POSITIVITY, SUMS OF SQUARES AND THE MOMENT PROBLEM

1. Two Representation Problems

(I) Positive Semidefinite Polynomials and Sums of Squares.

Let $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$ denote the polynomial $\mathbb{R}$-algebra.

Let $f \in \mathbb{R}[\underline{X}]$ be *positive semidefinite* (Psd), i.e. f is non-negative on $\mathbb{R}^n$.

• Is $f \in \sum \mathbb{R}[\underline{X}]^2$ (SOS)?

For $d, n \geq 1$ let $P_{d,n}$: = positive semidefinite forms of degree d in n variables, and $\sum_{d,n} \subseteq P_{d,n}$ the subset consisting of sums of squares.

• **Hilbert** (1888) proved: *For d even, $P_{d,n} = \sum_{d,n}$ if and only if $n \leq 2$ or $d = 2$ or ($n = 3$ and $d = 4$).*

• **Hilbert's 17th Problem:** Let $f \in \mathbb{R}[\underline{X}]$ be Psd, is f SOS of rational functions?

• **Artin-Schreier** (1927) give a positive solution.

• **Tarski** (1930) publishes his Transfer Principle.

• **Tarski-Seidenberg:** *The projection of a semi-algebraic set is semi-algebraic.*

• **Krivine** (1964) and **Stengle** (1974) **Positivstellensatz**: use Tarski-Transfer to give a more precise representation of positive polynomials on semialgebraic sets.

Let $K \subseteq \mathbb{R}^n$ and let $\text{Psd}(K)$ denote the set of nonnegative polynomials on K.

$K \subseteq \mathbb{R}^n$ is *basic closed semialgebraic* if there exists a finite set of polynomials $S = \{g_1, \dots, g_s\}$ such that

$$K = K_S := \{x \in \mathbb{R}^n \ : \ g_i(x) \geq 0, \ i = 1, \dots, s\} .$$

Such a finite S is a *description* of K.

A subset $C \subseteq \mathbb{R}[\underline{X}]$ is *convex* if for every $x, y \in C$ and $\lambda \in [0,1]$, $\lambda x + (1 - \lambda) y \in C$.

A subset $C \subseteq \mathbb{R}[\underline{X}]$ is a *cone* if $C + C \subseteq C$ and $\mathbb{R}^+ C \subseteq C$. A cone is convex.

A cone M of $\mathbb{R}[\underline{X}]$ is a *quadratic module* if $1 \in M$, and for each $h \in \mathbb{R}[\underline{X}]$, $h^2 M \subseteq M$.

For $S = \{g_1, \dots, g_s\}$, let

$$M_S := \{\sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[\underline{X}]^2 \text{ for } i = 0, \dots, s \text{ and } g_0 = 1\}.$$

M_S is the smallest (here, finitely generated) quadratic module of $\mathbb{R}[\underline{X}]$ containing S. Clearly $M_S \subseteq \text{Psd}(K_S)$.

• **Positivstellensatz:** *Let* $S \subset \mathbb{R}[\underline{X}]$ *finite,* K_S *and* M_S *as above,* $f \in \mathbb{R}[\underline{X}]$*. Then:* $f > 0$ *on* K *if and only if there exist* $p, q \in M_S$ *such that* $pf = 1 + q$*.*

• **Putinar's Archimedean Positivstellensatz:** (1993) Let K be a *compact* basic closed semialgebraic set. Let S be a description of K containing the inequality $N - \sum x_i^2 \geq 0$ expressing that $K := K_S$ is bounded, for some $N \in \mathbb{N}$. In this case: $f > 0$ on K_S implies $f \in M_S$.

• **Jacobi-Prestel** (2001) generalize the Archimedean Positivstellensatz: $\sum \mathbb{R}[\underline{X}]^2$ is replaced by the (proper) cone of sums of $2d$-powers, $\sum \mathbb{R}[\underline{X}]^{2d}$, for any integer $d \geq 1$, and quadratic modules by $\sum \mathbb{R}[\underline{X}]^{2d}$-modules.

The above results have direct applications to the **multi-dimensional moment problem** for semialgebraic sets.

(II) Positive Semidefinite Linear Functionals and Positive Borel Measures.

• Given a closed set $K \subseteq \mathbb{R}^n$, the K-moment problem is the question of when a linear functional $\ell : \mathbb{R}[\underline{X}] \to \mathbb{R}$ is representable as integration with respect to a positive Borel measure on K.

A necessary condition is that $\ell(f) \geq 0$, for $f \in \text{Psd}(K)$.

• **Haviland** (1935) proved this is also sufficient:

For a linear function $\ell : \mathbb{R}[\underline{X}] \to \mathbb{R}$ and a closed set $K \subseteq \mathbb{R}^n$, the following are equivalent:

(i) There exists a positive regular Borel measure μ on K such that,

$$\forall f \in \mathbb{R}[\underline{X}] \quad \ell(f) = \int_K f \; d\mu.$$

(ii) $\forall f \in Psd(K)$ $\ell(f) \geq 0$.

• The main challenge in applying Haviland's Theorem is verifying its condition (ii), indeed in general $\text{Psd}(K)$ is not finitely generated, so Haviland's result may be impractical.

• If K is compact, it follows from Archimedean PSS that nonnegativity of ℓ on $\text{Psd}(K_S)$ is ensured once nonnegativity of ℓ on M_S. Thus one is reduced to checking $s+2$ many systems of inequalities:

$$(1) \qquad \begin{array}{ll} \ell(h^2 g_i) \geq 0 & \text{for } h \in \mathbb{R}[\underline{X}],\ i = 0, \ldots, s+1, \\ g_0 := 1, & g_{s+1} := (N - \sum x_i^2). \end{array}$$

• Thus the K_S - moment problem is "solvable by finitely many SDP-probems". This can be summarized in a **single topological statement**.

2. Locally Convex Topologies.

Biduals are closures.

Set $V := \mathbb{R}[\underline{X}]$. A *locally convex* topology τ on V is a vector space topology which admits a neighbourhood basis of convex open sets at each point.

For a topological vector space (V, τ) denote the set of all τ**-continuous** linear functionals $\ell : V \to \mathbb{R}$ by V^*.

For $C \subseteq V$, let

$$C_\tau^\vee = \{\ell \in V^* : \ell \geq 0 \text{ on } C\}$$

be the *first dual* of C and define the *bidual* of C by

$$C_\tau^{\vee\vee} = \{a \in V : \forall \ell \in C_\tau^\vee,\ \ell(a) \geq 0\}.$$

Separation for Cones: *Suppose that A and B are disjoint nonempty convex sets in V. If A is open, then there exists $\ell \in V^*$ and $\gamma \in \mathbb{R}$ such that $\ell(x) < \gamma \leq \ell(y)$ for every $x \in A$ and $y \in B$. Moreover, if B is a cone, then γ can be taken to be* 0.

Duality: *For any nonempty cone C in (V, τ), $\overline{C}^\tau = C_\tau^{\vee\vee}$.*

Finest locally convex topology: V is of countable infinite dimension. We define the (direct limit) topology φ on V as follows: $U \subseteq V$ is open if and only if $U \cap W$ is open in W for each finite dimensional subspace W of V.

• Then φ is the finest lc topology on V and all linear functionals are φ-continuous.

Back to Putinar:

$$\mathrm{Psd}(K_S) \subseteq \overline{M_S}^\varphi$$

so every linear functional nonnegative on M_S is integration w.r.t. a measure on K_S.

Generalizing to arbitrary locally convex topologies on $\mathbb{R}[\underline{X}]$:

The setting is now a threefold statement about a locally convex topology τ, a closed subset K of $\mathbb{R}^n$, and a cone C in $\mathbb{R}[\underline{X}]$: If

$$\mathrm{Psd}(K) \subseteq \overline{C}^{\tau}$$

then any τ-continuous functional, nonnegative on C, is integration with respect to a positive Borel measure on K.

Berg et Al (1976) for example considered the ℓ_1-norm (in terms of coefficients) on $\mathbb{R}[\underline{X}]$ and showed

$$\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \mathrm{Psd}([-1,1]^n).$$

Thus every ℓ_1**- continuous** linear functional, which is *positive semidefinite* (i.e. $\ell(h^2) \geq 0$ for every $h \in \mathbb{R}[\underline{X}]$) is representable as integration with respect to a positive Borel measure on $\mathrm{Psd}([-1,1]^n)$.

This generalizes to ℓ_p-norms.

Theorem 2.1. *For* $1 \leq p \leq \infty$, $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_p} = Psd([-1,1]^n$.

Corollary 2.2. *Let* $1 \leq p \leq \infty$, *and let* $\ell : \mathbb{R}[\underline{X}] \to \mathbb{R}$ *be a linear functional on* $\mathbb{R}[\underline{X}]$ *such that* $\|(\ell(\underline{X}^\alpha))_{\alpha\in\mathbb{N}^n}\|_q < \infty$ *where* q *is the conjugate of* p. *If* ℓ *is positive semidefinite, then there exists a positive Borel measure* μ *on* $[-1,1]^n$ *such that* $\forall f \in \mathbb{R}[\underline{X}] \quad \ell(f) = \int_{[-1,1]^n} f \ d\mu$.

• And more generally to **weighted** $\ell_{p,r}$**-norm** (in terms of coefficients) on $\mathbb{R}[\underline{X}]$:

Let $r = (r_1, \dots, r_n)$ be a n-tuple of positive real numbers.

For $1 \leq p < \infty$, define

$$\|s\|_{p,r} = (\sum_{\alpha\in\mathbb{N}^n} |s(\alpha)|^p r_1^{\alpha_1} \dots r_n^{\alpha_n})^{\frac{1}{p}}$$

.

For $p = \infty$ define

$$\|s\|_{\infty,r} = \sup_{\alpha\in\mathbb{N}^n} |s(\alpha)| r_1^{\alpha_1} \dots r_n^{\alpha_n}$$

.

Theorem 2.3. *Let* $1 \leq p \leq \infty$. *Then:*

(1) *For* $1 \leq p < \infty$, $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_{p,r}} = Psd(\prod_{i=1}^n [-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}])$.
(2) $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_{\infty,r}} = Psd(\prod_{i=1}^n [-r_i, r_i])$.

Compare to Putinar....

Generalization to Cone of Sums of $2d$-Powers:

Using Jacobi-Prestel Archimedean PSS, we generalize the above Theorem 2.3 with the $\sum \mathbb{R}[\underline{X}]^2$ cone replaced by the cone of sums of $2d$-powers, $\sum \mathbb{R}[\underline{X}]^{2d}$. So the nonnegativity of the linear functional ought to be checked on the strictly smaller cone $\sum \mathbb{R}[\underline{X}]^{2d}$.

Lasserre's Topology:

The above setting has been recently exploited. Lasserre defines the following norm $\|\cdot\|_w$:

$$\|\sum_{s\in\mathbb{N}^n} f_s \underline{X}^s\|_w = \sum_{s\in\mathbb{N}^n} |f_s| w(s),$$

where

$$w(s) = (2\lceil |s|/2 \rceil)!$$

and

$$|s| = |(s_1, \ldots, s_n)| = s_1 + \cdots + s_n$$

.

He proves that for any finite S,

$$\overline{M_S}^{\|\cdot\|_w} = \mathrm{Psd}(K_S)$$

always holds.

Closure of the cone of sums of $2d$-powers in real topological algebras.

We consider the above in a more abstract general setting.

Let R be a commutative $\mathbb{R}$-algebra with 1 and

$$K \subseteq \text{ Hom } (R, \mathbb{R})$$

closed with respect to the product topology. We consider R endowed with the topology T_K, induced by the family of seminorms $\rho_\alpha(a) := |\alpha(a)|$, for $\alpha \in K$ and $a \in R$. In case K is compact, we also consider the topology induced by $\|a\|_K := \sup_{\alpha \in K} |\alpha(a)|$ for $a \in R$. If K is Zariski dense, then those topologies are Hausdorff.

We prove that the closure of the cone of sums of $2d$-powers, with respect to those two topologies is equal to $\text{Psd}K := \{a \in R : \alpha(a) \geq 0, \text{ for all } \alpha \in K\}$. In particular, any continuous linear functional L on the polynomial ring $R = \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$ with $L(h^{2d}) \geq 0$ for each $h \in \mathbb{R}[\underline{X}]$ is integration with respect to a positive Borel measure supported on K. Finally we give necessary and sufficient conditions to ensure the continuity of a linear functional with respect to those two topologies.