Brouwer Fixed Point Theorem in $\left(L^0\right)^d$

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ABSTRACT

The classical Brouwer fixed point theorem states that in a real vector space every continuous function from a compact set on itself has a fixed point. For an arbitrary probability space, let $L^0 = L^0(\Omega, \mathcal{A}, P)$ be the set of random variables. We consider $(L^0)^d$ as an L^0-module and show that local, sequentially continuous functions on closed and bounded subsets have a fixed point which is measurable by construction.

KEYWORDS: Conditional Simplex, Fixed Point in $(L^0)^d$, Brouwer

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1 Introduction

The Brouwer fixed point theorem states that a continuous function from a compact and convex set in $\mathbb{R}^d$ to itself has a fixed point. This result and its extensions play a central role in Analysis, Optimization and Economic Theory among others. To show the result one approach is to consider functions on simplexes first and use Sperner's Lemma.

Recently, Cheridito, Kupper, and Vogelpoth [3], inspired by the theory developed by Filipović, Kupper, and Vogelpoth [7] and Guo [9], studied $(L^0)^d$ as an L^0-module, discussing concepts like linear independence, σ-stability, locality and L^0-convexity. Based on this, we define affine independence and conditional simplexes in $(L^0)^d$. Showing first a result similar to Sperner's Lemma, we obtain a fixed point for local, sequentially continuous functions on conditional simplexes. From the measurable structure of the problem, it turns out that we have to work with local, measurable labeling functions. To cope with this difficulty and to maintain some uniform properties, we subdivide the conditional simplex barycentrically. We then prove the existence of a measurable completely labeled conditional simplex, contained in the original one, which turns out to be a suitable σ-combination of elements of the barycentric subdivision along a partition of Ω. Thus, we can construct a sequence of conditional simplexes converging to a point. By applying always the same rule of labeling using the locality of the function, we show that this point is a fixed point. Due to the measurability of the labeling function the fixed point is measurable by construction. Hence, even though we follow the approach of $\mathbb{R}^d$ (cf. [2]) we do not need any measurable selection argument.

In Probabilistic Analysis theory the problem of finding random fixed points of random operators is an important issue. Given $\mathcal{C}$, a compact convex set of a Banach space, a continuous random operator is a function $R\colon \Omega \times \mathcal{C} \to \mathcal{C}$ satisfying

(i) $R(.,x)\colon \Omega \to \mathcal{C}$ is a random variable for any fixed $x \in \mathcal{C}$,

(ii) $R(\omega,.)\colon \mathcal{C} \to \mathcal{C}$ is a continuous function for any fixed $\omega \in \Omega$. [1]

For R there exists a random fixed point which is a random variable $\xi\colon \Omega \to \mathcal{C}$ such that $\xi(\omega) = R(\omega, \xi(\omega))$ for any ω (cf. [1], [10], [6]). In contrast to this ω-wise consideration, our approach is completely within the theory of L^0. All objects and properties are therefore defined in that language and proofs are done with L^0-methods. Moreover, the connection between continuous random operators on $\mathbb{R}^d$ and sequentially continuous functions on $(L^0)^d$ is not entirely clear.

The present paper is organized as follows. In the first chapter we present the basic concepts concerning $(L^0)^d$ as an L^0-module. We define conditional simplexes and examine their basic properties. In the second chapter we define measurable labeling functions and show the Brouwer fixed point theorem for conditional simplexes via a construction in the spirit of Sperner's lemma. In the third chapter, we show a fixed point result for L^0-convex, bounded and sequentially closed sets in $(L^0)^d$. With this result at hand, we present the topological implications known from the real-valued case. On the one hand, the impossibility of contracting a ball to a sphere in $(L^0)^d$ and on the other hand, an intermediate value theorem in L^0.

2 Conditional Simplex

For a probability space $(\Omega, \mathcal{A}, P)$, let $L^0 = L^0(\Omega, \mathcal{A}, P)$ be the space of all $\mathcal{A}$-measurable random variables, where P-almost surely equal random variables are identified. In particular, for $X, Y \in L^0$, the relations $X \geq Y$ and $X > Y$ have to be understood P-almost surely. The set L^0 with the almost everywhere order is a lattice ordered ring, and every nonempty subset $\mathcal{C} \subseteq L^0$ has a least upper bound $\operatorname{ess\,sup} \mathcal{C}$ and a greatest lower bound $\operatorname{ess\,inf} \mathcal{C}$ (cf.[8]). For $m \in \mathbb{R}$, we denote the constant random variable $m \cdot 1_\Omega$ by m. Further, we define the sets $L^0_+ = \{X \in L^0\colon X \geq 0\}$, $L^0_{++} = \{X \in L^0\colon X > 0\}$ and $\mathcal{A}_+ = \{A \in \mathcal{A}\colon P(A) > 0\}$. The set of random variables which can only take values in a set $M \subseteq \mathbb{R}$ is denoted by $M(\mathcal{A})$. For example, $\{1, \ldots, r\}(\mathcal{A})$ is the set of $\mathcal{A}$-measurable functions with values in $\{1, \ldots, r\} \subseteq \mathbb{N}$, $[0,1](\mathcal{A}) = \{Z \in L^0\colon 0 \leq Z \leq 1\}$ and $(0,1)(\mathcal{A}) = \{Z \in L^0\colon 0 < Z < 1\}$.

The *convex hull* of $X_1, \ldots, X_N \in (L^0)^d$, $N \in \mathbb{N}$, is defined as

$$\operatorname{conv}(X_1, \ldots, X_N) = \{\sum_{i=1}^N \lambda_i X_i \colon \lambda_i \in L^0_+, \sum_{i=1}^N \lambda_i = 1\}.$$

An element $Y \in \operatorname{conv}(X_1, \ldots, X_N)$ such that $\lambda_i > 0$ for all $i \in I \subseteq \{1, \ldots, N\}$ is called a *strict convex combination* of $(X_i\colon i \in I)$.

The *σ-stable hull* of a set $\mathcal{C} \subseteq (L^0)^d$ is defined as

$$\sigma(\mathcal{C}) = \{\sum_{i\in\mathbb{N}} 1_{A_i} X_i \colon X_i \in \mathcal{C}, (A_i)_{i\in\mathbb{N}} \text{ is a } \textit{partition}\},$$

[1] There exist versions in which $\mathcal{C}$ depends on ω with the property $\omega \mapsto \mathcal{C}(\omega)$ is measurable.

where a partition is a countable family $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}$ such that $P(A_i \cap A_j) = 0$ for $i \neq j$ and $P(\bigcup_{i\in\mathbb{N}} A_i) = 1$. We call a nonempty set $\mathcal{C}$ *σ-stable* if it is equal to $\sigma(\mathcal{C})$. For a σ-stable set $\mathcal{C} \subseteq (L^0)^d$ a function $f\colon \mathcal{C} \to (L^0)^d$ is called *local* if $f(\sum_{i\in\mathbb{N}} 1_{A_i} X_i) = \sum_{i\in\mathbb{N}} 1_{A_i} f(X_i)$ for every partition $(A_i)_{i\in\mathbb{N}}$ and $X_i \in \mathcal{C}$, $i \in \mathbb{N}$. For $\mathcal{X}, \mathcal{Y} \subseteq (L^0)^d$, we call a function $f\colon \mathcal{X} \to \mathcal{Y}$ *sequentially continuous* if for every sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ converging to $X \in \mathcal{X}$ P-almost-surely it holds that $f(X_n)$ converges to $f(X)$ P-almost surely. Further, the *L^0-scalar product* and *L^0-norm* on $(L^0)^d$ are defined as

$$\langle X, Y\rangle = \sum_{i=1}^{d} X_i Y_i \quad \text{and} \quad \|X\| = \langle X, X\rangle^{\frac{1}{2}}.$$

We call $\mathcal{C} \subseteq (L^0)^d$ *bounded* if $\operatorname{ess\,sup}_{X\in\mathcal{C}} \|X\| \in L^0$ and *sequentially closed* if it contains all P-almost sure limits of sequences in $\mathcal{C}$. Further, the diameter of $\mathcal{C} \subseteq (L^0)^d$ is defined as $\operatorname{diam}(\mathcal{C}) = \operatorname{ess\,sup}_{X,Y\in\mathcal{C}} \|X - Y\|$.

Definition 2.1. Elements $X_1, \dots, X_N$ of $(L^0)^d$, $N \in \mathbb{N}$, are said to be *affinely independent*, if either $N = 1$ or $N > 1$ and $\{X_i - X_N\}_{i=1}^{N-1}$ are *linearly independent*, that is

$$\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0 \quad \text{implies} \quad \lambda_1 = \cdots = \lambda_{N-1} = 0, \tag{2.1}$$

where $\lambda_1, \dots, \lambda_{N-1} \in L^0$.

The definition of affine independence is equivalent to

$$\sum_{i=1}^{N} \lambda_i X_i = 0 \;\text{ and }\; \sum_{i=1}^{N} \lambda_i = 0 \quad \text{implies} \quad \lambda_1 = \cdots = \lambda_N = 0. \tag{2.2}$$

Indeed, first, we show that (2.1) implies (2.2). Let $\sum_{i=1}^N \lambda_i X_i = 0$ and $\sum_{i=1}^N \lambda_i = 0$. Then, $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = 0$. By assumption (2.1), $\lambda_1 = \cdots = \lambda_{N-1} = 0$, thus also $\lambda_N = 0$. To see that (2.2) implies (2.1), let $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0$. With $\lambda_N = -\sum_{i=1}^{N-1} \lambda_i$, it holds $\sum_{i=1}^N \lambda_i X_i = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = \sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0$. By assumption (2.2), $\lambda_1 = \cdots = \lambda_N = 0$.

Remark 2.2. We observe that if $(X_i)_{i=1}^N \subseteq (L^0)^d$ are affinely independent then $(\lambda X_i)_{i=1}^N$, for $\lambda \in L^0_{++}$, and $(X_i + Y)_{i=1}^N$, for $Y \in (L^0)^d$, are affinely independent. Moreover, if a family $X_1, \dots, X_N$ is affinely independent then also $1_B X_1, \dots, 1_B X_N$ are affinely independent on $B \in \mathcal{A}_+$, which means from $\sum_{i=1}^N 1_B \lambda_i X_i = 0$ and $\sum_{i=1}^N 1_B \lambda_i = 0$ always follows $1_B \lambda_i = 0$ for all $i = 1, \dots, N$. ♦

Definition 2.3. A *conditional simplex* in $(L^0)^d$ is a set of the form

$$\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$$

such that $X_1, \dots, X_N \in (L^0)^d$ are affinely independent. We call $N \in \mathbb{N}$ the dimension of $\mathcal{S}$.

Remark 2.4. In a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$, the coefficients of convex combinations are unique in the sense that

$$\sum_{i=1}^{N} \lambda_i X_i = \sum_{i=1}^{N} \mu_i X_i \text{ and } \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \mu_i = 1 \quad \text{implies} \quad \lambda_i = \mu_i \text{ for all } i = 1, \dots, N. \tag{2.3}$$

Indeed, assume the given convex combinations. Then $\sum_{i=1}^{N} (\lambda_i - \mu_i) X_i = 0$ with $\sum_{i=1}^{N} (\lambda_i - \mu_i) = 0$, and hence, by (2.2), $\lambda_i - \mu_i = 0$ for all i since $X_1, \dots, X_N$ are affinely independent.♦

Since a conditional simplex is a convex hull it is in particular σ-stable. In contrast to a simplex in $\mathbb{R}^d$ the representation of $\mathcal{S}$ as a convex hull of affinely independent elements is unique but up to σ-stability.

Proposition 2.5. *Let $(X_i)_{i=1}^{N}$ and $(Y_i)_{i=1}^{N}$ be families in $(L^0)^d$ such that $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. Then $\operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$. Moreover, $(X_i)_{i=1}^{N}$ are affinely independent if and only if $(Y_i)_{i=1}^{N}$ are affinely independent.*

If $\mathcal{S}$ is a conditional simplex such that $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$, then it holds $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$.

Proof. Suppose $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. For $i = 1, \dots, N$, it holds

$$X_i \in \sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N) \subseteq \operatorname{conv}(Y_1, \dots, Y_N).$$

Therefore, $\operatorname{conv}(X_1, \dots, X_N) \subseteq \operatorname{conv}(Y_1, \dots, Y_N)$ and the reverse inclusion holds analogously.

Now, let $(X_i)_{i=1}^{N}$ be affinely independent and $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. We want to show that $(Y_i)_{i=1}^{N}$ are affinely independent. To that end, we define the affine hull

$$\operatorname{aff}(X_1, \dots, X_N) = \{\sum_{i=1}^{N} \lambda_i X_i \colon \lambda_i \in L^0, \sum_{i=1}^{N} \lambda_i = 1\}.$$

First, let $Z_1, \dots, Z_M \in (L^0)^d$, $M \in \mathbb{N}$, such that $\sigma(X_1, \dots, X_N) = \sigma(Z_1, \dots, Z_M)$. We show that if $1_A \operatorname{aff}(X_1, \dots, X_N) \subseteq 1_A \operatorname{aff}(Z_1, \dots, Z_M)$ for $A \in \mathcal{A}_+$ and $X_1, \dots, X_N$ are affinely independent then $M \geq N$. Since $X_i \in \sigma(X_1, \dots, X_N) = \sigma(Z_1, \dots, Z_M) \subseteq \operatorname{aff}(Z_1, \dots, Z_M)$, we have $\operatorname{aff}(X_1, \dots, X_N) \subseteq \operatorname{aff}(Z_1, \dots, Z_M)$. Further, it holds that $X_1 = \sum_{i=1}^{M} 1_{B_i^1} Z_i$ for a partition $(B_i^1)_{i=1}^{M}$ and hence there exists at least one $B_{k_1}^1$ such that $A_{k_1}^1 := B_{k_1}^1 \cap A \in \mathcal{A}_+$, and $1_{A_{k_1}^1} X_1 = 1_{A_{k_1}^1} Z_{k_1}$. Therefore,

$$1_{A_{k_1}^1} \operatorname{aff}(X_1, \dots, X_N) \subseteq 1_{A_{k_1}^1} \operatorname{aff}(Z_1, \dots, Z_M) = 1_{A_{k_1}^1} \operatorname{aff}(\{X_1, Z_1, \dots, Z_M\} \setminus \{Z_{k_1}\}).$$

For $X_2 = \sum_{i=1}^{M} 1_{A_i^2} Z_i$ we find a set A_k^2, such that $A_{k_2}^2 = A_k^2 \cap A_{k_1}^1 \in \mathcal{A}_+$, $1_{A_{k_2}^2} X_2 = 1_{A_{k_2}^2} Z_{k_2}$ and $k_1 \neq k_2$. Assume to the contrary $k_2 = k_1$, then there exists a set $B \in \mathcal{A}_+$, such that

$1_B X_1 = 1_B X_2$ which is a contradiction to the affine independence of $(X_i)_{i=1}^N$. Hence, we can again substitute Z_{k_2} by X_2 on $A^2_{k_2}$. Inductively, we find $k_1, \dots, k_N$ such that

$$1_{A_{k_N}} \operatorname{aff}(X_1, \dots, X_N) \subseteq 1_{A_{k_N}} \operatorname{aff}(\{X_1, \dots, X_N, Z_1, \dots, Z_M\} \setminus \{Z_{k_1}, \dots Z_{k_N}\})$$

which shows $M \geq N$. Now suppose $Y_1, \dots, Y_N$ are not affinely independent. This means, there exist $(\lambda_i)_{i=1}^N$ such that $\sum_{i=1}^N \lambda_i Y_i = \sum_{i=1}^N \lambda_i = 0$ but not all coefficients λ_i are zero, without loss of generality, $\lambda_1 > 0$ on $A \in \mathcal{A}_+$. Thus, $1_A Y_1 = -1_A \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i$ and it holds $1_A \operatorname{aff}(Y_1, \dots, Y_N) = 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. To see this, consider $1_A Z = 1_A \sum_{i=1}^N \mu_i Y_i \in 1_A \operatorname{aff}(Y_1, \dots, Y_N)$ which means $1_A \sum_{i=1}^N \mu_i = 1_A$. Thus, inserting for Y_1,

$$1_A Z = 1_A \left[\sum_{i=2}^N \mu_i Y_i - \mu_1 \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i \right] = 1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1} \right) Y_i \right].$$

Moreover,

$$1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1} \right) \right] = 1_A \left[\sum_{i=2}^N \mu_i \right] + 1_A \left[-\frac{\mu_1}{\lambda_1} \sum_{i=2}^N \lambda_i \right] = 1_A(1 - \mu_1) + 1_A \frac{\mu_1}{\lambda_1} \lambda_1 = 1_A.$$

Hence, $1_A Z \in 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. It follows $1_A \operatorname{aff}(X_1, \dots, X_N) = 1_A \operatorname{aff}(Y_1, \dots, Y_N) = 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. This is a contradiction to the former part of the proof (because $N - 1 \ngeq N$).

Next, we characterize *extremal* points in $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$. To this end, we show $X \in \sigma(X_1, \dots, X_N)$ if and only if there do not exist Y and Z in $\mathcal{S} \setminus \{X\}$ and $\lambda \in (0,1)(\mathcal{A})$ such that $\lambda Y + (1-\lambda) Z = X$. Consider $X \in \sigma(X_1, \dots, X_N)$ which is $X = \sum_{k=1}^N 1_{A_k} X_k$ for a partition $(A_k)_{k \in \mathbb{N}}$. Now assume to the contrary that we find $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ in $\mathcal{S} \setminus \{X\}$ such that $X = \lambda Y + (1-\lambda) Z$. This means that $X = \sum_{k=1}^N (\lambda \lambda_k + (1-\lambda)\mu_k) X_k$. Due to uniqueness of the coefficients (cf. (2.3)) in a conditional simplex we have $\lambda \lambda_k + (1-\lambda)\mu_k = 1_{A_k}$ for all $k = 1 \dots, N$. By means of $0 < \lambda < 1$, it holds that $\lambda \lambda_k + (1-\lambda)\mu_k = 1_{A_k}$ if and only $\lambda_k = \mu_k = 1_{A_k}$. Since the last equality holds for all k it follows that $Y = Z = X$. Therefore, we cannot find Y and Z in $\mathcal{S} \setminus \{X\}$ such that X is a strict convex combination of them. On the other hand, consider $X \in \mathcal{S}$ such that $X \notin \sigma(X_1, \dots, X_N)$. This means, $X = \sum_{k=1}^N \nu_k X_k$, such that there exist ν_{k_1} and ν_{k_2} and $B \in \mathcal{A}_+$ with $0 < \nu_{k_1} < 1$ on B and $0 < \nu_{k_2} < 1$ on B. Define $\varepsilon := \operatorname{ess\,inf}\{\nu_{k_1}, \nu_{k_2}, 1 - \nu_{k_1}, 1 - \nu_{k_2}\}$. Then define $\mu_k = \lambda_k = \nu_k$ if $k_1 \neq k \neq k_2$ and $\lambda_{k_1} = \nu_{k_1} - \varepsilon$, $\lambda_{k_2} = \nu_{k_2} + \varepsilon$, $\mu_{k_1} = \nu_{k_1} + \varepsilon$ and $\mu_{k_2} = \nu_{k_2} - \varepsilon$. Thus, $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ fulfill $0.5Y + 0.5Z = X$ but both are not equal to X by construction. Hence, X can be written as a strict convex combination of elements in $\mathcal{S} \setminus \{X\}$. To conclude, consider $X \in \sigma(X_1, \dots, X_N) \subseteq \mathcal{S} = \operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$. Since $X \in \sigma(X_1, \dots, X_N)$ it is not a strict convex combinations of elements in $\mathcal{S} \setminus \{X\}$, in particular, of elements in $\operatorname{conv}(Y_1, \dots, Y_N) \setminus \{X\}$. Therefore, X is also in $\sigma(Y_1, \dots, Y_N)$. Hence, $\sigma(X_1, \dots, X_N) \subseteq \sigma(Y_1, \dots, Y_N)$. With the same argumentation the other inclusion follows. $\square$

As an example consider $[0,1](\mathcal{A})$. For an arbitrary $A \in \mathcal{A}$, it holds that 1_A and 1_{A^c} are affinely independent and $\operatorname{conv}(1_A, 1_{A^c}) = \{\lambda 1_A + (1-\lambda) 1_{A^c} \colon 0 \leq \lambda \leq 1\} = [0,1](\mathcal{A})$. Thus, the conditional simplex $[0,1](\mathcal{A})$ can be written as a convex combination of different affinely independent elements of L^0. This is due to the fact that $\sigma(0,1) = \{1_B : B \in \mathcal{A}\} = \sigma(1_A, 1_{A^c})$ for any $A \in \mathcal{A}$.

Remark 2.6. In $(L^0)^d$, let e_i be the random variable which is 1 in the i-th component and 0 in any other. Then the family $0, e_1, \ldots, e_d$ is affinely independent and $(L^0)^d = \operatorname{aff}(0, e_1, \ldots, e_d)$. Hence, the maximal number of affinely independent elements in $(L^0)^d$ is $d+1$. ♦

The characterization of $X \in \sigma(X_1, \ldots, X_N)$ leads to the following definition.

Definition 2.7. Let $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ be a conditional simplex. We define the set of *extremal points* $\operatorname{ext}(\mathcal{S}) = \sigma(X_1, \ldots, X_N)$. For an index set I and a collection $\mathscr{S} = (\mathcal{S}_i)_{i \in I}$ of conditional simplexes we denote $\operatorname{ext}(\mathscr{S}) = \sigma(\operatorname{ext}(\mathcal{S}_i) \colon i \in I)$.

Remark 2.8. Let $\mathcal{S}^j = \operatorname{conv}(X_1^j, \ldots, X_N^j)$, $j \in \mathbb{N}$, be conditional simplexes of the same dimension N and $(A_j)_{j \in \mathbb{N}}$ a partition. Then $\sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j$ is again a conditional simplex. To that end, we define $Y_k = \sum_{j \in \mathbb{N}} 1_{A_j} X_k^j$ and recognize $\sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j = \operatorname{conv}(Y_1, \ldots, Y_N)$. Indeed,

$$\sum_{k=1}^N \lambda_k Y_k = \sum_{k=1}^N \lambda_k \sum_{j \in \mathbb{N}} 1_{A_j} X_k^j = \sum_{j \in \mathbb{N}} 1_{A_j} \sum_{k=1}^N \lambda_k X_k^j \in \sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j, \tag{2.4}$$

shows $\operatorname{conv}(Y_1, \ldots, Y_N) \subseteq \sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j$. Considering $\sum_{k=1}^N \lambda_k^j X_k^j$ in $\mathcal{S}^j$ and defining $\lambda_k = \sum_{j \in \mathbb{N}} 1_{A_j} \lambda_k^j$ yields the other inclusion. To show that $Y_1, \ldots, Y_N$ are affinely independent consider $\sum_{k=1}^N \lambda_k Y_k = 0 = \sum_{k=1}^N \lambda_k$. Then by (2.4), it holds $1_{A_j} \sum_{k=1}^N \lambda_k X_k^j = 0$ and since $\mathcal{S}^j$ is a conditional simplex, $1_{A_j} \lambda_k = 0$ for all $j \in \mathbb{N}$ and $k = 1, \ldots, N$. From the fact that $(A_j)_{j \in \mathbb{N}}$ is a partition, it follows that $\lambda_k = 0$ for all $k = 1, \ldots, N$. ♦

We will prove the Brouwer fixed point theorem in our setting using an analogue version of Sperner's Lemma. As in the unconditional case we have to subdivide a conditional simplex in smaller ones. For our argumentation we cannot use arbitrary subdivisions and need very special properties of the conditional simplexes in which we subdivide. This leads to the following definition.

Definition 2.9. Let $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ be a conditional simplex and $\mathrm{S_N}$ the group of permutations of $\{1, \ldots, N\}$. Then for $\pi \in \mathrm{S_N}$ we define

$$\mathcal{C}_\pi = \operatorname{conv}\left(X_{\pi(1)}, \frac{X_{\pi(1)} + X_{\pi(2)}}{2}, \ldots, \frac{X_{\pi(1)} + \cdots + X_{\pi(k)}}{k}, \ldots, \frac{X_{\pi(1)} + \cdots + X_{\pi(N)}}{N} \right).$$

We call $(\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$ the *barycentric subdivision* of $\mathcal{S}$, and denote $Y_k^\pi = \frac{1}{k} \sum_{i=1}^k X_{\pi(i)}$.

Lemma 2.10. *The barycentric subdivision is a collection of finitely many conditional simplexes satisfying the following properties*

(i) $\sigma(\bigcup_{\pi \in \mathrm{S_N}} \mathcal{C}_\pi) = \mathcal{S}$.

(ii) $\mathcal{C}_\pi$ *has dimension* N, $\pi \in \mathrm{S_N}$.

(iii) $\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$ *is a conditional simplex of dimension* $r \in \mathbb{N}$ *and* $r < N$ *for* $\pi, \overline{\pi} \in \mathrm{S_N}$, $\pi \neq \overline{\pi}$.

(iv) *For* $s = 1, \dots, N-1$, *let* $\mathcal{B}_s := \mathrm{conv}(X_1, \dots, X_s)$. *All conditional simplexes* $\mathcal{C}_\pi \cap \mathcal{B}_s$, $\pi \in \mathrm{S_N}$, *of dimension* s *subdivide* $\mathcal{B}_s$ *barycentrically.*

Proof. We show the affine independence of $Y_1^\pi, \dots, Y_N^\pi$ in $\mathcal{C}_\pi$. It holds

$$\lambda_{\pi(1)} X_{\pi(1)} + \lambda_{\pi(2)} \frac{X_{\pi(1)} + X_{\pi(2)}}{2} + \dots + \lambda_{\pi(N)} \frac{\sum_{k=1}^N X_{\pi(k)}}{N} = \sum_{i=1}^N \mu_i X_i,$$

with $\mu_i = \sum_{k=\pi^{-1}(i)}^N \frac{\lambda_{\pi(k)}}{k}$. Since $\sum_{i=1}^N \mu_i = \sum_{i=1}^N \lambda_i$, the affine independence of $Y_1^\pi, \dots, Y_N^\pi$ is obtained by the affine independence of $X_1, \dots, X_N$. Therefore all $\mathcal{C}_\pi$ are conditional simplexes.

The intersection of two conditional simplexes $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ can be expressed in the following manner. Let $J = \{j \colon \{\pi(1), \dots, \pi(j)\} = \{\overline{\pi}(1), \dots, \overline{\pi}(j)\}\}$ be the set of indexes up to which both π and $\overline{\pi}$ have the same set of images. Then,

$$\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}} = \mathrm{conv}\left(\frac{\sum_{k=1}^j X_{\pi(k)}}{j} \colon j \in J \right). \tag{2.5}$$

To show $\supseteq$, let $j \in J$. It holds that $\frac{\sum_{k=1}^j X_{\pi(k)}}{j}$ is in both $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ since $\{\pi(1), \dots, \pi(j)\} = \{\overline{\pi}(1), \dots, \overline{\pi}(j)\}$. Since the intersection of convex sets is convex, we get this implication.

For the reverse inclusion, let $X \in \mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$. From $X \in \mathcal{C}_\pi \cap C_{\overline{\pi}}$, it follows that $X = \sum_{i=1}^N \lambda_i (\sum_{k=1}^i \frac{X_{\pi(k)}}{i}) = \sum_{i=1}^N \mu_i (\sum_{k=1}^i \frac{X_{\overline{\pi}(k)}}{i})$. Consider $j \notin J$. By definition of J, there exist $p, q \le j$ with $\overline{\pi}^{-1}(\pi(p)), \pi^{-1}(\overline{\pi}(q)) \notin \{1, \dots, j\}$. By (2.3), the coefficients of $X_{\pi(p)}$ are equal: $\sum_{i=p}^N \frac{\lambda_i}{i} = \sum_{i=\overline{\pi}^{-1}(\pi(p))}^N \frac{\mu_i}{i}$. The same holds for $X_{\pi(q)}$: $\sum_{i=q}^N \frac{\mu_i}{i} = \sum_{i=\pi^{-1}(\overline{\pi}(q))}^N \frac{\lambda_i}{i}$. Put together

$$\sum_{i=j+1}^N \frac{\mu_i}{i} \le \sum_{i=q}^N \frac{\mu_i}{i} = \sum_{i=\pi^{-1}(\overline{\pi}(q))}^N \frac{\lambda_i}{i} \le \sum_{i=j+1}^N \frac{\lambda_i}{i} \le \sum_{i=p}^N \frac{\lambda_i}{i} = \sum_{i=\overline{\pi}^{-1}(\pi(p))}^N \frac{\mu_i}{i} \le \sum_{i=j+1}^N \frac{\mu_i}{i}$$

which is only possible if $\mu_j = \lambda_j = 0$ since $p, q \le j$.

Furthermore, if $\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$ is of dimension N by (2.5) follows that $\pi = \overline{\pi}$. This shows (iii).

As for Condition (i), it clearly holds $\sigma(\cup_{\pi \in \mathrm{S_N}} \mathcal{C}_\pi) \subseteq \mathcal{S}$. On the other hand, let $X = \sum_{i=1}^N \lambda_i X_i \in \mathcal{S}$. Then, cf. [4], we find a partition $(A_n)_{n \in \mathbb{N}}$ such that on every A_n the indexes are completely ordered which is $\lambda_{i_1^n} \ge \lambda_{i_2^n} \ge \dots \ge \lambda_{i_N^n}$ on A_n. This means, that $X \in 1_{A_n} \mathcal{C}_{\pi^n}$ with $\pi^n(j) = i_j^n$. Indeed, we can rewrite X on A_n as

$$X = (\lambda_{i_1^n} - \lambda_{i_2^n}) X_{i_1^n} + \dots + (N-1)(\lambda_{i_{N-1}^n} - \lambda_{i_N^n}) \frac{\sum_{k=1}^{N-1} X_{i_k^n}}{N-1} + N \lambda_{i_N^n} \frac{\sum_{k=1}^N X_{i_k^n}}{N},$$

which shows that $X \in \mathcal{C}_{\pi^n}$ on A_n.

Further, for $\mathcal{B}_s = \mathrm{conv}(X_1, \dots, X_s)$ the elements $\mathcal{C}_{\pi'} \cap \mathcal{B}_s$ of dimension s are exactly the ones with $\{\pi(i) \colon i = 1, \dots, s\} = \{1, \dots, s\}$. Therefore, $(\mathcal{C}_{\pi'} \cap \mathcal{B}_s)_{\pi'}$ is exactly the barycentric subdivision of $\mathcal{B}_s$, which has been shown to fulfill the properties (i)-(iii). $\square$

Remark 2.11. If we subdivide the conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ barycentrically, we can consider an arbitrary $\mathcal{C}_\pi = \operatorname{conv}(Y_1^\pi, \ldots, Y_N^\pi)$, $\pi \in \mathrm{S_N}$. Then

$$\operatorname{diam}(\mathcal{C}_\pi) \le \operatorname*{ess\,sup}_{i=1,\ldots,N} \|Y_i^\pi - Y_N^\pi\| \le \frac{1}{N} \operatorname*{ess\,sup}_{i=1,\ldots,N} \left\| \sum_{k=1}^N (X_i^n - X_k^n) \right\| \le \frac{N-1}{N} \operatorname{diam}(\mathcal{S}).$$

If we now subdivide $\mathcal{C}_\pi$ barycentrically and continue in that way, we obtain a chain of conditional simplexes $\mathcal{S}^n$, with $\mathcal{S}^0 = \mathcal{S}$. For the diameter of $\mathcal{S}^n$, it holds that $\operatorname{diam}(\mathcal{S}^n) \le \left(\frac{N-1}{N}\right)^n \operatorname{diam}(\mathcal{S})$. Since $\operatorname{diam}(\mathcal{S}) < \infty$ and $\left(\frac{N-1}{N}\right)^n \to 0$, for $n \to \infty$, it follows that $\operatorname{diam}(\mathcal{S}^n) \to 0$, $n \to \infty$. ♦

3 Brouwer Fixed Point Theorem for Conditional Simplexes

Definition 3.1. Let $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ be a conditional simplex, barycentrically subdivided in $\mathscr{S} = (\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$. A local function $\phi \colon \operatorname{ext}(\mathscr{S}) \to \{1, \ldots, N\}(\mathcal{A})$ is called a *labeling function* of $\mathcal{S}$. For fixed $X_1, \ldots, X_N \in \operatorname{ext}(\mathcal{S})$ with $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$, the labeling function is called *proper*, if for any $Y \in \operatorname{ext}(\mathscr{S})$ it holds that

$$P\left(\{\phi(Y) = i\} \subseteq \{\lambda_i > 0\}\right) = 1,$$

for $i = 1, \ldots, N$, where $Y = \sum_{i=1}^N \lambda_i X_i$. A conditional simplex $\mathcal{C} = \operatorname{conv}(Y_1, \ldots, Y_N) \subseteq \mathcal{S}$, with $Y_j \in \operatorname{ext}(\mathscr{S}), j = 1, \ldots, N$, is said to be *completely labeled* by ϕ if this is a proper labeling function of $\mathcal{S}$ and

$$P\left(\bigcup_{j=1}^N \{\phi(Y_j) = i\}\right) = 1,$$

for all $i \in \{1, \ldots, N\}$.

Lemma 3.2. *Let $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ be a conditional simplex and $f \colon \mathcal{S} \to \mathcal{S}$ a local function. Let $\phi \colon \operatorname{ext}(\mathscr{S}) \to \{0, \ldots, N\}(\mathcal{A})$ be a local function such that*

(i) $P\left(\{\phi(X) = i\} \subseteq \{\lambda_i > 0\} \cap \{\lambda_i \ge \mu_i\}\right) = 1$, *for all* $i = 1, \ldots, N$,

(ii) $P\left(\bigcup_{i=1}^N \left(\{\lambda_i > 0\} \cap \{\lambda_i \ge \mu_i\}\right) \subseteq \bigcup_{i=1}^N \{\phi(X) = i\}\right) = 1$,

where $X = \sum_{i=1}^N \lambda_i X_i$ and $f(X) = \sum_{i=1}^N \mu_i X_i$. Then, ϕ is a proper labeling function.

Moreover, the set of functions fulfilling these properties is non-empty.

Proof. First we show that ϕ is a labeling function. Since ϕ is local we just have to prove that ϕ actually maps to $\{1, \ldots, N\}$. Due to (ii), we have to show that $P(\bigcup_{i=1}^N \{\lambda_i \ge \mu_i \colon \lambda_i > 0\}) = 1$. Assume to the contrary, $\mu_i > \lambda_i$ on $A \in \mathcal{A}_+$, for all λ_i with $\lambda_i > 0$ on A. Then it holds that $1 = \sum_{i=1}^N \lambda_i 1_{\{\lambda_i > 0\}} < \sum_{i=1}^N \mu_i 1_{\{\mu_i > 0\}} = 1$ on A which yields a contradiction. Thus, ϕ is a labeling function. Moreover, due to (i) it holds that $P\left(\{\phi(X) = i\} \subseteq \{\lambda_i > 0\}\right) = 1$ which shows that ϕ is proper.

To prove the existence, for $X \in \operatorname{ext}(\mathscr{S})$ with $X = \sum_{i=1}^N \lambda_i X_i$, $f(X) = \sum_{i=1}^N \mu_i$ let $B_i := \{\lambda_i > 0\} \cap \{\lambda_i \geq \mu_i\}$, $i = 1, \dots, N$. Then we define the function ϕ at X as $\{\phi(X) = i\} = B_i \setminus (\bigcup_{k=1}^{i-1} B_k)$, $i = 1, \dots, N$. It has been shown that ϕ maps to $\{1, \dots, N\}(\mathcal{A})$ and is proper. It remains to show that ϕ is local. To this end, consider $X = \sum_{j \in \mathbb{N}} 1_{A_j} X^j$ where $X^j = \sum_{i=1}^N \lambda_i^j X_i$ and $f(X^j) = \sum_{i=1}^N \mu_i^j X_i$. Due to uniqueness of the coefficients in a conditional simplex it holds that $\lambda_i = \sum_{j \in \mathbb{N}} 1_{A_j} \lambda_i^j$ and due to locality of f it follows that $\mu_i = \sum_{j \in \mathbb{N}} 1_{A_j} \mu_i^j$. Therefore it holds that $B_i = \bigcup_{j \in \mathbb{N}} \left(\{\lambda_i^j > 0\} \cap \{\lambda_i^j \geq \mu_i^j\} \cap A_j \right) = \bigcup_{j \in \mathbb{N}} (B_i^j \cap A_j)$. Hence, $\phi(X) = i$ on $B_i \setminus (\bigcup_{k=1}^{i-1} B_k) = [\bigcup_{j \in \mathbb{N}} (B_i^j \cap A_j)] \setminus [\bigcup_{k=1}^{i-1} (\bigcup_{j \in \mathbb{N}} B_k^j \cap A_j)] = \bigcup_{j \in \mathbb{N}} [(B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j) \cap A_j]$. On the other hand, we see that $\sum_{j \in \mathbb{N}} 1_{A_j} \phi(X^j)$ is i on any $A_j \cap \{\phi(X^j) = i\}$, hence it is i on $\bigcup_{j \in \mathbb{N}} (B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j) \cap A_j$. Thus, $\sum_{j \in \mathbb{N}} 1_{A_j} \phi(X^j) = \phi(\sum_{j \in \mathbb{N}} 1_{A_j} X^j)$ which shows that ϕ is local. □

The reason to demand locality of a labeling function is exactly because we want to label by the rule explained in Lemma 3.2 and hence keep local information with it. For example consider a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, X_2, X_3, X_4)$ and $\Omega = \{\omega_1, \omega_2\}$. Let $Y \in \operatorname{ext}(\mathscr{S})$ be given by $Y = \frac{1}{3} \sum_{i=1}^3 X_i$. Now consider a function f on $\mathcal{S}$ such that

$$f(Y)(\omega_1) = \frac{1}{4} X_1(\omega_1) + \frac{3}{4} X_3(\omega_1); \quad f(Y)(\omega_2) = \frac{2}{5} X_1(\omega_2) + \frac{2}{5} X_2(\omega_2) + \frac{1}{5} X_4(\omega_2).$$

If we label Y by the rule explained in Lemma 3.2, ϕ takes the values $\phi(\omega_1) \in \{1, 2\}$ and $\phi(\omega_2) = 3$. Therefore, we can really express on which set $\lambda_i \geq \mu_i$ and on which not. Using a deterministic labeling of Y, we would loose this information. For example bearing the label 3 would not mean anything on ω_1 for Y. Moreover, it would be impossible to label properly by a deterministic labeling function following the rule of the last lemma since there is no i such that $\lambda_i \geq \mu_i$.

Theorem 3.3. *Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex in $(L^0)^d$. Let $f\colon \mathcal{S} \to \mathcal{S}$ be a local, sequentially continuous function. Then there exists $Y \in \mathcal{S}$ such that $f(Y) = Y$.*

Proof. We consider the barycentric subdivision $(\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$ of $\mathcal{S}$ and a proper labeling function ϕ on $\operatorname{ext}(\mathscr{S})$. First, we show that we can find a completely labeled conditional simplex in $\mathcal{S}$. By induction on the dimension of $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$, we show that there exists a partition $(A_k)_{k=1,\dots,K}$ such that on any A_k there is an odd number of completely labeled $\mathcal{C}_\pi$. The case $N = 1$ is clear, since a point can be labeled with the constant index 1, only.

Suppose the case $N - 1$ is proven. Since the number of Y_i^π of the barycentric subdivision is finite and ϕ can only take finitely many values, it holds for all $V \in (Y_i^\pi)_{i=1,\dots,N, \pi \in \mathrm{S_N}}$ there exists a partition $(A_k^V)_{k=1,\dots,K}$, $K < \infty$, where $\phi(V)$ is constant on any A_k^V. Therefore, we find a partition $(A_k)_{k=1,\dots,K}$, such that $\phi(V)$ on A_k is constant for all V and A_k. Fix A_k now.

In the following, we denote by $\mathcal{C}_{\pi^b}$ these conditional simplexes for which $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1}$ are $N-1$-dimensional (cf. Lemma 2.10 (iv)), therefore $\pi^b(N) = N$. Further we denote by $\mathcal{C}_{\pi^c}$ these conditional simplexes which are not of the type $\mathcal{C}_{\pi^b}$, that is $\pi^c(N) \neq N$. If we use $\mathcal{C}_\pi$ we mean a conditional simplex of arbitrary type. We define

- $\mathscr{C} \subseteq (\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$ to be the set of $\mathcal{C}_\pi$ which are completely labeled on A_k.

- $\mathscr{A} \subseteq (\mathcal{C}_\pi)_{\pi\in \mathrm{S_N}}$ to be the set of the almost completely labeled $\mathcal{C}_\pi$, that is

$$\{\phi(Y_k^\pi), k=1,\dots,N\} = \{1,\dots,N-1\} \quad \text{on } A_k.$$

- $\mathscr{E}_\pi$ to be the set of the intersections $(\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l})_{\pi_l\in \mathrm{S_N}}$ which are $N-1$-dimensional and completely labeled on A_k.[2]

- $\mathscr{B}_\pi$ to be the set of the intersections $\mathcal{C}_\pi \cap \mathcal{B}_{N-1}$ which are completely labeled on A_k.

It holds that $\mathscr{E}_\pi \cap \mathscr{B}_\pi = \emptyset$ and hence $|\mathscr{E}_\pi \cup \mathscr{B}_\pi| = |\mathscr{E}_\pi| + |\mathscr{B}_\pi|$. Since $\mathcal{C}_{\pi^c} \cap \mathcal{B}_{N-1}$ is at most $N-2$-dimensional, it holds that $\mathscr{B}_{\pi^c} = \emptyset$ and hence $|\mathscr{B}_{\pi^c}| = 0$. Moreover, we know that $\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l}$ is $N-1$-dimensional on A_k if and only this holds on whole Ω (cf. Lemma 2.10 (ii)) and $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1} \neq \emptyset$ on A_k if and only if this also holds on whole Ω (cf. Lemma 2.10 (iv)). So it does not play any role if we look at these sets which are intersections on A_k or on Ω since they are exactly the same sets.

If $\mathcal{C}_{\pi^c} \in \mathscr{C}$ then $|\mathscr{E}_{\pi^c}| = 1$ and if $\mathcal{C}_{\pi^b} \in \mathscr{C}$ then $|\mathscr{E}_{\pi^b} \cup \mathscr{B}_{\pi^b}| = 1$. If $\mathcal{C}_{\pi^c} \in \mathscr{A}$ then $|\mathscr{E}_{\pi^c}| = 2$ and if $\mathcal{C}_{\pi^b} \in \mathscr{A}$ then $|\mathscr{E}_{\pi^b} \cup \mathscr{B}_{\pi^b}| = 2$. Therefore it holds $\sum_{\pi\in\mathrm{S_N}} |\mathscr{E}_\pi \cup \mathscr{B}_\pi| = |\mathscr{C}| + 2\,|\mathscr{A}|$.

If we pick an $E_\pi \in \mathscr{E}_\pi$ we know there always exists another π_l such that $E_\pi \in \mathscr{E}_{\pi_l}$ (Lemma 2.10(ii)). Therefore $\sum_{\pi\in\mathrm{S_N}} |\mathscr{E}_\pi|$ is even. Moreover $(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})_{\pi^b}$ subdivides $\mathcal{B}_{N-1}$ barycentrically[3] and hence we can apply the hypothesis (on $\mathrm{ext}(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})$). This means that the number of completely labeled conditional simplexes is odd on a partition of Ω but since ϕ is constant on A_k it also has to be odd there. This means that $\sum_{\pi^b} |\mathscr{B}_{\pi^b}|$ has to be odd. Hence, we also have that $\sum_\pi |\mathscr{E}_\pi \cup \mathscr{B}_\pi|$ is the sum of an even and an odd number and thus odd. So we conclude $|\mathscr{C}| + 2\,|\mathscr{A}|$ is odd and hence also $|\mathscr{C}|$. Thus, we find for any A_k a completely labeled $\mathcal{C}_{\pi_k}$.

We define $\mathcal{S}^1 = \sum_{k=1}^K 1_{A_k}\mathcal{C}_{\pi_k}$ which by Remark 2.8 is indeed a conditional simplex. Due to σ-stability of $\mathcal{S}$ it holds $\mathcal{S}^1 \subseteq \mathcal{S}$. By Remark 2.11 $\mathcal{S}^1$ has a diameter which is less then $\frac{N-1}{N}\,\mathrm{diam}(\mathcal{S})$ and since ϕ is local $\mathcal{S}^1$ is completely labeled on whole Ω.

This holds for any proper labeling function hence also for a ϕ of the type as in Lemma 3.2.

Now, we extract a chain $(\mathcal{S}^n)_{n\in\mathbb{N}}$ of completely labeled conditional simplexes contained in $\mathcal{S}$, fulfilling the diameter property $\mathrm{diam}(\mathcal{S}^n) \to 0$ as in Remark 2.11. By [3, Theorem 4.8]) it holds that $\bigcap_{n\in\mathbb{N}} \mathcal{S}^n \neq \emptyset$. The intersection consists of one element $Y = \sum_{l=1}^N \alpha_l X_l$ by the diameter property. Let $f(Y) = \sum_{l=1}^N \beta_l X_l$. Thus, all sequences of elements in $\mathrm{ext}(\mathcal{S}^n)$ also converge P-almost surely to Y, which then preserves the properties of the index function. That is, for each $i = 1,\dots,N$, there exist $V_k^n \in \mathrm{ext}(C_\pi^n)$ of $\mathcal{S}^n$, $k = 1,\dots,N, \pi \in \mathrm{S_N}$, with $P(\{\phi(V_k^n) = i\} \subseteq \{\lambda_i^{n,k} \geq \mu_i^{n,k}\}) = 1$ (cf. Lemma 3.2), where $V_k^n = \sum_{i=1}^N \lambda_i^{n,k} X_i$ and $f(V_k^n) = \sum_{i=1}^N \mu_i^{n,k} X_i$. Then $P(\bigcap_{n\in\mathbb{N}}\{\lambda_i^{n,k} \geq \mu_i^{n,k}\} \subseteq \{\alpha_i \geq \beta_i\}) = 1$ for all $k = 1,\dots,N$ by locality of f, $V^n \to Y$ P-almost surely, and by sequential continuity, $f(V^n) \to f(Y)$ P-almost surely. But, $P(\bigcup_{k=1}^N \bigcap_{n\in\mathbb{N}}\{\lambda_i^{n,k} \geq \mu_i^{n,k}\}) = P(\bigcap_{n\in\mathbb{N}} \bigcup_{k=1}^N \{\lambda_i^{n,k} \geq \mu_i^{n,k}\}) = 1$ by the complete labeling of $\mathcal{S}^n$. Hence, $\alpha_i \geq \beta_i$ for all $i = 1,\dots,N$. This is possible only if $\alpha_i = \beta_i$ for all $i = 1,\dots,N$ which is the condition of a fixed point. $\square$

[2] That is bearing exactly the label $1,\dots,N-1$ on A_k.

[3] The boundary of $\mathcal{S}$ is a σ-stable set so if it is partitioned by the labeling function into A_k we know that $\mathcal{B}_{N-1}(\mathcal{S}) = \sum_{k=1}^K 1_{A_k}\mathcal{B}_{N-1}(1_{A_k}\mathcal{S})$ and by Lemma 2.10 (iv) we can apply the induction hypothesis also on A_k.

Corollary 3.4. *Let $(\mathcal{S}^n)_{n\in\mathbb{N}}$ be conditional simplexes, $(A_n)_{n\in\mathbb{N}}$ be a partition of Ω and $\mathcal{S} := \sum_{n\in\mathbb{N}} 1_{A_n}\mathcal{S}^n$. Then a local, sequentially continuous function $f\colon \mathcal{S}\to\mathcal{S}$ has a fixed point.*

Proof. Since f is local, we have $f(\mathcal{S}) = \sum_{n\in\mathbb{N}} 1_{A_n} f(\mathcal{S}^n)$ and f restricted on $\mathcal{S}^n$ is still sequentially continuous. Therefore we find $Y_n \in \mathcal{S}^n$ with $1_{A_n} f(Y_n) = 1_{A_n} Y_n$. Defining $Y = \sum_{n\in\mathbb{N}} 1_{A_n} Y_n$ we have

$$f(Y) = f\left(\sum_{n\in\mathbb{N}} 1_{A_n} Y_n\right) = \sum_{n\in\mathbb{N}} 1_{A_n} f(Y_n) = \sum_{n\in\mathbb{N}} 1_{A_n} Y_n = Y. \qquad \square$$

Remark 3.5. The $\mathcal{S}^n$ which appear can be of different dimension. If $\mathcal{S}^n = \operatorname{conv}(Y_1^n,\dots,Y_{N_n}^n)$ is of dimension N_n, the object $\mathcal{S}$ as the conditional dimension $\sum_{n\in\mathbb{N}} 1_{A_n} N_n$. This conditional dimension is hence in $\mathbb{N}(\mathcal{A})$, in particular a measurable object, (c.f. [5]). ♦

4 Applications

4.1 Fixed point theorem for sequentially closed and bounded sets in $(L^0)^d$

Proposition 4.1. *Let $\mathcal{K}$ be an L^0-convex, sequentially closed and bounded subset of $(L^0)^d$ and $f\colon \mathcal{K}\to\mathcal{K}$ a sequentially continuous function. Then f has a fixed point.*

Proof. Since $\mathcal{K}$ is bounded, there exists a conditional simplex $\mathcal{S}$ such that $\mathcal{K}\subseteq\mathcal{S}$. Now define the function $h\colon \mathcal{S}\to\mathcal{K}$ by

$$h(X) = \begin{cases} X, & \text{if } X\in\mathcal{K},\\ \arg\min\{\|X-Y\| : Y\in\mathcal{K}\}, & \text{else.}\end{cases}$$

This means, that h is the identity on $\mathcal{K}$ and a projection towards $\mathcal{K}$ for the elements in $\mathcal{S}\setminus\mathcal{K}$. Due to [3, Corollary 5.5] this minmium exists and is unique. Therefore h is well-defined.

We can characterize h by

$$Y = h(X) \Leftrightarrow \langle X-Y, Z-Y\rangle \le 0, \quad \text{for all } Z\in\mathcal{K}. \tag{4.1}$$

Indeed, let $\langle X-Y, Z-Y\rangle \le 0$ for all $Z\in\mathcal{K}$. Then

$$\begin{aligned}\|X-Z\|^2 &= \|(X-Y)+(Y-Z)\|\\ &= \|X-Y\|^2 + 2\langle X-Y, Y-Z\rangle + \|Y-Z\|^2 \ge \|X-Y\|^2,\end{aligned}$$

which shows the minimizing property of h. On the other hand, let $Y = h(X)$. Since $\mathcal{K}$ is convex, $\lambda Z + (1-\lambda)Y\in\mathcal{K}$ for any $\lambda\in(0,1](\mathcal{A})$ and $Z\in\mathcal{K}$. By standard calculation,

$$\|X-(\lambda Z+(1-\lambda)Y)\|^2 \ge \|X-Y\|^2$$

yields $0 \geq -2\lambda\langle X, -Y\rangle + (2\lambda - \lambda^2)\langle Y, Y\rangle + 2\lambda\langle X, Z\rangle - \lambda^2 \|Z\|^2 - 2\lambda(1-\lambda)\langle Z, Y\rangle$. Any term can be divided by $\lambda > 0$. We do so and let $\lambda \downarrow 0$ afterwards. We obtain

$$0 \geq -2\langle X, -Y\rangle + 2\langle Y, Y\rangle + 2\langle X, Z\rangle - 2\langle Z, Y\rangle = 2\langle X - Y, Z - Y\rangle,$$

which is the claim.

Furthermore, for any $X, Y \in \mathcal{S}$ holds

$$\|h(X) - h(Y)\| \leq \|X - Y\|.$$

Indeed,

$$X - Y = (h(X) - h(Y)) + X - h(X) + h(Y) - Y =: (h(X) - h(Y)) + c$$

which means

$$\|X - Y\|^2 = \|h(X) - h(Y)\|^2 + \|c\|^2 + 2\langle c, h(X) - h(Y)\rangle. \tag{4.2}$$

Since

$$\langle c, h(X) - h(Y)\rangle = -\langle X - h(X), h(Y) - h(X)\rangle - \langle Y - h(Y), h(X) - h(Y)\rangle,$$

by (4.1), it follows that $\langle c, h(X) - h(Y)\rangle \geq 0$ and (4.2) yields $\|X - Y\|^2 \geq \|h(X) - h(Y)\|^2$. Hence, h is sequentially continuous, since if $\|X_n - X\| \to 0$ then also $\|h(X_n) - h(X)\| \to 0$.

The function $f \circ h$ is a sequentially continuous function mapping from $\mathcal{S}$ to $\mathcal{S}$, more precisely to $\mathcal{K}$. Hence, there exists a fixed point $f \circ h(Z) = Z$. But since $f \circ h$ maps to $\mathcal{K}$, this Z has to be in $\mathcal{K}$. Therefore we know $h(Z) = Z$ and hence $f(Z) = Z$ which ends the proof. □

Remark 4.2. In [5] a concept of conditional compactness is introduced and it is shown that there is an equivalence between conditional compactness and conditional closed- and boundedness in $(L^0)^d$. In this concept we can formulate the conditional Brouwer fixed point theorem as follows. A sequentially continuous function $f\colon \mathcal{K} \to \mathcal{K}$ such that $\mathcal{K}$ is a conditionally compact and L^0-convex subset of $(L^0)^d$ has a fixed point. ♦

4.2 Applications in Analysis on $(L^0)^d$

Working in $\mathbb{R}^d$ the Brouwer fixed point theorem can be used to prove several topological properties and is even equivalent to some of them. In the theory of $(L^0)^d$ we will shown that the conditional Brouwer fixed point theorem has several implications as well.

Define the *unit ball* in $(L^0)^d$ by $\mathcal{B}(d) = \{X \in (L^0)^d\colon \|X\| \leq 1\}$. Then by the former theorem any local, sequentially continuous function $f\colon \mathcal{B}(d) \to \mathcal{B}(d)$ has a fixed point. The *unit sphere* $\mathcal{S}(d-1)$ is defined as $\mathcal{S}(d-1) = \{X \in (L^0)^d\colon \|X\| = 1\}$.

Definition 4.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $(L^0)^d$. An L^0-*homotopy* of two local, sequentially continuous functions $f, g\colon \mathcal{X} \to \mathcal{Y}$ is a jointly local, sequentially continuous function $H\colon \mathcal{X} \times [0,1](\mathcal{A}) \to \mathcal{Y}$ such that $H(X, 0) = f(X)$ and $H(X, 1) = g(X)$. Jointly local means

$H(\sum_{j\in\mathbb{N}} 1_{A_j} X_j, \sum_{j\in\mathbb{N}} 1_{A_j} t_j) = \sum_{j\in\mathbb{N}} 1_{A_j} H(X_j, t_j)$ for any partition $(A_j)_{j\in\mathbb{N}}$, $(X_j)_{j\in\mathbb{N}}$ in $\mathcal{X}$ and $(t_j)_{j\in\mathbb{N}}$ in $[0,1](\mathcal{A})$. Sequential continuity of H is therefore $H(X_n, t_n) \to H(X,t)$ whenever $X_n \to X$ and $t_n \to t$ both P-almost surely for $X_n, X \in \mathcal{X}$ and $t_n, t \in [0,1](\mathcal{A})$.

Lemma 4.4. *The identity function of the sphere is not L^0-homotop to a constant function.*

The proof is a consequence of the following lemma.

Lemma 4.5. *There does not exist a local, sequentially continuous function $f\colon \mathcal{B}(d) \to \mathcal{S}(d-1)$ which is the identity on $\mathcal{S}(d-1)$.*

Proof. Suppose there is this local, sequentially continuous function f. Define $g\colon \mathcal{S}(d-1) \to \mathcal{S}(d-1)$ by $g(X) = -X$. Then the composition $g \circ f\colon \mathcal{B}(d) \to \mathcal{B}(d)$, which actually maps to $\mathcal{S}(d-1)$, is local and sequentially continuous. Therefore, this has a fixed point Y which has to be in $\mathcal{S}(d-1)$, since this is the image of $g \circ f$. But we know $f(Y) = Y$ and $g(Y) = -Y$ and hence $g \circ f(Y) = -Y$. Therefore, Y cannot be a fixed point (since $0 \notin \mathcal{S}(d-1)$) which is a contradiction. □

Directly follows that the identity on the sphere is not L^0-homotop to a constant function. In the case $d = 1$ we get the following result which is the L^0-version of an L^0-intermediate value theorem.

Lemma 4.6. *Let $X, \overline{X} \in L^0$ with $X \leq \overline{X}$. Let $\left[X, \overline{X}\right] = \left\{Z \in L^0\colon X \leq Z \leq \overline{X}\right\}$ and $f\colon \left[X, \overline{X}\right] \to L^0$ be a local, sequentially continuous function. Define $A = \{f(X) \leq f\left(\overline{X}\right)\}$. Then for every $Y \in \left[1_A f(X) + 1_{A^c} f\left(\overline{X}\right), 1_A f\left(\overline{X}\right) + 1_{A^c} f(X)\right]$ there exists $\overline{Y} \in \left[X, \overline{X}\right]$ with $f\left(\overline{Y}\right) = Y$.*

Proof. Since f is local, it is sufficient to prove the case for $f(X) \leq f\left(\overline{X}\right)$ which is $A = \Omega$. For the general case we would consider A and A^c separately, obtain $1_A f\left(\overline{Y}_1\right) = 1_A Y$, $1_{A^c} f\left(\overline{Y}_2\right) = 1_{A^c} Y$ and by locality we have $f\left(1_A \overline{Y}_1 + 1_{A^c} \overline{Y}_2\right) = Y$. So suppose $Y \in \left[f\left(X\right), f\left(\overline{X}\right)\right]$ in the rest of the proof.

Let first $f(X) < Y < f\left(\overline{X}\right)$. Define the function $g\colon \left[X, \overline{X}\right] \to \left[X, \overline{X}\right]$ by

$$g(V) := p(V - f(V) + Y) \quad \text{with} \quad p(Z) = 1_{\{Z \leq X\}} X + 1_{\{X \leq Z \leq \overline{X}\}} Z + 1_{\{\overline{X} \leq Z\}} \overline{X}.$$

Therefore g is local and continuous and hence has a fixed point $\overline{Y}$. If $\overline{Y} = X$, it must hold $X - f(X) + Y \leq X$ which means $Y \leq f(X)$ which is a contradiction. If $\overline{Y} = \overline{X}$, it follows $f\left(\overline{X}\right) \leq Y$, which is also a contradiction. Hence, $\overline{Y} = \overline{Y} - f\left(\overline{Y}\right) + Y$ which means $f\left(\overline{Y}\right) = Y$.

If $Y = f(X)$ on B and $Y = f\left(\overline{X}\right)$ on C, it holds that $f(X) < Y < f\left(\overline{X}\right)$ on $(B \cup C)^c =: D$. Then we find $\overline{Y}$ such that $f\left(\overline{Y}\right) = Y$ on D. In total $f\left(1_B X + 1_{C\setminus B} \overline{X} + 1_D \overline{Y}\right) = 1_B f(X) + 1_{C\setminus B} f\left(\overline{X}\right) + 1_D f\left(\overline{Y}\right) = Y$. This shows the claim for general $Y \in \left[f(X), f(\overline{X})\right]$. □

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