since $v(a) > 0$ and $a \neq 0$, it follows that

$$bd \equiv \sum_{i=0}^{n} \frac{(a+c)^i}{i!} \mod a^n I_v \; .$$

Since $v(a+c) = \min(v(a), v(c))$ by the assumption of our lemma, we have $v(a) = v(a+c)$, and the above equivalence is thus nothing else than $Q_n(a+c, bd)$.

Let $r, s > 0$ be arbitrary natural numbers. Replacing both a and c by $\frac{1}{s}a$ we obtain from (9) by induction up to s:

$$\left(\sum_{i=0}^{n} \frac{(\frac{r}{s}a)^i}{i!}\right)^s \equiv \left(\sum_{i=0}^{n} \frac{(ra)^i}{i!}\right) \equiv \left(\sum_{i=0}^{n} \frac{a^i}{i!}\right)^r \mod a^{n+1}R_v \ .$$

Since the sums have v–value 0, it follows that

$$\left(\sum_{i=0}^{n} \frac{(\frac{r}{s}a)^i}{i!}\right) \equiv \left(\sum_{i=0}^{n} \frac{a^i}{i!}\right)^{\frac{r}{s}} \mod a^{n+1}R_v$$

and, by virtue of $Q_n(a,b)$ and $v(a) > 0$, $a \neq 0$,

$$\left(\sum_{i=0}^{n} \frac{(\frac{r}{s}a)^i}{i!}\right) \equiv b^{\frac{r}{s}} \mod a^n I_v \ .$$

This proves $Q_n(qa, b^q)$ for all positive rationals q. Finally, it remains to show that $Q_n(a,b)$ implies $Q_n(-a, b^{-1})$. From (9), for $c = -a$ we obtain that

$$\left(\sum_{i=0}^{n} \frac{a^i}{i!}\right) \cdot \left(\sum_{i=0}^{n} \frac{(-a)^i}{i!}\right) \equiv 1 \mod a^{n+1}R_v \ .$$

Again, since the sums have v–value 0, this yields

$$\left(\sum_{i=0}^{n} \frac{(-a)^i}{i!}\right) \equiv \left(\sum_{i=0}^{n} \frac{a^i}{i!}\right)^{-1} \mod a^{n+1}R_v$$

and, by virtue of $Q_n(a,b)$ and $v(a) > 0$, $a \neq 0$,

$$\left(\sum_{i=0}^{n} \frac{(-a)^i}{i!}\right) \equiv b^{-1} \mod a^n I_v \ .$$

This proves $Q_n(-a, b^{-1})$ and consequently, $Q_n(qa, b^q)$ for all rationals $q \neq 0$. $\square$

2.1 Left exponentials and the value group

In this section, we will consider the growth axioms $T_L(f,x)$ for exponentials f on the left, that is, for $v(x) < 0$. More generally, we may also consider a left exponential f which is only defined on a group complement of R_v. In any case, we only have to deal with elements $x \in K$ of negative value. We will first show that in this situation, ($\mathrm{L3}_n(f,x)$) and ($\mathrm{L4}_n(f,x)$) are always satisfied, for all $n \in I\!N$. Giving a valuation theoretical interpretation for ($\mathrm{L1}_n(f,x)$) and ($\mathrm{L2}_n(f,x)$), we will then derive a simple assertion which is equivalent to $T_L(f,x)$, and we will express it by a condition on the map φ_f induced by f on the value group $G = v(K^\times)$.

Assume that $v(a) < 0$. Then

$$v\left(E_n(a)\right) = v\left(\frac{a^n}{n!}\right) = v(a^n) \tag{10}$$

and

$$v\left(E_n(a) - \frac{a^n}{n!}\right) = v\left(E_{n-1}(a)\right) > v\left(E_n(a)\right) \ , \tag{11}$$

hence by (5),

$$E_n(a) > 0 \iff \frac{a^n}{n!} > 0 \ .$$

Note that the latter holds if and only if $a > 0$ or n is even. Consequently, for $a < 0$ and $v(a) < 0$ we have

$$E_n(a) > 0 \iff n \text{ is even.} \tag{12}$$

Now let f be an exponential or a left exponential on K and $a \in \operatorname{dom}(f)$. If $a > 0$ then $f(a) > 1$ and $v(f(a)) \le 0$. If $a < 0$, then $f(a) < 1$ and $v(f(a)) \ge 0$. Hence, if $a < 0$ and $v(a) < 0$, then for $n \ge 1$,

$$v(a^n) < 0 \le v(f(a)) \tag{13}$$

and in view of (10),

$$v\left(E_n(a)\right) < 0 \le v(f(a)) \tag{14}$$

and

$$v\left(f(a) - E_n(a)\right) = \min\left\{v(f(a)), v\left(E_n(a)\right)\right\} = v(a^n) \ .$$

Lemma 2.3 *Let f be an exponential or a left exponential on K and $a \in \operatorname{dom}(f)$. Assume that $v(a) < 0$ and $a < 0$. Then:*

i) $f(a) > a^n$ holds if and only if a^n is negative, that is, if and only if n is odd. On the other hand, $f(a) < a^n$ holds if and only if a^n is positive, that is, if and only if n is even.

ii) $f(a) > E_n(a)$ holds for every odd n. On the other hand, $f(a) < E_n(a)$ holds for every even n.

Consequently, f satisfies $(\mathrm{L3}_n(f,a))$ and $(\mathrm{L4}_n(f,a))$ for every $n \in I\!N$.

Proof: i): From (13), we infer $v(a^n) < v(f(a))$. Since $f(a)$ is positive, this yields that $f(a) > a^n$ if a^n is negative, and that $f(a) < a^n$ if a^n is positive.

ii): Using (14) in the place of (13), we now obtain that $f(a) > E_n(a)$ if a^n is negative, and that $f(a) < E_n(a)$ if a^n is positive. Our assertion thus follows from (12). $\square$

Lemma 2.4 *Assume that $v(a) < 0$.*

i) If $b > 0$, then $v(b) < v(a^n)$ implies $b > a^n$ and $b > E_n(a)$.

ii) Assume that $a > 0$ and $b > 0$. If $b > a^{n+1}$ or if $b > E_{n+1}(a)$, then $v(b) < v(a^n)$.

Proof: i): If $b > 0$, then $v(b) < v(a^n)$ implies $b > a^n$, and in view of (10), it also implies $b > E_n(a)$.

ii): If $a > 0$ and $b > 0$, then $b > a^{n+1}$ implies $v(b) \le v(a^{n+1})$, and $b > E_{n+1}(a)$ implies $v(b) \le v(E_{n+1}(a))$. In view of (10) and $v(a) < 0$, we obtain $v(b) \le v(a^{n+1}) < v(a^n)$ in both cases. $\square$

From this lemma together with Lemma 2.3, we obtain:

Corollary 2.5 *Let f be any partial function from K to $K^{>0}$ and $a \in \mathrm{dom}\,(f)$ such that $v(a) < 0$ and $a > 0$. If $v(f(a)) < v(a^n)$, then f satisfies $(L1_n(f,a))$ and $(L2_n(f,a))$. Conversely, if f satisfies $(L1_{n+1}(f,a))$ or $(L2_{n+1}(f,a))$, then $v(f(a)) < v(a^n)$.*

If in addition f is an exponential or a left exponential on K, then

$$T_{L,n+1}(f,a) \implies v(f(a)) < v(a^n) \tag{15}$$

$$v(f(a)) < v(a^n) \implies T_{L,n}(f,a) \tag{16}$$

and $T_L(f,a)$ is equivalent to

$$\forall n \in I\!N : \; v(f(a)) < v(a^n) \tag{17}$$

which in turn may be expressed as

$$v_G(v(f(a))) < v_G(v(a)) \, . \tag{18}$$

Note that assertion (18) holds for a if and only if it holds for $-a$. This is true because $v(a) = v(-a)$ and $v_G(v(f(-a))) = v_G(v(f(a)^{-1})) = v_G(-v(f(a))) = v_G(v(f(a)))$. But this invariance does not hold for assertion (17). Hence, both assertions are not equivalent in the case of an element a with $v(a) < 0$ and $a < 0$.

Now let (G, φ) be an exponential group and consider the following condition on φ:

$$\forall g \in G^{<0} : \; \varphi(g) < v_G(g) \, . \tag{19}$$

By the definition of φ_f (cf. Remark 3.21 of [K]) and by the foregoing corollary, we obtain the main result of this section:

Theorem 2.6 *Let f be an exponential or a left exponential on K. Then the following assertions are equivalent:*

a) $\varphi = \varphi_f$ *satisfies condition (19)*

b) $\forall x \in \mathrm{dom}\,(f) : \; v(x) < 0 \to v_G(v(f(x))) < v_G(v(x))$

c) f *satisfies* $T_L(f,x)$ *for all* $x \in \mathrm{dom}\,(f)$ *with* $v(x) < 0$

d) f *satisfies* $f(x) > x^n$ *for all positive* $x \in \mathrm{dom}\,(f)$ *with* $v(x) < 0$ *and all* $n \in I\!N$

e) f *satisfies* $f(x) > E_n(x)$ *for all positive* $x \in \mathrm{dom}\,(f)$ *with* $v(x) < 0$ *and all* $n \in I\!N$.

We will say that an exponential group (G, φ) is a *strong exponential group* if φ satisfies (19). Hence, we have the following

Theorem 2.7 *For every exponential (resp. left exponential) f on K,*

$$f \text{ is left strong} \iff (G, \varphi_f) \text{ is a strong exponential group.}$$

Let us also note:

Lemma 2.8 *Let f be an exponential or a left exponential on K and $a \in \mathrm{dom}\,(f)$. Assume that $v(a) < 0$ and $a < 0$. Then $P_n(a, f(a))$ and thus also $P'_n(a, f(a))$ holds for all $n \geq 0$.*

Proof: The assertion is trivial for $n = 0$ because of $f(a) < 1$. This also yields $f(a)-1-a < -a$ and since $-a > 0$ by hypothesis and $f(a) > 1+a$ by the foregoing corollary, this gives our assertion in the case of $n = 1$. Now the remaining cases (where $n \geq 2$) may be proved by induction:

$$\begin{aligned} |f(a) - E_n(a)| &= \left| f(a) - E_{n-1}(a) - \frac{a^n}{n!} \right| \\ &\leq |f(a) - E_{n-1}(a)| + \left| \frac{a^n}{n!} \right| \\ &< |a^{n-1}| + \left| \frac{a^n}{n!} \right| < 2 \left| \frac{a^n}{n!} \right| \leq |a^n| \end{aligned}$$

in view of

$$0 > v(a^{n-1}) > v(a^n) = v(\frac{a^n}{n!}) .$$

□

The following theorem gives a condition on the value group G which enables "improving" a left exponential to a left strong exponential. This condition on G is to have the *lifting property*, that is, that every automorphism σ of the rank of G lifts to an order automorphism τ of G (which means that τ induces σ on the rank).

Theorem 2.9 *Suppose that K admits a left exponential h and that its value group G is a strong exponential group having the lifting property. Then K admits also a left strong exponential with the same domain and range as h.*

G has the lifting property if $(K, +, 0, <)$ (or G itself) admits a valuation basis or if it is maximally valued (e.g. if K is a power series field).

Proof: Assume the decompositions given in (1) and (2) with $h : \mathbf{A} \to \mathbf{B}$. By Theorem 3.8 of [K], $\mathbf{B}$ is isomorphic to G. Hence, $\mathbf{B}$ has the same properties that we have assumed for G. Following Remark 3.21 of [K], the isomorphism h induces an isomorphism $\varphi_h : G^{<0} \cong \Gamma^-$. By our hypothesis that G be a strong exponential group, there exists

$$\varphi : G^{<0} \cong \Gamma^- \text{ such that } \forall g : \varphi(g) < v_G(g) .$$

Then $\varphi \circ \varphi_h^{-1}$ is an order isomorphism on Γ^-. By our hypothesis on $G \cong \mathbf{B}$, it is induced by an isomorphism τ of $\mathbf{B}$. We set $f_L = \tau \circ h$. Then $f_L : \mathbf{A} \to \mathbf{B}$ is an order isomorphism, and we have $\varphi_{f_L} = \varphi$. Hence,

$$v_G(v(f_L(a))) = \varphi(v(a)) < v_G(v(a))$$

which by virtue of Theorem 2.6 yields that f_L satisfies $T_L(f_L, x)$ on $\mathbf{A}$.

Concerning the lifting property of G, let us consider the case where $(K, +, 0, <)$ admits a valuation basis. Then $\mathbf{A}$ admits a valuation basis (cf. Lemma 2.18 of [K]), and via the isomorphism h, the same is true for $\mathbf{B}$ which in turn is isomorphic to G. That is, G is a Hahn sum over its skeleton and as such, admits a canonical lifting of every automorphism of its rank. This also holds for Hahn products, and indeed, if $(K, +, 0, <)$ is maximally valued, then by Lemma 2.17 of [K], the same is true for $\mathbf{A}$ and thus by Corollary 2.8 of [K], $\mathbf{A}$ and also $\mathbf{B}$ and G are Hahn products. □

Note that also certain intermediate groups between Hahn sum and Hahn product have the lifting property, namely the κ–bounded Hahn products (the subgroups of all elements of a Hahn product whose support has cardinality $< \kappa$). But it does not seem likely that every value group of an exponential field has the lifting property; however, we do not know of an example (which necessarily would have to be uncountable, as it will turn out in chapter 3).

To close this section, let us note that the preceding theorem is not true if we drop the assumption "K admits a left exponential". In fact, if K is a nonarchimedean power series field such that G admits a valuation basis, then K admits no left exponential at all. This is a consequence of Proposition 3.26 of [K].

2.2 Middle exponentials and the residue field

In this section, we will consider the axiom scheme

$$\forall x \in K: \ v(x) = 0 \ \rightarrow \ T(f, x)$$

and determine its relation to the corresponding axiom scheme $\forall x \in \overline{K}: \ T(\overline{f}, x)$. For the first basic results, f need not be an exponential; the only hypothesis required is that f be a (partial) map on K which induces a well defined map $\overline{f}$ on $\overline{K}$ through $\overline{f}(\overline{a}) = \overline{f(a)}$ for all $a \in \operatorname{dom}(f) \cap R_v$. In this case, the following facts hold in view of (6). If $\overline{f}(\overline{a}) > \overline{a}^n$, then $f(a) > a^n$. If $\overline{f}(\overline{a}) > E_n(\overline{a})$, then $f(a) > E_n(a)$, and if $\overline{f}(\overline{a}) < E_n(\overline{a})$, then $f(a) < E_n(a)$. If $|\overline{f}(\overline{a}) - E_n(\overline{a})| < |\overline{a}^{n+1}|$, then $|f(a) - E_n(a)| < |a^{n+1}|$. Hence:

Lemma 2.10 *Let f be as above. If $\overline{f}$ satisfies $\forall x \in \overline{K}: \ T(\overline{f}, x)$, then f satisfies $\forall x \in \operatorname{dom}(f): \ v(x) = 0 \ \rightarrow \ T(f, x)$.*

Now we have to ask for the converse. Again by (6), the following facts hold. If $f(a) > a^n$, then $\overline{f}(\overline{a}) \geq \overline{a}^n$. If $f(a) > E_n(a)$, then $\overline{f}(\overline{a}) \geq E_n(\overline{a})$, and if $f(a) < E_n(a)$, then $\overline{f}(\overline{a}) \leq E_n(\overline{a})$. If $|f(a) - E_n(a)| < |a^{n+1}|$, then $|\overline{f}(\overline{a}) - E_n(\overline{a})| \leq |\overline{a}^{n+1}|$. In other words, in passing from the field K to its residue field, the strictness of the inequalities is lost. In particular, if f satisfies $T(f, x)$ for all $x \in \operatorname{dom}(f)$ with $v(x) = 0$, then $\overline{f}$ satisfies the corresponding "weak version of T" on $\overline{K}$. However, the axiom scheme

$$x > 0 \rightarrow f(x) \geq E_n(x) \qquad (n \in I\!N)$$

is equivalent to the axiom scheme

$$x > 0 \rightarrow f(x) > E_n(x) \qquad (n \in I\!N) \ ,$$

and for $|x| < 1$ and n_0 as big as to satisfy $(n_0 + 1)! > (1 - |x|)^{-1}$, the axiom scheme

$$|f(x) - E_n(x)| \leq |x^{n+1}| \qquad (n_0 \leq n \in I\!N)$$

is equivalent to

$$|f(x) - E_n(x)| < |x^{n+1}| \qquad (n_0 \leq n \in I\!N) \ .$$

But for the other axiom schemes of T, such an equivalence is not immediately seen, even if f is an exponential. Nevertheless, we may use the important additional information that $\overline{K}$ is an archimedean field since it is the residue field of the natural valuation on K.

Lemma 2.11 *Let f be an exponential or a middle exponential on K. If f satisfies the axiom scheme*

$$\forall x \in \mathrm{dom}\,(f) \setminus I_v : \ |x| < 1 \to |f(x) - E_n(x)| \leq |x^{n+1}| \quad (\forall n \in I\!N)\ , \tag{20}$$

then $\overline{f}$ coincides with the usual exponential exp *and is thus a strong exponential on $\overline{K}$.*

Proof: By what we have said preceding to this lemma, the axiom scheme (20) implies that $\overline{f}$ satisfies the axiom scheme

$$\forall x \in \overline{K} : \ |x| < 1 \to |\overline{f}(x) - E_n(x)| \leq |x^{n+1}| \quad (\forall n \in I\!N)\ .$$

But for $\overline{a} \in \overline{K}$ with $|\overline{a}| < 1$ this yields that $\overline{f}(\overline{a})$ is a limit of the series $\sum_{i=0}^{\infty} \frac{\overline{a}^i}{i!}$. Since $\overline{K}$ is archimedean, it is the only limit and thus equal to $\exp(\overline{a})$.

We have shown that $\overline{f}$ coincides with exp on the interval $(-1,1)$. But for every $\overline{a} \in \overline{K}$ there is some $n \in I\!N$ such that $\overline{a}/n \in (-1,1)$ (using again that $\overline{K}$ is archimedean). By the homomorphism property of $\overline{f}$ and exp, it follows that

$$\overline{f}(\overline{a}) = \overline{f}(\overline{a}/n)^n = \exp(\overline{a}/n)^n = \exp(\overline{a})\ ,$$

showing that $\overline{f}$ coincides with exp on all of $\overline{K}$. $\square$

As a corollary to the foregoing two lemmata, we obtain:

Theorem 2.12 *Let f be an exponential or a middle exponential on K. Then the following assertions are equivalent:*

a) f *satisfies* $\forall x \in \mathrm{dom}\,(f) : \ v(x) = 0 \ \to \ T(f,x)$,

b) $\overline{f}$ *is a strong exponential (and thus coincides with* exp*),*

c) f *satisfies (20).*

2.3 Right exponentials

Lemma 2.13 *Let f be any partial function from K to K and $a \in I_v$ an element of* $\mathrm{dom}\,(f)$*. Then:*

i) $Q_{n+1}(a, f(a))$ *implies* $T_{R,n}(f,a)$

ii) $(\mathrm{R4}_{n+1}(f,a))$ *implies* $Q_n(a, f(a))$.

Proof: i): Assume the hypothesis. Then by virtue of part iii) of Lemma 2.1, $Q_m(a, f(a))$ holds for $1 \leq m \leq n+1$. By part i) of Lemma 2.1, it follows that $f(a) > E_m(a)$ whenever m is odd or $a > 0$, and that $f(a) < E_m(a)$ whenever m is even and $a < 0$. This shows that $(\mathrm{R1}_n(f,a))$, $(\mathrm{R2}_n(f,a))$ and $(\mathrm{R3}_n(f,a))$ hold. Finally, by part ii) of Lemma 2.1, $P'_m(a, f(a))$ holds for $m \leq n$, showing that $(\mathrm{R4}_n(f,a))$ holds.

ii): If $(\mathrm{R4}_{n+1}(f,a))$ holds, then in particular, $P'_{n+1}(a,f(a))$ holds. Now the assertion follows from part iv) of Lemma 2.1. $\square$

Corollary 2.14 *Let f be any partial function from K to K and $a \in I_v$ an element of* $\mathrm{dom}\,(f)$. *Then $T_R(f,a)$ is equivalent to*

$$\forall n \in I\!N : \; Q_n(a, f(a)) \, , \tag{21}$$

and this in turn is equivalent to $\forall n \in I\!N : (R4_n(f,a))$.

As an important consequence of the last corollary, we have:

Theorem 2.15 *For every exponential (resp. right exponential) f on K, the following assertions are equivalent:*
a) f satisfies $\forall x \in I_v : \; Q_n(x, f(x)$ for all $n \in I\!N$,
b) f is right strong,
c) f satisfies $\forall x \in I_v : \; |f(x) - E_n(x)| < |x^{n+1}|$ for all $n \in I\!N$.

To close this section, we will have a look at the map ψ_f induced by f on $G^{>0}$ (cf. Remark 3.21 of [K]).

Lemma 2.16 *Suppose that K admits an exponential f.*
i) If $f(\varepsilon) \geq 1+\varepsilon$ for all positive $\varepsilon \in I_v$, then $\psi_f(g) \leq g$ for all $g \in G^{>0}$. Conversely, if $\psi_f(g) < g$ for all $g \in G^{>0}$, then $f(\varepsilon) > 1+\varepsilon$ for all positive $\varepsilon \in I_v$.
ii) If $\forall x \in I_v : Q_1(x, f(x))$ holds, then $\psi_f = \mathrm{id}$.

Proof: i): $\Rightarrow$: Assume that $f(\varepsilon) \geq 1+\varepsilon$ for all positive $\varepsilon \in I_v$. Let $g \in G^{>0}$ and $\varepsilon > 0$ such that $v(\varepsilon) = g$. We know that $f(\varepsilon)-1 \geq \varepsilon > 0$. Hence, $v(f(\varepsilon)-1) \leq v(\varepsilon)$ by the convexity of the valuation.
Now assume that $\psi_f(g) < g$ for all $g \in G^{>0}$. Let $0 < \varepsilon \in I_v$. If $f(\varepsilon) \leq 1+\varepsilon$, then $f(\varepsilon) - 1 \leq \varepsilon$. But $\varepsilon > 0$ implies $f(\varepsilon) - 1 > 0$. In view of the convexity of the valuation, this shows that $v(f(\varepsilon) - 1) \geq v(\varepsilon)$, contradicting our hypothesis.
ii): By hypothesis, for $v(a) > 0$ we have

$$v(f(a) - (1+a)) > v(a) \, ;$$

but then, we must have $v(f(a) - 1) = v(a)$. $\square$

2.4 (De)composing exponentials

From Theorems 2.7, 2.12, 2.15 and Lemma 2.13 we obtain the following criterion for strong exponentials:

Theorem 2.17 *Let f be an exponential on K. Then f is a strong exponential if and only if*
1) (G, φ_f) is a strong exponential group,
2) $\overline{f}$ is a strong exponential,
3) f satisfies $\forall x \in I_v : Q_n(x, f(x))$ for all $n \in I\!N$.

More generally, the following holds. If f satisfies $\forall x \in K : T_n(f,x)$, then 1) and 2) hold and f satisfies $\forall x \in I_v : Q_n(x, f(x))$. Conversely, if 1) and 2) hold and f satisfies $\forall x \in I_v : Q_{n+1}(x, f(x))$, then f satisfies $\forall x \in K : T_n(f,x)$.

Also from Theorems 2.7, 2.12, 2.15 together with Theorem 2.6, we obtain the following information on our system of growth and Taylor axioms:

Theorem 2.18 *The following theories are equivalent:*
a) $\forall x : T(f,x)$,
b) $\forall x : (x > 0 \to f(x) > E_n(x)) \wedge (|x| \le 1 \to |f(x) - E_n(x)| < |x^{n+1}|)\ (n \in I\!N)$,
c) $\forall x : (x \ge n^2 \to f(x) > x^n) \wedge (|x| \le 1 \to |f(x) - E_n(x)| < |x^{n+1}|) \quad (n \in I\!N)$.

For the construction of (more or less) strong exponentials, we need the following lemma which puts left, middle and right exponentials together:

Lemma 2.19 *Suppose that $\mathbf{A}$ (resp. $\mathbf{B}$) is a group complement to R_v (resp. to $\mathcal{U}_v^{>0}$) in $(K,+,0,<)$ (resp. in $(K^{>0},\cdot,1,<)$), and that $\mathbf{A}'$ (resp. $\mathbf{B}'$) is a group complement to I_v (resp. to $1+I_v$) in R_v (resp. in $\mathcal{U}_v^{>0}$), so that we have the decompositions (1) and (2). Suppose moreover that*
1) $f_L : \mathbf{A} \to \mathbf{B}$ is a left strong exponential,
2) $e' : \mathbf{A}' \to \mathbf{B}'$ is a middle exponential satisfying $\forall x \in \mathbf{A}' : T(e',x)$,
3) $f_R : I_v \to 1 + I_v$ is a right exponential satisfying $\forall x \in I_v : T_{R,n}(f_R, x)$.
Let

$$\begin{array}{rcl} f = f_L \amalg e' \amalg f_R : (K,+,0,<) & \to & (K^{>0},\cdot,1,<) \\ a + a' + \varepsilon & \mapsto & f_L(a)\cdot e'(a')\cdot f_R(\varepsilon) \end{array}$$

(where $a \in \mathbf{A}$, $a' \in \mathbf{A}'$, $\varepsilon \in I_v$). Then f is an exponential satisfying $\forall x : T_n(f,x)$, and $\overline{f}$ is equal to the exponential e which is canonically induced by e' on $\overline{K}$. If moreover, f_R is right strong, then f is a strong exponential.

Proof: The (finite) lexicographic sum of order preserving isomorphisms is again an order preserving isomorphism, so f is indeed an exponential on K. Since on $\mathbf{A}'$ it coincides with e', it induces e on $\overline{K}$.

Assuming hypothesis 1), 2) and 3), we want to show that f satisfies $\forall x : T_n(f,x)$. By 3), it satisfies $T_{R,n}(f,x)$ on I_v since there it coincides with f_R (similarly, if f_R satisfies $T_R(f_R,x)$ on I_v, then also f satisfies $T_R(f,x)$ on I_v). Since e' satisfies $T(e',x)$ on $\mathbf{A}'$, we know by Theorem 2.12 that e is a strong exponential on $\overline{K}$. Applying Theorem 2.12 a second time, we find that f satisfies $T(f,x)$ for all $x \in R_v \setminus I_v$. Further, since f_L satisfies $T_L(f_L,x)$ on $\mathbf{A}$, Theorem 2.7 shows that (G, φ_{f_L}) is a strong exponential group. But since f coincides with f_L on $\mathbf{A}$, we have $\varphi_f = \varphi_{f_L}$, and Theorem 2.7 now shows that f satisfies $T_L(f,x)$ for all x with

$v(x) < 0$. All in all, we find that f satisfies $\forall x : T_n(f,x)$, and that it even satisfies $\forall x : T(f,x)$ if f_R satisfies $T_R(f_R,x)$ on I_v. $\square$

Let us mention that this lemma may be generalized by replacing T_L by $T_{L,n}$. In view of (15) and (16) of Corollary 2.5, we obtain that f satisfies $T_{L,n}(f,x)$ for all x with $v(x) < 0$ if f_L satisfies $T_{L,n+1}(f_L,x)$ for all $x \in \mathbf{A}$.

As a corollary to the last lemma and Theorem 2.12, we obtain

Theorem 2.20 *Let K be an ordered field, root closed for positive elements. If there exist*

1) a left exponential f_L on K which is left strong,

2) a strong exponential e on $\overline{K}$,

3) a right exponential f_R on K which satisfies $\forall x \in I_v : T_{R,n}(f_R,x)$,

then K admits an exponential f which satisfies $\forall x \in K : T_n(f,x)$ and such that $\overline{f} = e$. If in addition, f_R is right strong, then f may be chosen to be a strong exponential on K.

Proof: The only thing that is left to show is the existence of a middle exponential e' as required in Lemma 2.19. Since K is assumed to be root closed for positive elements, there is a decomposition (2), according to Theorem 3.8 of [K]. (Since the additive group of every field is divisible, the decomposition (1) does always exist.) By Remark 3.21 of [K], e induces a middle exponential $e' : \mathbf{A}' \to \mathbf{B}'$. If e is a strong exponential on $\overline{K}$, then by Theorem 2.12, e' satisfies $T(e',x)$ on $\mathbf{A}'$. $\square$

We can now extend Theorem 2.9 as follows:

Theorem 2.21 *Suppose that K is an exponential field such that its value group G is a strong exponential group having the lifting property. Then K admits an exponential f which is left strong. If in addition, $\overline{K}$ admits a strong exponential e, then f may be chosen such that it also satisfies $T(f,x)$ for all x with $v(x) = 0$, and that $\overline{f} = e$.*

Proof: Let $\tilde{f}$ be any exponential on K. Then $\tilde{f}$ induces a right exponential $f_R : I_v \to 1 + I_v$. We may assume the decomposition (1), and setting $\mathbf{B} = \tilde{f}(\mathbf{A})$ and $\mathbf{B}' = \tilde{f}(\mathbf{A}')$, we also obtain a decomposition (2). If $\overline{K}$ admits a strong exponential e, then there exists a middle exponential $e' : \mathbf{A}' \to \mathbf{B}'$ that satisfies $T(e',x)$ for all $x \in \mathbf{A}'$ and that induces e, as we have already shown in the proof of Theorem 2.20. If $\overline{K}$ does not admit such an exponential then we take e' to be the restriction of f to $\mathbf{A}'$. From Theorem 2.9, we infer the existence of a left strong exponential $f_L : \mathbf{A} \to \mathbf{B}$. Now we put $f = f_L \amalg e' \amalg f_R$. It follows from the proof of Lemma 2.19 that f satisfies $T_L(f_L,x)$ for all x with $v(x) < 0$ and $T(e',x)$ for all x with $v(x) = 0$. That is, f satisfies $T(f,x)$ for all x with $v(x) \leq 0$, as required. $\square$

Corollary 2.22 *Let $E((G))$ be a power series field, root closed for positive elements. Let e be a strong exponential on E. Then $E((G))$ admits a strong exponential lifting e if and only if G is a strong exponential group in E and maximally valued.*

Proof: $\Rightarrow$: follows from Theorem 2.7 together with Proposition 3.26 of [K].
$\Leftarrow$: Since G is a strong exponential group in E and maximally valued, it follows by Corollary 3.27 of [K] that $E((G))$ admits an exponential. By Theorem 2.9, G has the lifting property. By Theorem 2.21, we obtain that $E((G))$ admits an exponential which is left strong and satisfies $\overline{f} = e$. As observed by Alling in [ALL], section 3, pp. 709-710, using a result due to B. H. Neumann (cf. [N]), it can be shown that every power series field admits a right exponential which is right strong. Our assertion now follows from Theorem 2.20. □

Remark 2.23 We can slightly improve the conclusion of the last theorem: In the case where $(K, +, 0, <)$ admits a valuation basis or is maximally valued, f may be constructed such that it satisfies in addition $\forall x \in I_v : \; x > 0 \to f(x) \geq 1 + x$. In fact, in that case every automorphism of the rank of I_v (i.e. every automorphism of $G^{>0}$) lifts to an automorphism of I_v itself. On the other hand, since K admits an exponential $\tilde{f}$, its value group G is divisible, so there exists an automorphism ψ of $G^{>0}$ such that $\psi <$ id (e.g. take $\psi(g) = g/2$). Consider now the automorphism $\psi \circ \psi_{\tilde{f}^{-1}}$ of $G^{>0}$, and let h be an automorphism of $1 + I_v$ inducing $\psi \circ \psi_{\tilde{f}^{-1}}$ (note that $1 + I_v$ has the same lifting property as I_v since $f_R : I_v \simeq 1 + I_v$). Now replace $\tilde{f}$ by $h \circ \tilde{f}_R$ in the proof of the preceding theorem. The resulting function f has now the additional property that $\psi_f = \psi$, so by part i) of Lemma 2.16, it satisfies $f(x) \geq 1 + x$ for all $x > 0$ with $v(x) > 0$.

If f satisfies $\forall x : \; v(x) < 0 \to T_L(f, x)$ and $\forall x : \; v(x) = 0 \to T(f, x)$, then it follows that $f(x) \geq 1 + x$ whenever $v(x) \leq 0$, and we have $\forall x > 0 : \; f(x) \geq 1 + x$.

In the next chapter it will be shown that in the case where K is countable, the conclusions of the theorem can be improved in a much better way, i.e. so that at least we have

$$\forall x : \; f(x) > 1 + x \; .$$

3 The structure theory for countable exponential fields

In [K], the second author has shown that a countable divisible ordered Abelian group G is an exponential group in A (where A is a countable divisible archimedean ordered Abelian group) if and only if $G \simeq \coprod_{\mathbb{Q}} A$ (cf. Proposition 3.33 of [K]).

Proposition 3.1 *Suppose that $A \neq 0$ is any countable divisible archimedean ordered Abelian group, then $\coprod_{\mathbb{Q}} A$ is a strong exponential group. Hence, every nontrivial countable divisible exponential group is a strong exponential group.*

Proof: We shall in fact show more: Suppose that $\{A(q); q \in \mathbb{Q}\}$ is a family of nontrivial countable dense archimedean ordered Abelian groups and set $G = \coprod_{q \in \mathbb{Q}} A(q)$ (hence also G is dense). Then we will show the existence of an isomorphism

$$\varphi : G^{<0} \longrightarrow \mathbb{Q}$$

such that $\forall g : \; \varphi(g) < v_G(g)$.

Let Φ be the family of all maps ϕ which are order preserving isomorphisms of a finite subset of $G^{<0}$ onto a finite subset of $\mathbb{Q}$ such that $\forall g:\ \phi(g) < v_G(g)$.

We show that Φ is a nonempty Karpian family. Once we have shown that, φ is obtained by a back and forth argument, using induction on countable enumerations of $G^{<0}$ and $\mathbb{Q}$. We use the fact that $\mathbb{Q}$ is dense and without endpoints.

Φ is nonempty: let $g \in G^{<0}$ and $q \in \mathbb{Q}$ such that $q < v_G(g)$. Set $\phi(g) = q$, then $\phi \in \Phi$. Now let $\phi \in F$ and $\operatorname{dom}\phi = \{g_0, \ldots, g_n\}$ with $g_0 < \ldots < g_n$, and $\operatorname{range}\phi = \{q_0, \ldots, q_n\}$ with $q_0 < \ldots < q_n$, and $\phi(g_i) = q_i$ for $0 \le i \le n$.

Φ has the back property: Let $g \in G^{<0}$, $g \notin \operatorname{dom}\phi$.
If $g < g_0$, let $q < \min\{v_G(g), q_0\}$.
If $g > g_n$, we have $q_n < v_G(g_n) \le v_G(g)$; so let $q_n < q < v_G(g)$.
If $g_i < g < g_{i+1}$, we have $q_i < v_G(g_i) \le v_G(g)$; so let $q_i < q < \min\{v_G(g), q_{i+1}\}$.
In all cases, set $\phi(g) = q$.

Φ has the forth property: Let $q \notin \operatorname{range}\phi$, $q \in \mathbb{Q}$.
If $q < q_0$, let $q' \in \mathbb{Q}$ such that $q < q' < v_G(g_0)$.
If $q > q_n$, let $q' > \max\{v_G(g_n), q\}$.
If $q_i < q < q_{i+1}$ then $v_G(g_i) \le v_G(g_{i+1})$ and $q < v_G(g_{i+1})$. Assume first that $v_G(g_i) < v_G(g_{i+1})$, then choose $q' \in \mathbb{Q}$ s.t. $\max\{v_G(g_i), q\} < q' < v_G(g_{i+1})$.
Now let $g \in G^{<0}$ such that $v_G(g) = q'$, in the above three cases. Finally, if $v_G(g_i) = v_G(g_{i+1})$, choose $g \in G^{<0}$ such that $g_i < g < g_{i+1}$ (here, we have to use that G is dense).
In all cases, set $\phi(g) = q$. $\square$

Remark 3.2 In a straightforward manner, the above procedure may be changed such that it produces a map

$$\tilde{\varphi} : G^{<0} \longrightarrow \mathbb{Q} \quad \text{satisfying} \quad \forall g:\ \tilde{\varphi}(g) > v_G(g)\ .$$

On the other hand, we can also achieve that "$<$" holds at some part of the group, while "$>$" holds at some other part. Indeed, we may partition $\mathbb{Q}$ into countably many disjoint open intervals I_j with $j \in J \subset \mathbb{N}$, and construct $\tilde{\varphi}$ such that for all $g \in G$ with $v_G(g) \in I_j$, $\tilde{\varphi}(g) > v_G(g)$ whenever j is odd and $\tilde{\varphi}(g) < v_G(g)$ whenever j is even.

Corollary 3.3 *Assume that K admits a left exponential and that $G = v(K^\times)$ is countable. Then K admits a left exponential which is left strong.*

Proof: If $G = 0$ then $\mathbf{A} = 0 = \mathbf{B}$ and there is nothing to prove. If $G \neq 0$ then it is of the form $G \simeq \coprod_{\mathbb{Q}} A$ as we have remarked above. Hence, it admits a valuation basis (cf. Corollary 2.5 of [K]), and by the preceding proposition, it is a strong exponential group. The conclusion now follows from Theorem 2.9. $\square$

If also K is supposed to be countable, we may replace the condition that it admits a left exponential by the condition that G is an exponential group in $(\overline{K}, +, 0, <)$.

Corollary 3.4 *Assume K to be a countable nonarchimedean ordered field, root closed for positive elements. If G is an exponential group in $(\overline{K},+,0,<)$, then K admits a left exponential which is left strong.*

Proof: By part b) of Corollary 3.34 of [K], K admits a left exponential. Since K is countable, so is G. Now the argument is the same as in the proof of the preceding corollary. $\square$

It now remains to consider right exponentials in the countable case. By Corollary 3.16 of [K], we know that there exists an isomorphism $f: I_v \to 1+I_v$ if K is countable and root closed for positive elements. But we want to realize additional conditions for that isomorphism.

Lemma 3.5 *Suppose that K is root closed for positive elements. Let $\mathcal{B}$ be a subset of I_v, $n \geq 1$ and*

$$\mathcal{B}_n = \{E_n(a) \mid a \in \mathcal{B}\} \subset 1 + I_v \; .$$

Then $\mathcal{B}$ is valuation independent in I_v if and only if $\mathcal{B}_n$ is valuation independent in $1+I_v$, and $\mathcal{B}$ is maximal with this property if and only if $\mathcal{B}_n$ is. If $\mathcal{B}$ is valuation independent, then the map

$$\tilde{f}_n : \mathcal{B} \ni a \mapsto E_n(a) \in \mathcal{B}_n$$

extends additively to an order preserving isomorphism f_n from $\langle\mathcal{B}\rangle$ onto $\langle\mathcal{B}_n\rangle$ satisfying $Q_n(a, f_n(a))$ for all $a \in \langle\mathcal{B}\rangle$.

Proof: Taking over the notation from [K], for "the smallest convex subgroup containing x" resp. "the biggest convex subgroup not containing x" we will write C_x resp. D_x for elements x in the additive group of K, and $\mathbf{C}_x$ resp. $\mathbf{D}_x$ for x in the multiplicative group of positive elements of K.

First note that for all $n \geq 1$ and all $a \in \mathcal{B}$,

$$w(1+a) = w\left(E_n(a)\right) \quad (\text{hence, } \mathbf{D}_{1+a} = \mathbf{D}_{E_n(a)}\,)$$

and moreover,

$$(1+a)\cdot \mathbf{D}_{1+a} = E_n(a)\cdot \mathbf{D}_{1+a} \quad (\text{i.e., } w\left(\frac{1+a}{E_n(a)}\right) > w(1+a)\,) \; .$$

In fact, by Corollary 3.13 of [K],

$$w(1+a) = v(a) = v\left(\sum_{i=1}^{n}\frac{a^i}{i!}\right) = w\left(1+\sum_{i=1}^{n}\frac{a^i}{i!}\right) = w(E_n(a))$$

and

$$\begin{aligned} w\left(\frac{1+a}{E_n(a)}\right) &= v\left(\frac{1+a}{E_n(a)}-1\right) = v\left(\frac{1+a-E_n(a)}{E_n(a)}\right) \\ &= v\left(\sum_{i=2}^{n}\frac{a^i}{i!}\right) - v\left(E_n(a)\right) = 2v(a) \\ &> v(a) = w(1+a) \; . \end{aligned}$$

So by Proposition 2.10 of [K] we see that for all $n \geq 1$, $\mathcal{B}_n$ is valuation independent in $1+I_v$ if and only if $\mathcal{B}_1$ is. Similarly, by Corollary 2.11 of [K] we see that $\mathcal{B}_n$ is maximal with this property if and only if $\mathcal{B}_1$ is. So in order to prove the first assertion of our present corollary, we may assume w.l.o.g. that $n=1$.

Now note that for given $a_1, \ldots, a_m \in \mathcal{B}$ we have: $v(a_1) = \ldots = v(a_m)$ if and only if $w(1+a_1) = \ldots = w(1+a_m)$. Moreover, in that case we have: $a_1 + D_{a_1}, \ldots, a_m + D_{a_m}$ are $\mathbb{Q}$–independent in C_{a_1}/D_{a_1} if and only if $(1+a_1)\cdot \mathbf{D}_{a_1}, \ldots, (1+a_m)\cdot \mathbf{D}_{a_m}$ are $\mathbb{Q}$–independent in $\mathbf{C}_{1+a_1}/\mathbf{D}_{1+a_1}$. Indeed, this last statement is true because the map

$$\begin{array}{rcl} \phi_{1+a_1} : \mathbf{C}_{1+a_1}/\mathbf{D}_{1+a_1} & \to & C_{a_1}/D_{a_1} \\ c \cdot \mathbf{D}_{1+a_1} & \mapsto & (c-1) + D_{a_1} \end{array}$$

is an order preserving isomorphism (cf. Lemma 3.14 of [K]). Hence, Proposition 2.10 of [K] shows that $\mathcal{B}$ is valuation independent if and only if $\mathcal{B}_1$ is, and Corollary 2.11 of [K] shows that $\mathcal{B}$ is maximal with this property if and only if $\mathcal{B}_1$ is.

Finally, assume that $\mathcal{B}$ is valuation independent. Since $v(a) = w\left(E_n(a)\right)$, we know by Lemma 2.13 of [K] that f_n extends linearly to a valuation preserving isomorphism $\tilde{f}_n : \langle \mathcal{B} \rangle \to \langle \mathcal{B}_n \rangle$. By Lemma 2.2, $Q_n(a, \tilde{f}_n(a))$ holds for all $a \in \langle \mathcal{B} \rangle$. Consequently, if $a \in \langle \mathcal{B} \rangle$, $a > 0$, then by virtue of part i) of Lemma 2.1, we have $f_n(a) > E_{n-1}(a) \geq 1$, which shows that f_n preserves the order. $\square$

Theorem 3.6 *Suppose that K is a countable field, root closed for positive elements. Then for $n = 1, 2$, there exists a right exponential $f_n : I_v \to 1 + I_v$ and a valuation basis $\{a_j \mid j \in \mathbb{N}\}$ of I_v such that:*

(1) $f_n(a_j) = E_n(a_j)$ for all $j \in \mathbb{N}$,

(2) $\{f_n(a_j) \mid j \in \mathbb{N}\}$ is a valuation basis of $1 + I_v$,

(3) $Q_n(x, f_n(x))$ holds for all $x \in I_v$, $x \neq 0$ and thus, f_n satisfies $\forall x \in I_v : T_{R,n-1}(f_n, x)$.

If in addition, K is henselian for its natural valuation v (which in particular is the case if K is real closed), then there is such a right exponential f_n for every n.

Proof: We will construct the required isomorphism by a back and forth procedure. Since K is assumed to be countable, I_v admits a countable basis $\{a'_j \mid j \in \mathbb{N}\}$ and $1 + I_v$ admits a countable basis $\{b'_j \mid j \in \mathbb{N}\}$. Given $m \geq 0$, assume that we have already constructed an isomorphism $f_{n,m}$ between an m–dimensional subvector space U_m of I_v and a subvector space V_m of $1 + I_v$ and a v–valuation basis $\{a_1, \ldots, a_m\}$ of U_m such that

$(1)_m$ $f_{n,m}(a_j) = E_n(a_j)$ for $1 \leq j \leq m$,

$(2)_m$ $\{f_{n,m}(a_j) \mid 1 \leq j \leq m\}$ is a w–valuation basis of V_m,

$(3)_m$ $Q_n(x, f_{n,m}(x))$ holds for all $x \in U_m$, $x \neq 0$.

By Lemma 3.5, conditions $(2)_m$ and $(3)_m$ both follow from $(1)_m$. By convention, for $m = 0$ we set $U_0 = \{0\}$ and $V_0 = \{1\}$. If m is even, then let a be the basis element a'_j of smallest index j such that $a'_j \notin U_m$. If m is odd, then let b be the basis element b'_j of smallest index j such that $b'_j \notin V_m$. We want to construct a prolongation $f_{n,m+1}$ of $f_{n,m}$ to $U_m + \mathbb{Q}a$ resp. of $f_{n,m}^{-1}$ to $V_m \cdot b^{\mathbb{Q}}$ still satisfying condition $(1)_{m+1}$ on the new domains. We may use the following well known fact: *if W is a finite dimensional subvector space of a valued K–vector space and $a \in V$, then every valuation basis $\mathcal{B}$ of W can be extended to a valuation basis of $W + Ka$* (cf. Lemma 2.12 of [K]). So we may extend the v–valuation basis $\{a_1, \ldots, a_m\}$ of U_m to a v–valuation basis $\{a_1, \ldots, a_m, a_{m+1}\}$ of $U_m + \mathbb{Q}a$ (respectively, the w–valuation basis $\{f_{n,m}(a_1), \ldots, f_{n,m}(a_m)\}$ of V_m to a w–valuation basis $\{f_{n,m}(a_1), \ldots, f_{n,m}(a_m), b_{m+1}\}$ of $V_m \cdot b^{\mathbb{Q}}$).

If m is even, we take $f_{n,m+1}(a_{m+1})$ to be equal to $E_n(a_{m+1})$, which we will call d. If m is odd, we are looking for an element $c \in I_v$ such that $E_n(c) = b_{m+1}$. For $n = 1$, this is just $c = b_{m+1} - 1$. For $n = 2$, our task requires to solve an equation

$$X^2 + 2X + 2(1 - b_{m+1}) = 0$$

which is always solvable in the root–closed field K since $1 - 2(1 - b_{m+1}) = 2b_{m+1} - 1 > 0$ in view of $b_{m+1} \in 1 + I_v$. If $n > 2$, the equation

$$\sum_{i=0}^{n} \frac{1}{i!} X^i - b_{m+1} = 0$$

is still solvable if the natural valuation on K is henselian. Indeed, $v(1 - b_{m+1}) > 0$ and hence, the above equation will then admit a root c whose residue is 0, that is, $c \in I_v$.

Now note that by Lemma 3.5, $\{a_1, \ldots, a_m, c\}$ is again v–valuation independent (resp. $\{f_{n,m}(a_1), \ldots, f_{n,m}(a_m), d\}$ is again w–valuation independent). Hence, we set $a_{m+1} := c$ resp. $b_{m+1} := d$ and obtain in both cases that $f_{n,m}(a_{m+1}) = b_{m+1}$. So indeed, we are able to extend $f_{n,m}$ to an isomorphism $f_{n,m+1}$ such that $a \in \operatorname{dom} f_{n,m+1}$ resp. $b \in \operatorname{im} f_{n,m+1}$.

We set $f_n = \bigcup_{m \in \mathbb{N}} f_{n,m}$. Since by our back and forth construction, every a'_j is contained in some U_m and every b'_j is contained in some V_m, we find that f_n is an isomorphism from I_v onto $1 + I_v$. Since for every $m \in \mathbb{N}$, the isomorphism $f_{n,m}$ has the properties $(1)_{m+1}$, $(2)_{m+1}$ and $(3)_{m+1}$, the induced isomorphism $f_{n,m}$ has the properties (1), (2) and (3). Finally, since every finite subset $\{a_1, \ldots, a_m\}$ of $\{a_j \mid j \in \mathbb{N}\}$ is a valuation basis of U_m, the set $\{a_j \mid j \in \mathbb{N}\}$ itself is a valuation basis of $I_v = \bigcup_{m \in \mathbb{N}} U_m$. $\square$

Finally, let us put left, middle and right together.

Theorem 3.7 *Let K be a countable nonarchimedean ordered field, root closed for positive elements. Assume that its value group G is isomorphic to $\coprod_{\mathbb{Q}} (\overline{K}, +, 0, <)$ and that its residue field $\overline{K}$ admits a strong exponential e. Then K admits an exponential f satisfying $\forall x : T_1(f, x)$ and $\overline{f} = e$.*

If in addition, K is henselian for its natural valuation, then for every fixed $n \geq 1$, K admits an exponential f satisfying $\forall x : T_n(f, x)$ and $\overline{f} = e$.

Conversely, if K admits an exponential satisfying $\forall x : T_n(f, x)$ for some $n \geq 1$, then $G \simeq \coprod_{\mathbb{Q}} (\overline{K}, +, 0, <)$ and $\overline{K}$ admits a strong exponential.

Proof: We may assume the decompositions (1) and (2) of the hypothesis of Lemma 2.19, the latter because by hypothesis, K is root closed for positive elements. If $G \simeq \coprod_{\mathbb{Q}} \overline{K}$, then by Corollary 3.4, K admits a left strong exponential

f_L. Further, the exponential e of $\overline{K}$ induces a middle exponential e' (cf. the proof of Theorem 2.20). Since e is a strong exponential, Theorem 2.12 yields that e' satisfies $\forall x \in R_v \setminus I_v : T(e', x)$. From Theorem 3.6 we infer the existence of a right exponential f_R which satisfies $\forall x \in I_v : T_{R,1}(f_n, x)$. Under the additional assumption that K be henselian for its natural valuation, we may replace $T_{R,1}$ by $T_{R,n}$ for an arbitrary fixed n. By virtue of Lemma 2.19, the resulting exponential $f = f_L \amalg e' \amalg f_R$ will satisfy $T_1(f, x)$ respect. $T_n(f, x)$ on K, as well as $\overline{f} = e$.

For the converse, assume that K admits an exponential f satisfying $\forall x : T_n(f, x)$ for some $n \geq 1$. Then f induces a left strong exponential and thus, G is a strong exponential group by Theorem 2.7. On the other hand, $\overline{f}$ is a strong exponential on $\overline{K}$ by virtue of Theorem 2.12. $\square$

Let us mention that the last theorem is the best that can be expected, in the following sense. If we would try to obtain a strong exponential, we would have to construct a right strong exponential, that is, an exponential f_R satisfying $\forall x \in I_v : Q_n(x, f_R(x))$ simultaneously for all $n \in \mathbb{N}$. With our approach, the existence would only be guaranteed if we would ensure some convergence, but this usually contradicts the countability condition. Certainly, countable fields with strong exponentials can be constructed in a different way, but not in the sense of giving a criterion for a countable field to admit a strong exponential.

4 Contraction groups

The disadvantage of the group exponential φ that we have used so far is that it is not a map from G to G. Instead, it is an isomorphism between $G^{<0}$ and the value set $\Gamma^- = v_G(G \setminus \{0\})$. But if we compose its inverse with the natural valuation $v_G : G \to v_G(G)$, then we obtain a map $\varphi^{-1} \circ v_G : G \setminus \{0\} \to G^{<0}$. This map is onto $G^{<0}$. Further, it contracts archimedean classes, that is, archimedean comparable elements are sent to the same element; this follows since v_G already has the same property by definition.

The map $\varphi^{-1} \circ v_G$ has an obvious disadvantage: its range is only $G^{<0}$, and it shows no symmetry between the positive and the negative part of G. Although this corresponds quite well to the behaviour of the exponential, it appears to us that the idea of a map contracting archimedean classes is better expressed if it shows symmetry between positive and negative. To achieve symmetry, we may set

$$\chi_\varphi(g) = \begin{cases} \varphi^{-1} \circ v_G(g) \in G^{<0} & \text{if } g \in G^{<0} \\ -\varphi^{-1} \circ v_G(g) \in G^{>0} & \text{if } g \in G^{>0} \\ 0 & \text{if } g = 0\,. \end{cases}$$

Now, we have $\chi_\varphi(-g) = -\chi_\varphi(g)$, and χ_φ is a surjective map from G onto G. On $G^{<0}$, the map v_G preserves $\leq$, and on $G^{>0}$, it reverses $\leq$. Since φ is order preserving and $\alpha \mapsto -\alpha$ is order reversing, it follows that χ_φ preserves $\leq$ on all of G. Note that $\chi_\varphi(g) = 0 \Leftrightarrow g = 0$.

Let us now extract an axiomatization from the structure that we have derived so far. Let G be an ordered abelian group and χ a map from G into G. Then χ will be called a *contraction* if it satisfies the following axioms:

(C0) $\chi(g) = 0 \Leftrightarrow g = 0$,
(C1) χ is surjective,
(C2) χ preserves $\leq$,
(C3) $\chi(-g) = -\chi(g)$,
(C4) if g is archimedean equivalent to g' and $\mathrm{sign}(g) = \mathrm{sign}(g')$, then $\chi(g) = \chi(g')$.
Note that by axioms (C2) and (C1) it follows that $\chi(G^{<0}) = G^{<0}$ and $\chi(G^{>0}) = G^{>0}$.

In the language of ordered abelian groups, the last axiom is not an elementary sentence, but it is equivalent to a (recursive) axiom scheme; in the presence of axiom (C3), it suffices to state it for the positive elements of G:

(C4$'$) $\forall x, y: \ \ x \geq y > 0 \wedge ny \geq x \ \rightarrow \ \chi(x) = \chi(y) \qquad (n \in I\!N)$

We will call χ a natural contraction if $\chi(x) = \chi(y)$ implies that x and y are archimedean equivalent. This notion is not elementary: by general model theory, it can be shown that in every $\aleph_0$–saturated model, the contraction will contract elements that are not archimedean equivalent.

The same definition of a contraction works for an ordered vector space. We have seen that the value group G of an exponential field K is actually an ordered $\overline{K}$–vector space. Here, $\overline{K}$ was an archimedean field, and in this case, axiom scheme (C4$'$) is equivalent to

(C4$''$) $\forall x, y: \ \ x \geq y > 0 \wedge ky \geq x \ \rightarrow \ \chi(x) = \chi(y) \qquad (k \in \overline{K})$

However, if exponential fields with a predicate for a (compatible) valuation are considered, then it is not possible to axiomatize elementarily that this valuation be the natural one, and the residue field will in general not be archimedean. In a multisorted language with a sort for the residue field, (C4$''$) will then express that χ contracts $\overline{K}$–archimedean classes (which are larger than the natural archimedean classes if, and only if, $\overline{K}$ is nonarchimedean). For this case, (C4$''$) turns out to be stronger than (C4$'$). The language with a sort for the residue field has the advantage that we can express that χ contracts *precisely* the $\overline{K}$–archimedean classes:

(C4$'''$) $\forall x, y: \ \ x \geq y > 0 \ \rightarrow \ (\, \chi(x) = \chi(y) \leftrightarrow \exists k \in \overline{K} : ky \geq x \,)$.

Since a contraction χ sends archimedean equivalent elements of the same sign to the same element, it induces a map ρ_χ from the rank of G into $G^{<0}$ through $v_G(g) \mapsto \chi(g)$ for $g \in G^{<0}$. Since χ is surjective, the same is true for ρ_χ. Further, ρ_χ is an isomorphism if and only if χ is a natural contraction. If ρ_χ is an isomorphism, then its inverse is a group exponential φ such that $\chi = \chi_\varphi$. On the other hand, if $\chi = \chi_\varphi$ is induced by the group exponential φ, then χ is a natural contraction such that $\rho_\chi^{-1} = \varphi$. Hence, the concept of a group exponential and that of a natural contraction are interchangeable. If $\chi = \chi_\varphi$ is induced by the group exponential $\varphi = \varphi_f$ which in turn is induced by the (left) exponential f, then χ_f may be given as follows: $\chi_f(0) = 0$ and

$$\chi_f(v(f(b))) = \begin{cases} v(b) & \text{if } b > 0 \\ -v(b) & \text{if } b < 0 \end{cases}$$

where b runs through all elements $b \in \mathrm{dom}(f)$ of value $v(b) < 0$.

More generally, we may consider the $\overline{K}$–natural valuation of an ordered $\overline{K}$–vector space whose value set is represented by the $\overline{K}$–archimedean classes; it may be viewed as a coarsening of the natural valuation. If a contraction on this vector space contracts precisely the $\overline{K}$–archimedean classes (i.e., it satisfies (C4$'''$)), then χ induces an isomorphism $\rho_\chi^{\overline{K}}$ between the value set of the $\overline{K}$–natural valuation and $G^{<0}$. Let us also mention that from every contraction χ, a valuation v_χ can be defined elementarily, having the property that two elements have the same v_χ–value if and only if they have the same image under χ, up a change of the sign. Again, this valuation may be considered a coarsening of the natural valuation.

Up to this point, we have not yet considered strong exponential groups. If the group exponential φ satisfies $\forall x \in G^{<0} : \varphi(x) < v_G(x)$, then it follows that

$$\forall x \in G^{<0} : x < \varphi^{-1} \circ v_G(x) = \chi(x)$$

which by axiom (C3) implies $\forall x \in G^{>0} : x > \chi(x)$. That is, χ maps towards the center of the ordered group (which is the element 0). This gives rise to the following definitions: a contraction χ is *centripetal*, if it satisfies
(CP) $\forall x \in G : \; |x| > |\chi(x)|$,
and it will be called *centrifugal*, if it satisfies
(CF) $\forall x \in G : \; |x| < |\chi(x)|$.

The proof of the following observation is straightforward:

Lemma 4.1 *Let (G,φ) be an exponential group and χ_φ the contraction induced by φ. Then:*

a) χ is centripetal if and only if (G,φ) is a strong exponential group.

b) χ is centrifugal if and only if φ satisfies

$$\forall x \in G^{<0} : \varphi(x) > v_G(x) \,. \tag{22}$$

In view of this lemma, Proposition 3.1 has shown that the group $\coprod_{\mathbb{Q}} A$ may be endowed with a group exponential inducing a centripetal contraction. By Remark 3.2, it also admits a group exponential inducing a centrifugal contraction, as well as a group exponential inducing a contraction which is neither centrifugal nor centripetal. This shows that axioms (CP) and (CF) are independent of the other contraction axioms.

Property (22) reflects a quite strange behaviour of an exponential. Indeed, if $\varphi = \varphi_f$ is the group exponential induced by an exponential (or a left exponential) f on K, then (22) is equivalent to

$$\forall x \in K : \;\; v(x) < 0 \;\rightarrow\; v_G(v(f(x))) > v_G(v(x)) \,.$$

This in turn means that for infinitely big $a \in K$ we have that $f(a) \lessdot a$, that is, $f(a)$ is smaller than any root of a. In the presence of a predicate for the natural valuation, it is possible to axiomatize such strange exponential fields where the exponential induces the usual exponential on the “finite” part of the field (the convex hull of $\mathbb{Q}$, which is the valuation ring of the natural valuation), but “reverses” its behaviour on some “infinite” part of the field (some infinitely big elements). However, since the natural valuation cannot be axiomatized elementarily, it is not possible to axiomatize elementarily a class of exponential fields where this reversed behaviour is shown at *all* infinitely big elements.

The model theory of divisible ordered abelian groups with contraction will is studied in [KF1] and [KF2].

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