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http://math.usask.ca/~skuhlman/slidesmaltsev2010.pdf

Valued Differential Fields.

Joint work with M. Matusinski

I. Motivation

I.1 Ax - Kochen Ershov Principles for Valued Fields.

Let K be a field and $(\Gamma, \preceq)$ a totally ordered abelian group (written multiplicatively). A surjetive map

$$v \;:\; K^{\times} \to \Gamma$$

is a **field valuation** if for all $a, b \in K^{\times}$:

$v(a.b) = v(a).v(b)$ (homomorphism)

$v(a+b) \preceq \max\{v(a), v(b)\}$ (ultrametric inequality).

$K_v := \{a \in K \mid v(a) \preceq 1\}$ is the **valuation ring** of K

$I_v := \{a \in K \mid v(a) \prec 1\}$ the **maximal ideal** of K_v.

$v(K) := \Gamma$ is the **value group** (also: monomials group)

$K_v/I_v := \overline{K}$ is the residue field.

$v(K)$ and $\overline{K}$ are important invariants of a valued field:

AKE Transfer Principle:

Let K and L be two valued fields
(*plus additional conditions*).
Assume that:

$\overline{K}$ is elementarily equivalent to $\overline{L}$

$v(K)$ is elementarily equivalent to $v(L)$.

Then K is elementarily equivalent to L (?)

If in addition L is an extension of K, one can replace: "elementarily equivalent" by "elementary substructure" or " existencially closed" in the above query.

I.2. Kaplansky Embedding Theorem for Valued Fields.

Theorem: Let K be a valued field with char (K)=char$(\overline{K})$. Then K is analytically isomorphic to a subfield of a suitable generalized series field.

Let k be a (coefficients) field and $(\Gamma, \preceq)$ a totally ordered abelian (monomials) group.

$K = k\left((\Gamma)\right)$ denotes the **generalised series field**. It is the set of maps

$$\begin{array}{rcccc} a & : & \Gamma & \rightarrow & k \\ & & \alpha & \mapsto & a_\alpha \end{array}$$

such that Supp $a = \{\alpha \in \Gamma \mid a_\alpha \neq 0\}$ is anti-well-ordered in Γ.

We write these maps $a = \sum_{\alpha \in \mathrm{Supp}\ a} a_\alpha \alpha$.

This set provided with component-wise sum and the following convolution product

$$\Big(\sum_{\alpha \in \text{Supp } a} a_\alpha \alpha\Big) \Big(\sum_{\beta \in \text{Supp } b} b_\beta \beta\Big) = \sum_{\gamma \in \Gamma} \Big(\sum_{\alpha\beta=\gamma} a_\alpha b_\beta\Big) \gamma$$

is a field.

For any series $0 \neq a$, we define its **leading monomial**:

$$\text{LM } (a) := \max (\text{Supp } a) \in \Gamma .$$

The map

$$\text{LM} \ : \ K^\times \to \Gamma$$

is the canonical valuation on K.

E.g. $\Gamma = \{x^z \ ; \ z \in \mathbb{Z}\}$ (respectively $\Gamma = \{x^z \ ; \ z \in \mathbb{R}\}$) gives:

$\mathbb{R}((\Gamma))$ the Laurent series field (respectively the Levi-Civita series field).

• We have classification invariants and universal domains.

• What if the valued fields carry additional structure? Additional structure induced on the value group and residue field. AKE in this framework?

• In particular, generalised series fields are suitable domains for the study of real algebra.

Are they suitable domains for the study of real differential algebra ?

This work is the first step in this project:

Endow $K := \mathbb{R}((\Gamma))$ *with derivations.*

I.3. Hardy fields. The set of germs at infinity of real valued functions of a real variable forms a ring under pointwise addition and multiplication of germs.

A **Hardy field** is a subfield closed under differentiation of germs.

A Hardy field H carries a natural valuation:

$$H_v := \{f \in H\ ;\ \lim_{x\to\infty} f \in \mathbb{R}\}$$

.

Hardy fields are prime examples of valued differential fields.

II. Defining Derivations.

II.1. Hahn groups as monomial groups. Let $(\Phi, \preceq)$ be a totally ordered set, that we call the set of **fundamental monomials**.

Consider the set Γ of formal products $\gamma \in \Gamma$ of the form

$$\gamma = \prod_{\phi \in \Phi} \phi^{\gamma_\phi}$$

where $\gamma_\phi \in \mathbb{R}$, and the support of γ

$$\operatorname{supp} \gamma := \{\phi \in \Phi \mid \gamma_\phi \neq 0\}$$

is an anti-well-ordered subset of Φ.

Multiplication of formal products is defined pointwise: for $\alpha, \beta \in \Gamma$

$$\alpha\beta = \prod_{\phi \in \Phi} \phi^{\alpha_\phi + \beta_\phi}$$

Γ is an abelian group with identity 1 (the product with empty support).

We endow Γ with the anti lexicographic ordering $\preceq$ which extends $\preceq$ of Φ:

$$\gamma \succ 1 \text{ if and only if } \gamma_\phi > 0\,, \text{ for } \phi := \max(\text{supp}\ \gamma)\,.$$

The **leading fundamental monomial** of $1 \neq \gamma \in \Gamma$ is $\mathrm{LF}(\gamma) := \max(\text{supp}\ \gamma)$.

Γ is a totally ordered abelian group, the **Hahn group of generalised monic monomials**.

Hahn's Embedding Theorem: Hahn groups are universal domains.

II.2. Summable Families of Series.

We want to differentiate

$$a = \sum_{\alpha \in \Gamma} a_\alpha \alpha$$

term by term.

There are two problems:

(i) we first have to know how to differentiate a monomial $\alpha \in \Gamma$,

(ii) then we have to make sense of

$$a' = \sum_{\alpha \in \Gamma} a_\alpha \alpha'$$

a possibly infinite sum of field elements.

sometimes it is possible, but it can go wrong. Easy examples.

Let I be an infinite index set and $\mathcal{F} = \{a_i\ ;\ i \in I\}$ be a family of series in K. $\mathcal{F}$ is said to be **summable** if:

(SF1) Supp $\mathcal{F} := \bigcup\limits_{i \in I}$ Supp a_i (the support of the family) is an anti-well-ordered subset of Γ.

(SF2) For any $\alpha \in$ Supp $\mathcal{F}$, the set

$$S_\alpha := \{i \in I \mid \alpha \in \text{Supp } a_i\} \subseteq I$$

is finite.

Write $a_i = \sum\limits_{\alpha \in \Gamma} a_{i,\alpha}\alpha$, and assume that $\mathcal{F} = (a_i)_{i \in I}$ is summable. Then

$$\sum_{i \in I} a_i := \sum_{\alpha \in \text{Supp } \mathcal{F}} \Big(\sum_{i \in S_\alpha} a_{i,\alpha} \Big)\alpha$$

is a well defined element of K that we call the **sum** of $\mathcal{F}$.

II.3 Series derivations.

Let

$$\begin{array}{rccc} d_\Phi & : \ \Phi & \rightarrow & K\backslash\{0\} \\ & \phi & \mapsto & \phi' \end{array}$$

be a map.

We say d_Φ **extends to a series derivation on** Γ if the following property holds:

(SD1) For any anti-well-ordered subset $E \subset \Phi$,

the family $\left(\frac{\phi'}{\phi}\right)_{\phi \in E}$ is summable.

Then the **series derivation** d_Γ on Γ (extending d_Φ) is defined to be the map

$$d_\Gamma : \ \Gamma \rightarrow K$$

obtained through the following axioms:

•[(D0)] $1' = 0$

• [(D1) Strong Leibniz rule:]

If $\alpha = \prod_{\phi \in \mathrm{supp}\ \alpha} \phi^{\alpha_\phi}$ then $(\alpha)' = \alpha \sum_{\phi \in \mathrm{supp} \alpha} \alpha_\phi \frac{\phi'}{\phi}$.

We say that a series derivation d_Γ on Γ **extends to a series derivation on** K if the following property holds:

(SD2) For any anti-well-ordered subset $E \subset \Gamma$,

the family $(\alpha')_{\alpha \in E}$ is summable.

Then the **series derivation** d on K (extending d_Γ) is defined to be the map

$$d \ : \ K \rightarrow K$$

obtained through the following axiom:

(D2) Strong linearity:

$$\text{If } a = \sum_{\alpha \in \mathrm{Supp}\ a} a_\alpha \alpha, \text{ then } a' = \sum_{\alpha \in \mathrm{Supp}\ a} a_\alpha \alpha'.$$

We now study necessary and sufficient condition on the map d_Φ so that properties (SD1) and (SD2) hold.

II.4 Sequential Characterization Summability.

We use the following two key observations:

(i) $\mathcal{F}$ is summable if and only if every countably infinite subfamily is summable.

(ii) (Infinite Ramsey.) Let Γ be a totally ordered set. Every sequence $(\alpha_n)_{n\in\mathbb{N}}$ in Γ has an infinite subsequence which is either constant, or strictly increasing, or strictly decreasing.

We isolate the following two crucial "bad" hypotheses:

(H1) There exists a strictly decreasing sequence $(\phi_n)_{n\in\mathbb{N}}$ in Φ and an increasing sequence $(\tau^{(n)})_{n\in\mathbb{N}}$ in Γ such that $\tau^{(n)} \in \operatorname{Supp} \dfrac{\phi'_n}{\phi_n}$ for all $n \in \mathbb{N}$.

(H2) There exist strictly increasing sequences $(\phi_n)_{n\in\mathbb{N}}$ in Φ and $(\tau^{(n)})_{n\in\mathbb{N}}$ in Γ such that $\tau^{(n)} \in \operatorname{Supp} \dfrac{\phi'_n}{\phi_n}$ and $\operatorname{LF}\left(\dfrac{\tau^{(n+1)}}{\tau^{(n)}}\right) \succeq \phi_{n+1}$, for all $n \in \mathbb{N}$,

Theorem A: A map $d_\Phi : \Phi \to K\backslash\{0\}$ extends to a series derivation on K if and only $(H1)$ and $(H2)$ fail.

III. Hardy Type Derivations.

Let K be a valued field.

Notation: For $a, b \in K$ set

$$a \preceq b \text{ if and only if } v(a) \leq v(b)$$

and

$$a \asymp b \text{ if and only if } v(a) = v(b)\ .$$

Assume that K contains a sub-field $\mathcal{C}$ isomorphic to its residue field $\overline{K}$.

Let d be a derivation on K.

$d\ :\ K \to K$ is a **Hardy type derivation** if :

• The **sub-field of constants** of K is $\mathcal{C}$:

$$\forall a \in K,\ a' = 0 \Leftrightarrow a \in \mathcal{C}\ .$$

• d verifies **l'Hospital's rule**:

$\forall a, b \in K\backslash\{0\}$ with a, b not aymptotic to 1, we have

$$a \preceq b \Leftrightarrow a' \preceq b'\ .$$

• The logarithmic derivative is **compatible with the valuation**:

$\forall a, b \in K\ \ with\ |a| \succ |b| \succ 1, we\ have \dfrac{a'}{a} \succeq \dfrac{b'}{b}.$

Set $\theta^{(\phi)}\ :=\ \ \mathrm{LM}\ (\phi'/\phi)$.

Theorem B: A series derivation d on K verifies l'Hospital rule and is compatible with the logarithmic derivative if and only if the following condition holds:

(H3') : $\forall \phi \prec \psi \in \Phi$, $\theta^{(\phi)} \prec \theta^{(\psi)}$ and $\ \mathrm{LF}\ \left(\dfrac{\theta^{(\phi)}}{\theta^{(\psi)}}\right) \prec \psi$.

IV. Example.

Take the following chain of infinitely increasing real germs at infinity (applying the usual comparison relations of germs):

$$\Phi := \{\exp_n(x)\ ;\ n \in \mathbb{Z}\}$$

where $\exp_n$ denotes for positive n, the n'th iteration of the real exponential function, for negative n, the $|n|$'s iteration of the logarithmic function, and for $n = 0$ the identity map.

Applying the usual derivation on real germs, we obtain:

$$\begin{cases} \dfrac{(\exp_n(x))'}{\exp_n(x)} = \Pi_{k=1}^{n-1} \exp_k(x) & \text{if } n \geq 2 \\ \dfrac{(\exp(x))'}{\exp(x)} = 1 & \\ \dfrac{(\exp_n(x))'}{\exp_n(x)} = \Pi_{k=0}^{n} \dfrac{1}{\exp_k(x)} & \text{if } n \leq 0 \end{cases}$$

So for any integers $m < n$, we have:

- $\exp_m(x) \prec \exp_n(x)$
- $\dfrac{(\exp_m(x))'}{\exp_m(x)} \prec \dfrac{(\exp_n(x))'}{\exp_n(x)}$
- $\exp_{n-1}(x) = \text{ LF } (\dfrac{(\exp_m(x))'/\exp_m(x)}{(\exp_n(x))'/\exp_n(x)}) \prec \exp_n(x).$

The map $\exp_n(x) \mapsto (\exp_n(x))'$ extends to a series derivation of Hardy type on K.

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