Duality for increasing convex functionals with countably many marginal constraints

D. Bartl* P. Cheridito† M. Kupper‡ L. Tangpi§

September 2015

Abstract. The main result of this paper is a convex dual representation for increasing convex functionals that are defined on a space of real-valued Borel measurable functions living on a countable product of metric spaces. Our principal assumption is that the functionals fulfill convex marginal constraints satisfying a tightness condition. In the special case where the marginal constraints are given by expectations or maxima of expectations, we obtain linear and sublinear versions of Kantorovich's transport duality and the recently discovered martingale transport duality on products of countably many metric spaces.

MSC 2010: Primary 47H07, 28C20; Secondary 60G42, 91G20.
Keywords: Representation theorem, increasing convex functional, marginal constraints, transport problem, martingale transport problem, Kantorovich duality, model-independent finance.

1 Introduction

We consider an increasing convex functional $\phi : B_b \to \mathbb{R}$, where B_b is the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ defined on a countable product of metric spaces $X = \prod_n X_n$. Under the assumption that there exist certain increasing convex functionals ϕ_n, defined on all bounded Borel measurable functions $g_n : X_n \to \mathbb{R}$, such that

$$\phi(f) \leq \sum_n \phi_n^+(g_n) \quad \text{whenever} \quad f(x) \leq \sum_n g_n(x_n) \text{ for all } x \in X,$$

we derive a convex dual representation of the form

$$\phi(f) = \max_{\mu \in ca^+} \left(\langle f, \mu \rangle - \phi^*_{C_b}(\mu) \right) \quad \text{for all } f \in C_b, \tag{1.1}$$

where ca^+ is the set of all finite Borel measures, C_b the set of bounded continuous functions on X, $\langle f, \mu \rangle$ the integral $\int f d\mu$, and $\phi^*_{C_b}$ the convex conjugate defined by

$$\phi^*_{C_b}(\mu) := \sup_{f \in C_b} \left(\langle f, \mu \rangle - \phi(f) \right).$$

*Department of Mathematics, University of Konstanz, 78464 Konstanz Germany.
†ORFE, Princeton University, Princeton, NJ 08544, USA.
‡Department of Mathematics, University of Konstanz, 78464 Konstanz Germany.
§Faculty of Mathematics, University of Vienna, 1090, Vienna Austria. Partially supported by Vienna Science and Technology Fund (WWTF) under grant MA 14-008.

We also provide equivalent conditions under which the representation (1.1) extends to all bounded upper semicontinuous functions $f : X \to \mathbb{R}$. In the special case, where the functionals ϕ_n are linear, our arguments can be generalized to cover functionals ϕ that are defined on spaces of unbounded functions $f : X \to \mathbb{R}$. This yields variants of the representation (1.1) for unbounded continuous and upper semicontinuous functions $f : X \to \mathbb{R}$.

As an application we derive versions of Kantorovich's transport duality and the recently introduced martingale transport duality in the case where the state space is a countable product of metric spaces. A standard Monge–Kantorovich transport problem consists in finding a probability measure on the product of two metric spaces with fixed marginals that minimizes the expectation of a given cost function. It is a linear optimization problem whose dual has the form of a subreplication problem (which, after changing the sign, becomes a superreplication problem). Kantorovich first showed that there is no duality gap between the two problems under compactness and continuity assumptions in the seminal paper [16]. Since then, the result has been generalized in various directions; see e.g. [18, 13, 19, 1] for an overview. We develop a linear and sublinear version of Kantorovich's duality for countable products of metric spaces and lower semicontinuous cost functions (corresponding to upper semicontinuous functions $f : X \to \mathbb{R}$ in our setup). It has been shown that in the case, where the state space is a finite product of Polish spaces, Kantorovich's duality even holds for Borel measurable cost functions; see e.g. [17, 4, 3]. However, we provide a counter-example illustrating that this is no longer true if the state space is a countable product of compact metric spaces.

Martingale transport duality was discovered by [2, 12] in the context of model-independent finance by noting that the superreplication problem in the presence of liquid markets for European call and put options can be viewed as the dual of a transport problem in which the optimization is carried out over the set of all martingale measures. While [2] considers a discrete-time model with finitely many marginal distributions, [12] works with a continuous-time model with just two marginal distributions. In this paper we obtain a martingale transport duality for countably many time periods and equally many marginal constraints (for martingale transport in continuous time, see e.g. [9, 14] and the references therein). Standard martingale transport duality describes a situation where a financial asset can be traded dynamically without transaction costs and any European derivative can efficiently be replicated with a static investment in European call and put options. From our general results, we obtain a sublinear generalization of the martingale transport duality corresponding to proportional transaction costs and incomplete markets of European call and put options. This extends the duality of [8] to a setup with countably many time periods and markets for European options with all maturities.

Our proofs differ from the standard arguments used in establishing Kantorovich duality and martingale transport duality in that they view the subreplication (or superreplication) problem as the primal problem and use the fact, observed in [7], that it follows from the Daniel–Stone theorem that an increasing convex functional on a function space has a max-representation with countably additive measures if it is continuous from above under point-wise decreasing sequences.

The structure of the paper is as follows: In Section 2 we derive two general representation results for increasing convex functionals satisfying countably many convex marginal constraints. In Section 3 we focus on the special cases where the constraints are sublinear and linear. In Section 4 we derive linear and sublinear versions of Kantorovich's transport duality and the martingale transport duality for countably many marginal constraints.

2 Convex marginal constraints

Let (X_n) be a countable (finite or countably infinite) family of metric spaces, and consider the product topology on $X = \prod_n X_n$. Denote by C_b, U_b and B_b all bounded functions $f : X \to \mathbb{R}$ that are continuous, upper semicontinuous and Borel measurable, respectively. Similarly, let $C_{b,n}$, $U_{b,n}$ and $B_{b,n}$ be all bounded functions $f : X_n \to \mathbb{R}$ that are continuous, upper semicontinuous and Borel measurable, respectively. By ca^+ we denote all finite Borel measures on X and by ca_n^+ all finite Borel measures on X_n. For a measure $\mu \in ca^+$, we define $\mu_n := \mu \circ \pi_n^{-1}$, where $\pi_n : X \to X_n$ is the projection on the n-th coordinate $x \mapsto \pi_n(x) := x_n$. For a sequence $g_n \in B_{b,n}^+$, where $B_{b,n}^+$ is the set of all bounded Borel measurable functions $f : X_n \to \mathbb{R}_+$, we denote $\oplus g := \sum_n g_n \circ \pi_n : X \to \mathbb{R}_+ \cup \{+\infty\}$. When we write $f_j \downarrow f$, we mean that f_j is a decreasing sequence of functions that converges point-wise to f.

Our goal in this section is to derive a dual representation for an increasing convex functional $\phi : B_b \to \mathbb{R}$, where by increasing we mean that $\phi(f) \geq \phi(g)$ whenever $f \geq g$ and the second inequality is understood point-wise. For every n, let $\phi_n : B_{b,n} \to \mathbb{R}$ be an increasing convex functional satisfying the following tightness condition: for all $m, \varepsilon \in \mathbb{R}_+ \setminus \{0\}$, there exists a compact $K_n \subseteq X_n$ such that

$$\phi_n(m 1_{K_n^c}) \leq \varepsilon. \tag{2.1}$$

(In the special case where ϕ_n is given by $\phi_n(f) = \sup_{\nu \in \mathcal{P}_n} \int f d\nu$ for a set of Borel probability measures $\mathcal{P}_n$ on X_n, (2.1) corresponds to tightness of $\mathcal{P}_n$. A related condition for convex risk measures was introduced in [11].)

In the rest of the paper we use the notation $\langle f, \mu \rangle := \int f d\mu$ and define the convex conjugates

$$\phi^*_{C_b}(\mu) : ca^+ \to \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad \phi^*_n : ca_n^+ \to \mathbb{R} \cup \{+\infty\}$$

by

$$\phi^*_{C_b}(\mu) := \sup_{f \in C_b} (\langle f, \mu \rangle - \phi(g)) \quad \text{and} \quad \phi^*_n(\mu_n) := \sup_{f \in C_{b,n}} (\langle f, \mu_n \rangle - \phi_n(g)).$$

Then the following holds:

Theorem 2.1. *Let $\phi : B_b \to \mathbb{R}$ be an increasing convex functional satisfying $\phi(f) \leq \sum_n \phi_n^+(g_n)$ for all $f \in B_b$ and $g_n \in B_{b,n}^+$ such that $f \leq \oplus g$. Then*

$$\phi(f) = \max_{\mu \in ca^+} (\langle f, \mu \rangle - \phi^*_{C_b}(\mu)) \quad \textit{for all } f \in C_b.$$

Proof. Fix $f \in C_b$ and let (f_j) be a sequence in C_b such that $f_j \downarrow 0$. Since $\alpha \mapsto \phi(\alpha f)$ is a real-valued convex function on $\mathbb{R}$, it is continuous. So, for a given constant $\varepsilon > 0$, one can choose $\alpha \in (0,1)$ small enough such that $(1-\alpha)\phi(f/(1-\alpha)) - \phi(f) \leq \varepsilon$. By assumption, there exist compact sets $K_n \subseteq X_n$ such that $\sum_n \phi_n^+(g_n) \leq \varepsilon$, where

$$g_n := \frac{2}{\alpha} \|f_1\|_\infty 1_{K_n^c}.$$

By Tychonoff's theorem, $K := \prod_n K_n \subseteq X$ is compact. Since the function

$$\tilde{\phi}(\cdot) := \phi(\cdot + f) - \phi(f) : B_b \to \mathbb{R}$$

is convex, one has

$$\tilde{\phi}(f_j) \leq \frac{\tilde{\phi}(2 f_j 1_K) + \tilde{\phi}(2 f_1 1_{K^c})}{2}.$$

By Dini's lemma, $f_j \to 0$ uniformly on the compact K. Since $\lim_{\alpha\to 0}\tilde{\phi}(\alpha 1) = 0$, one obtains by monotonicity that $\tilde{\phi}(2f_j 1_K) \to 0$. On the other hand, as $\frac{2}{\alpha} f_1 1_{K^c} \le \oplus g$, one has

$$\phi\left(\frac{2}{\alpha} f_1 1_{K^c}\right) \le \sum_n \phi_n^+(g_n) \le \varepsilon,$$

and therefore,

$$\tilde{\phi}(2f_1 1_{K^c}) \le \alpha\phi\Big(\frac{2}{\alpha} f_1 1_{K^c}\Big) + (1-\alpha)\phi\Big(\frac{f}{1-\alpha}\Big) - \phi(f) \le 2\varepsilon.$$

This shows that $\phi(f+f_j) \downarrow \phi(f)$. By the Hahn–Banach extension theorem, there exists a positive linear functional $\psi : C_b \to \mathbb{R}$ such that

$$\psi(g) \le \tilde{\phi}(g) = \phi(f+g) - \phi(f) \quad \text{for all } g \in C_b.$$

Since $\psi(g_j) \downarrow 0$ for every sequence (g_j) in $C_b(X)$ satisfying $g_j \downarrow 0$, one obtains from the Daniel-Stone theorem (see e.g. [10]) that there exists a $\nu \in ca^+$ such that $\psi(g) = \langle g, \nu\rangle$ for all $g \in C_b$. It follows that $\phi(f) + \phi^*_{C_b}(\nu) \le \langle f, \nu\rangle$, which together with $\phi(f) \ge \sup_{\mu\in ca^+}(\langle f,\mu\rangle - \phi^*_{C_b}(\mu))$ implies that

$$\phi(f) = \max_{\mu\in ca^+} \left(\langle f,\mu\rangle - \phi^*_{C_b}(\mu)\right).$$

□

The next result gives conditions under which the dual representation of Theorem 2.1 extends to the set of bounded upper semicontinuous functions U_b. We call a subset Λ of ca^+ sequentially compact if every sequence in Λ has a subsequence that converges to some $\mu \in \Lambda$ with respect to the topology $\sigma(ca^+, C_b)$.

Theorem 2.2. *Let $\phi : B_b \to \mathbb{R}$ be an increasing convex functional satisfying the assumption of Theorem 2.1. Then the lower level sets $\Lambda_a := \{\mu \in ca^+ : \phi^*_{C_b}(\mu) \le a\}$, $a \in \mathbb{R}$, are sequentially compact, and the following are equivalent:*

(i) $\phi(f) = \max_{\mu\in ca^+}(\langle f,\mu\rangle - \phi^*_{C_b}(\mu))$ *for all* $f \in U_b$

(ii) $\phi(f_j) \downarrow \phi(f)$ *for all* $f \in U_b$ *and every sequence* (f_j) *in* C_b *satisfying* $f_j \downarrow f$

(iii) $\phi(f) = \inf_{g\in C_b,\, g\ge f} \phi(g)$ *for all* $f \in U_b$

(iv) $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) := \sup_{f\in U_b}(\langle f,\mu\rangle - \phi(f))$ *for all* $\mu \in ca^+$.

Proof. It is clear that for all $a \in \mathbb{R}$, Λ_a is $\sigma(ca^+, C_b)$-closed. Moreover, for all $\mu \in ca^+$,

$$\phi^*_{C_b}(\mu) \ge \sup_{x\in\mathbb{R}_+}(\langle x1,\mu\rangle - \phi(x1)) = \gamma(\langle 1,\mu\rangle),$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}\cup\{+\infty\}$ is the increasing convex function given by

$$\gamma(y) := \sup_{x\in\mathbb{R}_+}(xy - \phi(x1)).$$

Since $\lim_{y\to+\infty}\gamma(y)/y = +\infty$, the right-continuous inverse $\gamma^{-1} : \mathbb{R} \to \mathbb{R}_+$ given by

$$\gamma^{-1}(x) := \sup\{y \in \mathbb{R}_+ : \gamma(y) \le x\} \quad \text{with} \quad \sup\emptyset := 0,$$

is increasing and satisfies $\lim_{x\to+\infty}\gamma^{-1}(x)/x = 0$. For every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ such that $(a+1)/m \le \varepsilon$ and compact sets $K_n \subseteq X_n$ so that $\sum_n \phi_n(m1_{K_n^c}) \le 1$. Since $m1_{K^c} \le \oplus g$ for the compact $K := \prod_n K_n$ and $g_n := m1_{K_n^c}$, one has $\phi(m1_{K^c}) \le \sum_n \phi_n(m1_{K_n^c}) \le 1$. As $m1_{K^c}$ is lower semicontinuous, there exists a sequence (g_j) in C_b such that $g_j \uparrow m1_{K^c}$. Since $\phi(g_j) \le \phi(m1_{K^c}) \le 1$, one has for all $\mu \in \Lambda_a$,

$$m\mu(K^c) = \sup_j \langle g_j, \mu \rangle \le \sup_j (\langle g_j, \mu \rangle - \phi(g_j) + 1) \le \phi^*_{C_b}(\mu) + 1 \le a + 1.$$

So $\mu(K^c) \le \varepsilon$ and $\mu(X) = \langle 1, \mu \rangle \le \gamma^{-1}\big(\phi^*_{C_b}(\mu)\big) \le \gamma^{-1}(a)$. Now one obtains from the first half of Prokhorov's theorem (see e.g. Theorem 5.1 in [5]) that Λ_a is sequentially compact.

(i) $\Rightarrow$ (ii): Fix $f \in U_b$ and a sequence (f_j) in C_b such that $f_j \downarrow f$. If (i) holds, there exists a sequence (μ_j) in ca^+ such that

$$\phi(f_j) = \langle f_j, \mu_j \rangle - \phi^*_{C_b}(\mu_j) \le \|f_1\|_\infty \langle 1, \mu_j \rangle - \phi^*_{C_b}(\mu_j) \le \|f_1\|_\infty \gamma^{-1}(\phi^*_{C_b}(\mu_j)) - \phi^*_{C_b}(\mu_j).$$

It follows that (μ_j) is in Λ_a for some $a \in \mathbb{R}$ large enough. Therefore, after possibly passing to a subsequence, μ_j converges to a measure $\mu \in \Lambda_a$ in $\sigma(ca^+, C_b)$. Clearly, $\phi^*_{C_b}$ is $\sigma(ca^+, C_b)$-lower semicontinuous, and so

$$\phi^*_{C_b}(\mu) \le \liminf_j \phi^*_{C_b}(\mu_j).$$

Moreover, for every $\varepsilon > 0$, there is a k such that $\langle f_k, \mu \rangle \le \langle f, \mu \rangle + \varepsilon$. Now choose $j \ge k$ such that $\langle f_k, \mu_j \rangle \le \langle f_k, \mu \rangle + \varepsilon$. Then

$$\langle f_j, \mu_j \rangle \le \langle f_k, \mu_j \rangle \le \langle f_k, \mu \rangle + \varepsilon \le \langle f, \mu \rangle + 2\varepsilon.$$

It follows that $\limsup_j \langle f_j, \mu_j \rangle \le \langle f, \mu \rangle$, and therefore,

$$\lim_j \phi(f_j) = \lim_j \langle f_j, \mu_j \rangle - \phi^*_{C_b}(\mu_j) \le \langle f, \mu \rangle - \phi^*_{C_b}(\mu) \le \phi(f),$$

showing that $\phi(f_j) \downarrow \phi(f)$.

(ii) $\Rightarrow$ (iii): Let $f \in U_b$. Since the product topology on X is metrizable, there exists a sequence (f_j) in C_b such that $f_j \downarrow f$. So it is clear that (iii) follows from (ii).

(iii) $\Rightarrow$ (vi): It is immediate from the definitions that $\phi^*_{U_b} \ge \phi^*_{C_b}$. On the other hand, if (iii) holds, then for every $f \in U_b$, there is a sequence (f_j) in $C_b(S)$ such that $f_j \ge f$ and $\phi(f_j) \downarrow \phi(f)$. In particular,

$$\sup_j (\langle f_j, \mu \rangle - \phi(f_j)) \ge \langle f, \mu \rangle - \phi(f),$$

from which one obtains $\phi^*_{C_b} \ge \phi^*_{U_b}$.

(iv) $\Rightarrow$ (i): Fix $f \in U_b$. It is a direct consequence of the definition of $\phi^*_{U_b}$ that

$$\phi(f) \ge \sup_{\mu \in ca^+} (\langle f, \mu \rangle - \phi^*_{U_b}(\mu)) = \sup_{\mu \in ca^+} (\langle f, \mu \rangle - \phi^*_{C_b}(\mu)).$$

On the other hand, there exists a sequence (f_j) in C_b such that $f_j \downarrow f$. Since

$$\langle f_j, \mu \rangle \le \langle f_1, \mu \rangle \le \|f_1\|_\infty \langle 1, \mu \rangle \le \|f_1\|_\infty \gamma^{-1}\big(\phi^*_{C_b}(\mu)\big),$$

it follows from Theorem 2.1 that one can choose $a \in \mathbb{R}$ large enough such that

$$\phi(f_j) = \langle f_j, \mu_j \rangle - \phi^*_{C_b}(\mu_j)$$

for a sequence (μ_j) in the sequentially compact set Λ_a. After passing to a subsequence, μ_j converges to a μ in $\sigma(ca^+, C_b)$. Then it follows as above that

$$\phi(f) \le \lim_j \phi(f_j) = \lim_j (\langle f_j, \mu_j \rangle - \phi^*_{C_b}(\mu_j)) \le \langle f, \mu \rangle - \phi^*_{C_b}(\mu),$$

from which one obtains $\phi(f) = \max_{\mu \in ca^+}(\langle f, \mu \rangle - \phi^*_{C_b}(\mu))$. □

3 Sublinear and linear marginal constraints

In this section we assume the X_n to be Polish spaces and the mappings $\phi_n : B^+_{b,n} \to \mathbb{R}$ to be of the form

$$\phi_n(g) = \sup_{\nu_n \in \mathcal{P}_n} \langle g, \nu_n \rangle ,$$

where $\mathcal{P}_n$ is a non-empty convex $\sigma(ca^+_n, C_{b,n})$-compact set of Borel probability measures on X_n. Then all ϕ_n are increasing and sublinear. Moreover, they have the translation property

$$\phi_n(g + m) = \phi_n(g) + m, \quad g \in B_{b,n},\ m \in \mathbb{R},$$

and it follows from Prokhorov's theorem that they satisfy the tightness condition (2.1); see e.g. [5]. By $\mathcal{P}$ we denote the set of Borel probability measures μ on $X = \prod_n X_n$ whose marginal distributions μ_n are in $\mathcal{P}_n$ for all n. Under these assumptions we obtain the following result:

Proposition 3.1. *Let $\phi : B_b \to \mathbb{R}$ be an increasing convex functional satisfying*

$$\phi(f) \le m + \sum\nolimits_n \phi_n(g_n) \tag{3.1}$$

whenever $f \le m + \oplus g$ for some $m \in \mathbb{R}$ and $g_n \in B^+_{b,n}$. Then

$$\phi(f) = \max_{\mu \in \mathcal{P}} \left(\langle f, \mu \rangle - \phi^*_{C_b}(\mu) \right) \quad \text{for all } f \in C_b. \tag{3.2}$$

If in addition, $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in \mathcal{P}$, the representation (3.2) *extends to all $f \in U_b$.*

Proof. Since $\phi_n(g_n) \ge 0$ for $g_n \in B^+_{b,n}$, one obtains from Theorem 2.1 that

$$\phi(f) = \max_{\mu \in ca^+} (\langle f, \mu \rangle - \phi^*_{C_b}(\mu)) \quad \text{for all } f \in C_b,$$

and from Theorem 2.2 that the representation holds for all $f \in U_b$ if $\phi^*_{C_b} = \phi^*_{U_b}$. So the proposition follows if we can show that $\phi^*_{C_b}(\mu) = +\infty$ for all $\mu \in ca^+ \setminus \mathcal{P}$. So fix a $\mu \in ca^+ \setminus \mathcal{P}$. If it is not a probability measure, then

$$\phi^*(\mu) \ge \sup_{m \in \mathbb{R}} (\langle m, \mu \rangle - \phi(m)) \ge \sup_{m \in \mathbb{R}} (\langle m, \mu \rangle - m) = +\infty.$$

On the other hand, if μ is a probability measure, but does not belong to $\mathcal{P}$, one obtains from the Hahn–Banach separation theorem that there exist n and $g_n \in C_{b,n}$ such that $\langle g_n, \mu \rangle > \phi_n(g_n)$. Moreover, since ϕ_n has the translation property, g_n can be shifted until it is non-negative. Then

$$\phi(m g_n \circ \pi_n) \le \phi_n(m g_n) = m \phi_n(g_n) \quad \text{for all } m \in \mathbb{R}_+,$$

and therefore,

$$\phi^*_{C_b}(\mu) \ge \sup_{m \in \mathbb{R}_+} (\langle m g_n \circ \pi_n, \mu \rangle - \phi(m g_n \circ \pi_n)) \ge \sup_{m \in \mathbb{R}_+} m(\langle g_n, \mu_n \rangle - \phi_n(g_n)) = +\infty.$$

□

In the next step we concentrate on the special case where every $\mathcal{P}_n$ consists of just one Borel probability measure ν_n on X_n. Then the mappings ϕ_n are of the form $\phi_n(g) = \langle g, \nu_n \rangle$. In particular, they are linear, and the representation (3.2) can be extended to unbounded functions f.

Let us denote by $\mathcal{P}(\nu)$ the set of all Borel probabilities on X with marginals $\mu_n = \nu_n$. Furthermore, let B be the space of all Borel measurable functions $f : X \to \mathbb{R}$, U the subset of upper semicontinuous functions $f : X \to \mathbb{R}$ and B_n^+ the set of all Borel measurable functions $f : X_n \to \mathbb{R}_+$. Consider the following sets:

$$\begin{aligned} G(\nu) &:= \left\{ \oplus g : (g_n) \in \prod\nolimits_n B_n^+ \text{ such that } \sum\nolimits_n \langle g_n, \nu_n \rangle < +\infty \right\} \\ B(\nu) &:= \{ f \in B : |f| \leq \oplus g \text{ for some } \oplus g \in G(\nu) \} \\ U(\nu) &:= \{ f \in U : f^+ \in B_b \text{ and } f^- \in B(\nu) \}. \end{aligned}$$

Note that $G(\nu)$ is not contained in $B(\nu)$ since a function $\oplus g \in G(\nu)$ can take on the value $+\infty$. But one has $\langle \oplus g, \mu \rangle = \sum_n \langle g_n, \nu_n \rangle < +\infty$ for all $\oplus g \in G(\nu)$ and $\mu \in \mathcal{P}(\nu)$. So $G(\nu)$ is contained in $L^1(\mu)$, and every $\oplus g \in G(\nu)$ is finite μ-almost surely.

Proposition 3.2. *Let $\phi : B(\nu) \to \mathbb{R}$ be increasing and convex such that*

$$\phi(f) \leq m + \sum\nolimits_n \langle g_n, \nu_n \rangle \tag{3.3}$$

if $f \leq m + \oplus g$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\nu)$. Moreover, assume that

$$\phi^*_{C_b}(\mu) = \phi^*_{U(\nu)}(\mu) := \sup_{f \in U(\nu)} (\langle f, \mu \rangle - \phi(f)) \quad \textit{for all } \mu \in \mathcal{P}(\nu).$$

Then

$$\phi(f) = \max_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \textit{for all } f \in B(\nu) \cap (U(\nu) + G(\nu)).$$

Proof. By Proposition 3.1, one has

$$\phi(f) = \max_{\mathcal{P}(\nu)} (\langle f, \mu \rangle - \phi^*_{C_b}(\mu)) \quad \text{for all } f \in C_b.$$

Furthermore, for given $f \in U(\nu)$, there exists a sequence (f_j) in C_b such that $f_j \downarrow f$, and it follows as in the proof of (iv) $\Rightarrow$ (i) in Theorem 2.2 that there exists a $\mu \in \mathcal{P}(\nu)$ such that $\phi(f) \leq \langle f, \mu \rangle - \phi^*_{C_b}(\mu)$. Since on the other hand,

$$\phi(f) \geq \sup_{\mu \in \mathcal{P}(\nu)} (\langle f, \mu \rangle - \phi^*_{U(\nu)}(\mu)) = \sup_{\mu \in \mathcal{P}(\nu)} (\langle f, \mu \rangle - \phi^*_{C_b}(\mu)),$$

one obtains

$$\phi(f) = \max_{\mu \in \mathcal{P}(\nu)} \Big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu) \Big).$$

Next, notice that it follows, as in the proof of Theorem 2.1, from the Hahn–Banach extension theorem that

$$\phi(f) = \max_{\psi \in B'(\nu)} (\psi(f) - \phi^*(\psi)) \quad \text{for all } f \in B(\nu),$$

where $B'(\nu)$ is the algebraic dual of $B(\nu)$ and $\phi^*(\psi) := \sup_{f \in B(\nu)} (\psi(f) - \phi(f))$, $\psi \in B'(\nu)$. For $\psi \in B'(\nu)$ with $\phi^*(\psi) < +\infty$, one has for all $\oplus g \in G(\nu) \cap B(\nu)$,

$$\psi(\oplus g) - \sum\nolimits_n \langle g_n, \nu_n \rangle \leq \psi(\oplus g) - \phi(\oplus g) \leq \phi^*(\psi) < +\infty,$$

and therefore, $\psi(\oplus g) \leq \sum_n \langle g_n, \nu_n \rangle$. Consequently, if one sets $g_n^N := g_n \wedge N$ for $n \leq N$ and $g_n^N := 0$ for $n > N$, then

$$\psi(N^2 - \oplus g^N) \leq N^2 - \sum_{n=1}^{N} \langle g_n \wedge N, \nu_n \rangle,$$

from which one obtains

$$\psi(\oplus g) \geq \lim_N \psi(\oplus g^N) \geq \lim_N \sum_{n=1}^N \langle g_n \wedge N, \nu_n \rangle = \sum_n \langle g_n, \nu_n \rangle .$$

This shows that $\psi(\oplus g) = \sum_n \langle g_n, \nu_n \rangle$ for all $\oplus g \in G(\nu) \cap B(\nu)$, and as a result,

$$\phi(f - \oplus g) = \max_{\psi \in B'(\nu)} (\psi(f - \oplus g) - \phi^*(\psi)) = \phi(f) - \sum_n \langle g_n, \nu_n \rangle$$

for all $f \in B(\nu)$ and $\oplus g \in G(\nu)$. Finally, let $f \in B(\nu)$ be of the form $f = \oplus g + h$ for $\oplus g \in G(\nu)$ and $h \in U(\nu)$. Then $f - \oplus g \in U(\nu)$ and $\oplus g \in G(\nu) \cap B(\nu)$. So

$$\begin{aligned}\phi(f) - \sum\nolimits_n \langle g_n, \nu_n \rangle &= \phi(f - \oplus g) = \max_{\mu \in \mathcal{P}(\nu)} \big(\langle f - \oplus g, \mu \rangle - \phi^*_{C_b}(\mu)\big) \\ &= \max_{\mu \in \mathcal{P}(\nu)} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\big) - \sum\nolimits_n \langle g_n, \nu_n \rangle,\end{aligned}$$

and hence, $\phi(f) = \max_{\mu \in \mathcal{P}(\nu)} \big(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\big)$. □

4 Generalized (martingale) transport dualities

As a consequence of the results in Section 3, one obtains generalized versions of the classical transport duality and the more recently introduced martingale transport duality.

4.1 Generalized transport dualities

As in Section 3, let X_n be Polish spaces. We first study the case where a probability measure ν_n is given on each X_n. For given $f \in B(\nu)$, consider the minimization problem

$$\phi(f) := \inf \Big\{ m + \sum\nolimits_n \langle g_n, \nu_n \rangle : m \in \mathbb{R}, \oplus g \in G(\nu) \text{ such that } m + \oplus g \geq f \Big\} . \tag{4.1}$$

Remark 4.1. Up to a different sign, (4.1) can be viewed as a generalized version of the dual of a transport problem. A standard transport problem in the sense of Kantorovich consists in finding a Borel probability measure μ on the product of two metric spaces $X_1 \times X_2$ with given marginals ν_1 and ν_2 that minimizes the expectation $\mathbb{E}^\mu c$ of a cost function $c : X_1 \times X_2 \to \mathbb{R}$. The (negative of the) dual problem is a minimization problem of the form

$$\inf \sum_{n=1}^2 \langle g_n, \nu_n \rangle , \tag{4.2}$$

where the infimum is taken over all $g_n \in L^1(\nu_n)$ such that $\oplus g \geq f := -c$. To relate (4.1) more closely to (4.2), note that if $X = \prod_n X_n$ is a countably infinite product, $\oplus g^1 - \oplus g^2$ is well-defined for all $\oplus g^1 \in G(\nu)$ and $\oplus g^2 \in G(\nu) \cap B(\nu)$. So instead of (4.1), we could have defined $\phi(f)$ equivalently as

$$\inf \Big\{ \sum\nolimits_n \langle g^1_n - g^2_n, \nu_n \rangle : \oplus g^1 \in G(\nu), \oplus g^2 \in G(\nu) \cap B(\nu) \text{ such that } \oplus g^1 - \oplus g^2 \geq f \Big\} .$$

Indeed, it is clear that $\phi(f)$ dominates the above infimum, and since $\lim_{N \to +\infty} \sum_{n=1}^N \langle g^2_n \wedge N, \nu_n \rangle = \sum_n \langle g^2_n, \nu_n \rangle$, it cannot exceed it.

As a consequence of the results in Section 3, one obtains the following version of Kantorovich's duality with countably many marginal distributions:

Corollary 4.2.

$$\phi(f) = \max_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \textit{for all } f \in B(\nu) \cap (U(\nu) + G(\nu)). \tag{4.3}$$

Proof. It is clear that $\phi(f) < +\infty$ for all $f \in B(\nu)$. On the other hand, since $\mathcal{P}(\nu)$ is non-empty (it contains the product measure $\otimes_n \nu_n$), one has

$$m + \sum_n \langle g_n, \nu_n \rangle \geq \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle > -\infty$$

for all $f \in B(\nu)$ such that $m + \oplus g \geq f$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\nu)$. It follows that $\phi : B(\nu) \to \mathbb{R}$ is an increasing sublinear functional such that

$$\phi(f) \geq \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \text{for all } f \in B(\nu).$$

In particular, $\phi(0) = 0$, and $\phi^*_{C_b}(\mu) = \phi^*_{U(\nu)}(\mu) = 0$ for all $\mu \in \mathcal{P}(\nu)$. So the duality (4.3) follows from Proposition 3.2. □

Remark 4.3. If X is a finite product of Polish spaces, it can be shown that

$$\phi(f) = \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \text{for all } f \in B_b;$$

see e.g. [17, 4, 3]. But for countably infinite products, there may arise a duality gap; that is, it may happen that

$$\phi(f) > \sup_{\mu \in \mathcal{P}(\nu)} \langle f, \mu \rangle \quad \text{for some } f \in B_b.$$

For instance, if X is the product of $X_n = \{0, 1\}$, $n \in \mathbb{N}$, and $\nu_n = \frac{1}{2}(\delta_0 + \delta_1)$ for all n, then $f := \liminf_{n \to \infty} \pi_n$ belongs to B_b, and it follows from Fatou's lemma that

$$\langle f, \mu \rangle \leq \liminf_{n \to \infty} \langle \pi_n, \mu \rangle = \frac{1}{2} \quad \text{for all } \mu \in \mathcal{P}(\nu).$$

On the other hand, assume $f \leq m + \oplus g$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\nu)$. Since

$$\frac{1}{2} \sum_n (g_n(0) + g_n(1)) = \sum_n \langle g_n, \nu_n \rangle < +\infty,$$

one has $\sum_n g_n(x_n) < +\infty$, for all $x \in X$, and therefore,

$$\inf_{k \in \mathbb{N}} \min_{(y_1, \dots, y_k) \in \{0,1\}^k} \left(\sum_{n \leq k} g_n(y_n) + \sum_{n > k} g_n(x_n) \right) = \sum_n \min_{y_n \in \{0,1\}} g_n(y_n) \leq \sum_n \langle g_n, \nu_n \rangle .$$

Consequently,

$$1 = \inf_{k \in \mathbb{N}} \min_{(y_1, \dots, y_k) \in \{0,1\}^k} f(y_1, \dots, y_k, 1, 1, \dots) \leq m + \sum\nolimits_n \langle g_n, \nu_n \rangle ,$$

from which it follows that $\phi(f) \geq 1$.

In the more general case, where the $\phi_n : B_{b,n} \to \mathbb{R}$ are sublinear functionals given by

$$\phi_n(g) = \sup_{\nu_n \in \mathcal{P}_n} \langle g, \nu_n \rangle$$

for non-empty convex $\sigma(ca_n^+, C_{b,n})$-compact sets of Borel probability measures $\mathcal{P}_n$ on X_n, we obtain a generalized Kantorovich duality with countably many sets of marginal distributions. As in Section 3, $\mathcal{P}$ denotes the set of probability distributions such that $\mu_n \in \mathcal{P}_n$ for all n. Compared to Corollary 4.2, one has to modify the definition of ϕ slightly:

$$\phi(f) := \inf \left\{ m + \sum\nolimits_n \phi_n(g_n) : m \in \mathbb{R},\, g_n \in B_{b,n}^+ \text{ such that } m + \oplus g \geq f \right\}. \tag{4.4}$$

Then, an application of Proposition 3.1 and essentially the same arguments as in the proof of Corollary 4.2 yield the following duality:

Corollary 4.4.

$$\phi(f) = \max_{\mu \in \mathcal{P}} \langle f, \mu \rangle \quad \textit{for all } f \in U_b. \tag{4.5}$$

Proof. As in the proof of Corollary 4.2 it is easy to see that $\phi : B_b \to \mathbb{R}$ is an increasing sublinear functional such that

$$\phi(f) \geq \sup_{\mu \in \mathcal{P}} \langle f, \mu \rangle \quad \text{for all } f \in B_b.$$

Since $\mathcal{P}$ is non-empty (it contains all product measures $\otimes_n \nu_n$ for $\nu_n \in \mathcal{P}_n$), it follows that $\phi(0) = 0$ and $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) = 0$ for all $\mu \in \mathcal{P}$. So (4.5) follows from Proposition 3.1. □

4.2 Generalized martingale transport dualities

As further applications of the results of Section 3 we deduce linear and sublinear versions of the martingale transport duality with countably many marginal constraints. We let X_n be non-empty closed subsets of $\mathbb{R}^d$ and model the discounted prices of d financial asset by $S_0 := s_0 \in \mathbb{R}^d$ and $S_n(x) := x_n$, $x \in X = \prod_n X_n$, $n \geq 1$. The corresponding filtration is given by $\mathcal{F}_n := \sigma(S_j : j \leq n)$.

We first assume that each space X_n carries a single Borel probability measure ν_n. Moreover, we suppose that money can be lent and borrowed at the same rate and European options with general discounted payoffs $g_n \in B_n^+$ can be bought at price $\langle g_n, \nu_n \rangle$ (we suppose they either exist as structured products or they can be synthesized with a mix of more standard options; see [6] for the form of ν_n if European call options exist with maturity n and all strikes). A function $\oplus g \in G(\nu)$ then corresponds to a static option portfolio costing $\sum_n \langle g_n, \nu_n \rangle$. In addition, the underlying can be traded dynamically. The set $\mathcal{H}$ of dynamic trading strategies consists of all finite sequences $h_1, \dots, h_N$ such that each h_n is an $\mathbb{R}^d$-valued $\mathcal{F}_{n-1}$-measurable function on X. An $h \in \mathcal{H}$ generates gains of the form

$$(h \cdot S)_N := \sum_{n=1}^{N} h_n \cdot (S_n - S_{n-1}).$$

A triple $(m, \oplus g, h) \in \Theta := \mathbb{R} \times G(\nu) \times \mathcal{H}$ describes a semi-static trading strategy with cost $m + \sum_n \langle g_n, \nu_n \rangle$ and outcome $m + \oplus g + (h \cdot S)_N$.

A strategy $(m, \oplus g, h) \in \Theta$ is said to be a model-independent arbitrage if

$$m + \sum\nolimits_n \langle g_n, \nu_n \rangle \leq 0 \quad \text{and} \quad m + \oplus g + (h \cdot S)_N > 0.$$

We denote by $\mathcal{M}(\nu)$ the set of probability measures $\mu \in \mathcal{P}(\nu)$ under which S is a d-dimensional martingale and introduce the superhedging functional

$$\phi(f) := \inf \left\{ m + \sum\nolimits_n \langle g_n, \nu_n \rangle : (m, \oplus g, h) \in \Theta \text{ such that } m + \oplus g + (h \cdot S)_N \geq f \right\}. \tag{4.6}$$

Remark 4.5. The static part of a semi-static strategy in Θ consists of a cash position and a portfolio of options with non-negative payoffs. But one could extend the set of strategies to include portfolios with outcomes $\oplus g^1 - \oplus g^2 + (h \cdot S)_N$ and prices $\sum_n \langle g_n^1 - g_n^2, \nu_n \rangle$ for $g^1 \in G(\nu)$, $g^2 \in G(\nu) \cap B(\nu)$ and $h \in \mathcal{H}$. It follow as in Remark 4.1 that this would neither change the definition of a model-independent arbitrage nor the superhedging functional (4.6).

The following corollary extends the superheding duality of [2] to a model with countably many time periods and contains a model-independent no-arbitrage result as a consequence. For $x \in X_n \subseteq \mathbb{R}^d$, denote by $|x|$ the Euclidean length of x.

Corollary 4.6. *Assume that $\int_{X_n} |x| \, d\nu_n(x) < \infty$ for all n. Then, the following are equivalent:*

(i) *There is no model-independent arbitrage*

(ii) $\mathcal{M}(\nu) \neq \emptyset$.

Moreover, if (i)–(ii) *hold, then*

$$\phi(f) = \max_{\mu \in \mathcal{M}(\nu)} \langle f, \mu \rangle \quad \text{for all } f \in B(\nu) \cap (U(\nu) + G(\nu)). \tag{4.7}$$

Proof. (ii) $\Rightarrow$ (i): If there exists a μ in $\mathcal{M}(\nu)$ and a strategy $(m, \oplus g, h) \in \Theta$ such that $m + \oplus g + (h \cdot S)_N > 0$, one has $\mathbb{E}^\mu (h \cdot S)_N^- \leq m^+ + \sum_n \langle g_n, \nu_n \rangle < +\infty$. So it follows that $(h \cdot S)_n$, $n = 1, \ldots, N$, is a martingale under μ (see e.g. [15]). In particular, $\mathbb{E}^\mu (h \cdot X)_N = 0$, and therefore,

$$m + \sum\nolimits_n \langle g_n, \nu_n \rangle = \langle m + \oplus g + (h \cdot S)_N, \mu \rangle > 0.$$

This shows that there exists no model-independent arbitrage.

(i) $\Rightarrow$ (ii): $\phi : B(\nu) \to \mathbb{R} \cup \{-\infty\}$ is an increasing sublinear functional with the property that $\phi(f) \leq m + \sum_{n \geq 1} \langle g_n, \nu_n \rangle$ whenever $f \leq m + \oplus g$ for some $m \in \mathbb{R}$ and $g \in G(\nu)$. If there is no model-independent arbitrage, one has $\phi(0) = 0$, from which it follows by subadditivity that $\phi(f) > -\infty$ for all $f \in B(\nu)$. Moreover, if

$$m + \oplus g + (h \cdot S)_N \geq f$$

for $(m, \oplus g, h) \in \Theta$ and $f \in B(\nu)$, one has for all $\mu \in \mathcal{M}(\nu)$,

$$\mathbb{E}^\mu (h \cdot S)_N^- \leq m^+ + \sum_n \langle g_n, \nu_n \rangle + \langle f^-, \mu \rangle < +\infty.$$

It follows as above that $\mathbb{E}^\mu (h \cdot S)_N = 0$, and therefore, $m + \sum_n \langle g_n, \nu_n \rangle \geq \langle f, \mu \rangle$. This implies that $\phi(f) \geq \langle f, \mu \rangle$, and consequently, $\phi^*_{C_b}(\mu) = \phi^*_{U(\nu)}(\mu) = 0$ for all $\mu \in \mathcal{M}(\nu)$. So if we can show that

$$\phi^*_{C_b}(\mu) = +\infty \quad \text{for all } \mu \in \mathcal{P}(\nu) \setminus \mathcal{M}(\nu), \tag{4.8}$$

we obtain from Proposition 3.2 that (4.11) holds, which in turn, implies that $\mathcal{M}(\nu)$ cannot be empty.

To show (4.12), let $\mu \in \mathcal{P}(\nu)$. If $\mathbb{E}^\mu S_1 = s_0$ and $\mathbb{E}^\mu [v(x_1, \ldots, x_n)(x_{n+1} - x_n)] = 0$ for all $n \geq 1$ and every bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}^d$, then S is a martingale under μ, and therefore, $\mu \in \mathcal{M}(\nu)$. So for $\mu \in \mathcal{P}(\nu) \setminus \mathcal{M}(\nu)$, there exists an $f \in C \cap B(\nu)$ with $\langle f, \mu \rangle > 0$ such that f is either of the form $f(x) = v \cdot (x_1 - s_0)$ for a vector $v \in \mathbb{R}^d$ or $f(x) = v(x_1, \ldots, x_n) \cdot (x_{n+1} - x_n)$ for some $n \geq 1$ and a bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}^d$. For $k \in \mathbb{N}$, $f^k := f \wedge k$ is bounded above and $f^k_k := f^k \vee (-k)$ bounded. By monotonicity, one has $\phi(f^k) \leq \phi(f) \leq 0$. Moreover,

$$f^k_k(x) = f^k(x) + (k + f(x))^- \leq f^k(x) + w^k(x),$$

where

$$w^k(x) := (c|x_n| - k/2)^+ + (c|x_{n+1}| - k/2)^+$$

and $c \in \mathbb{R}_+$ is a bound on $|v|$. Since $w^k(x)$ is in $G(\nu)$, one gets

$$\phi(w^k) \le \int_{X_n} (c|x_n| - k/2)^+ d\nu_n(x_n) + \int_{X_{n+1}} (c|x_{n+1}| - k/2)^+ d\nu_{n+1}(x_{n+1}) \to 0 \quad \text{for } k \to +\infty.$$

So for k large enough, one obtains from monotonicity and subadditivity that

$$\langle f_k^k, \mu \rangle - \phi(f_k^k) \ge \langle f^k, \mu \rangle - \phi(f^k) - \phi(w^k) \ge \langle f^k, \mu \rangle - \phi(w^k) > 0,$$

and as a result, $\phi^*_{C_b}(\mu) = +\infty$. □

Now, we extend the setting of Corollary 4.6 by adding friction and incompleteness. To simplify the presentation we assume that each X_n is a non-empty closed subset of $\mathbb{R}^d_+$. As above, $S_0 = s_0 \in \mathbb{R}^+_d$, $S_n(x) = x_n$, $x \in X$, and the set of dynamic trading strategies $\mathcal{H}$ is given by all finite sequences $h_1, \dots, h_N$ of $\mathcal{F}_{n-1}$-measurable mappings $h_n : X \to \mathbb{R}^d$. But now we assume that dynamic trading incurs proportional transaction costs. If the bid and ask prices of asset i are given by $(1 - \varepsilon_i) S^i_n$ and $(1 + \varepsilon_i) S^i_n$ for a constant $\varepsilon_i \ge 0$, a strategy $h \in \mathcal{H}$ leads to an outcome of

$$h(S) := \sum_{n=1}^N \sum_{i=1}^d h^i_n (S^i_n - S^i_{n-1}) - \varepsilon_i |h^i_n - h^i_{n-1}| S^i_{n-1}, \quad \text{where } h^i_0 := 0.$$

(We assume there are no initial asset holdings. So there is a transaction cost at time 0. On the other hand, asset holdings at time N are valued at $h_N \cdot S_N$ and do not have to be converted into cash.) Similarly, a European option with payoff $g_n \in B^+_n$ at time n is assumed to cost

$$\phi_n(g_n) = \sup_{\nu_n \in \mathcal{P}_n} \langle g_n, \nu_n \rangle,$$

where $\mathcal{P}_n$ is a non-empty convex $\sigma(ca^+_n, C_{b,n})$-compact sets of Borel probability measures $\mathcal{P}_n$ on X_n (non-linear prices $\phi_n(g_n)$ may arise if e.g. not enough liquidly traded vanilla options exist to exactly replicate the payoffs g_n, or there are positive bid-ask spreads in the vanilla options market; see e.g. [8]). Compared to the frictionless case, we now have to require a little bit more integrability of the option portfolio. We introduce the sets

$$\begin{aligned} G(\mathcal{P}) &:= \left\{ \oplus\, g : (g_n) \in \prod\nolimits_n B^+_n \text{ such that } \sum\nolimits_n \phi_n(g_n) < +\infty \right\} \\ B(\mathcal{P}) &:= \{ f \in B : |f| \le \oplus g \text{ for some } \oplus\, g \in G(\nu) \}, \end{aligned}$$

and consider option portfolios with payoffs $\oplus g$ for functions $g_n \in B^+_n$ such that $\sum_n \phi_n(g_n) < +\infty$. We still denote the set of corresponding strategies $(m, \oplus g, h)$ by Θ.

The corresponding superheding functional is given by

$$\phi(f) := \inf \left\{ m + \sum\nolimits_n \phi_n(g_n) : (m, \oplus g, h) \in \Theta \text{ such that } m + \oplus g + h(S) \ge f \right\}. \tag{4.9}$$

A model-independent arbitrage now consist of a strategy $(m, \oplus g, h) \in \Theta$ such that

$$m + \sum_n \phi_n(g_n) \le 0 \quad \text{and} \quad m + \oplus g + h(S) > 0,$$

and instead of all martingale measures, the correct dual object will turn out to be the set $\mathcal{M}$ of all measures $\mu \in \mathcal{P}$ satisfying

$$(1-\varepsilon_i)S_n^i \le \mathbb{E}^\mu[S_N^i \mid \mathcal{F}_n] \le (1+\varepsilon_i)S_n^i \quad \text{for all } i \text{ and } n. \tag{4.10}$$

The following is a variant of Corollary 4.6 with friction and incompleteness. It is similar to [8]. But it covers the case of countably many time periods and European options with all maturities.

Corollary 4.7. *Assume that* $\lim_{k\to\infty} \sup_{\nu_n \in \mathcal{P}_n} \int_{X_n} (|x|-k)^+ \, d\nu_n(x) = 0$ *for all* n*. Then, the following are equivalent:*

(i) *There is no model-independent arbitrage*

(ii) $\mathcal{M} \neq \emptyset$.

Moreover, if (i)–(ii) *hold, then*

$$\phi(f) = \max_{\mu \in \mathcal{M}} \langle f, \mu \rangle \quad \text{for all } f \in U_b. \tag{4.11}$$

Proof. (ii) $\Rightarrow$ (i): Assume there exists a μ in $\mathcal{M}$, and let $(m, \oplus g, h) \in \Theta$ be a strategy such that $m + \oplus g + h(S) > 0$. First note that for all i,

$$\begin{aligned}
& \sum_{n=1}^N h_n^i (S_n^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \\
= & \sum_{n=1}^N \sum_{k=1}^n (h_k^i - h_{k-1}^i)(S_n^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \\
= & \sum_{n=1}^N (h_n^i - h_{n-1}^i)(S_N^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i.
\end{aligned}$$

Moreover, if the conditional expectation is understood in the general sense of e.g. [15], one has

$$\begin{aligned}
& \mathbb{E}^\mu \left[(h_n^i - h_{n-1}^i)(S_N^i - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \mid \mathcal{F}_{n-1} \right] \\
= & (h_n^i - h_{n-1}^i)(\mathbb{E}^\mu[S_N^i \mid \mathcal{F}_{n-1}] - S_{n-1}^i) - \varepsilon_i |h_n^i - h_{n-1}^i| S_{n-1}^i \le 0.
\end{aligned}$$

Since

$$\mathbb{E}^\mu h(S)^- \le m^+ + \langle \oplus g, \mu \rangle \le m^+ + \sum_n \phi_n(g_n) < +\infty,$$

one obtains from similar arguments as in [15] that $h(S)$ is μ-integrable with $\mathbb{E}^\mu h(S) \le 0$. Therefore,

$$m + \sum_n \phi_n(g_n) \ge m + \sum_n \langle g_n, \nu_n \rangle \ge \mathbb{E}^\mu \left[m + \oplus g + h(S) \right] > 0,$$

which shows that $(m, \oplus g, h)$ cannot be a model-independent arbitrage.

(i) $\Rightarrow$ (ii): If there is no model-independent arbitrage, it follows as in the proof of Corollary 4.2 that ϕ is a real-valued increasing convex functional on $B(\mathcal{P})$ such that $\phi(0) = 0$ and $\phi(f) \le m + \sum_n \phi_n(g_n)$ whenever $f \le m + \oplus g$ for some $m \in \mathbb{R}$ and $\oplus g \in G(\mathcal{P})$. Moreover, if

$$m + \oplus g + h(S) \ge f$$

for a strategy $(m, \oplus g, h) \in \Theta$ and $f \in B(\mathcal{P})$, one has for all $\mu \in \mathcal{M}$,

$$\mathbb{E}^\mu h(S)^- \le m^+ + \sum_n \langle g_n, \nu_n \rangle + \langle f^-, \mu \rangle \le m^+ + \sum_n \phi_n(g_n) + \langle f^-, \mu \rangle < +\infty.$$

So it follows as above that $\mathbb{E}^\mu h(S) \le 0$, and therefore, $m + \sum_n \phi_n(g_n) \ge \langle f, \mu \rangle$. This implies that $\phi(f) \ge \langle f, \mu \rangle$, and consequently, $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) = 0$ for all $\mu \in \mathcal{M}$. It remains to show that

$$\phi^*_{C_b}(\mu) = +\infty \quad \text{for } \mu \in \mathcal{P} \setminus \mathcal{M}. \tag{4.12}$$

Then the corollary follows from Proposition 3.1. To show (4.12), fix $\mu \in \mathcal{P}$. If

$$(1 - \varepsilon_i) s^i_0 \le \mathbb{E}^\mu x^i_N \le (1 + \varepsilon_i) s^i_0$$

as well as

$$\mathbb{E}^\mu[v(x_1, \dots, x_n)(x^i_N - (1 + \varepsilon_i) x^i_n)] \le 0 \quad \text{and} \quad \mathbb{E}^\mu[v(x_1, \dots, x_n)((1 - \varepsilon_i) x^i_n - x^i_N)] \le 0,$$

for all i, n and every bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}_+$, then

$$(1 - \varepsilon_i) S^i_n \le \mathbb{E}^\mu[S^i_N \mid \mathcal{F}_n] \le (1 + \varepsilon_i) S^i_n \quad \text{for all } i \text{ and } n.$$

So for $\mu \in \mathcal{P} \setminus \mathcal{M}$, there exists an f with $\langle f, \mu \rangle > 0$, where f is of the form $f(x) = x^i_N - (1 + \varepsilon_i) s^i_0$, $f(x) = (1 - \varepsilon_i) s^i_0 - x^i_N$, $f(x) = v(x_1, \dots, x_n)(x^i_N - (1+\varepsilon_i) x^i_n)$ or $f(x) = v(x_1, \dots, x_n)((1-\varepsilon_i) x^i_n - x^i_N)$ for a bounded continuous function $v : \prod_{j=1}^n X_j \to \mathbb{R}_+$. For $k \in \mathbb{N}$, define $f^k := f \wedge k$ and $f^k_k := f^k \vee (-k)$. By monotonicity, one has $\phi(f^k) \le \phi(f) \le 0$. Moreover,

$$f^k_k(x) = f^k(x) + (k + f(x))^- \le f^k(x) + (c|x^i_n| - k/2)^+ + (c|x^i_N| - k/2)^+,$$

for $c \in \mathbb{R}_+$ large enough. Since $w^k(x) := (c|x^i_n| - k/2)^+ + (c|x^i_N| - k/2)^+$ belongs to $G(\mathcal{P})$ one gets

$$\phi(w^k) \le \phi_n((c|x_n| - k/2)^+) + \phi_N((c|x_N| - k/2)^+) \to 0 \quad \text{for } k \to +\infty$$

by our assumption on $\mathcal{P}_n$. So for k large enough, one has

$$\langle f^k_k, \mu \rangle - \phi(f^k_k) \ge \langle f^k, \mu \rangle - \phi(f^k) - \phi(w^k) \ge \langle f^k, \mu \rangle - \phi(w^k) > 0,$$

and therefore, $\phi^*_{C_b}(\mu) = +\infty$. □

References

[1] L. Ambrosio and N. Gigli (2011). A User's Guide to Optimal Transport. Lecture Notes. University of Maryland.

[2] M. Beiglböck, P. Henry-Labordère and F. Penkner (2013). Model-independent bounds for option prices: a mass transport approach. Finance Stoch. 17(3), 477–501.

[3] M. Beiglböck, C. Leonard and W. Schachermayer (2012). A general duality theorem for the Monge–Kantorovich transport problem. Studia Math. 209(2), 151–167.

[4] M. Beiglböck and W. Schachermayer (2011). Duality for Borel measurable cost functions. Trans. Amer. Math. Soc., 363, no. 8, 4203–4224.

[5] P. Billingsley (1999). Convergence of Probability Measures. Wiley Series in Probability and Statistics.

[6] D. Breeden and R. Litzenberger (1978). Prices of state-contingent claims implicit in option prices. J. Business 51, 621–651.

[7] P. Cheridito, M. Kupper and L. Tangpi (2015). Representation of convex increasing functionals with countably additive measures. arXiv Preprint.

[8] Y. Dolinsky and H.M. Soner (2014). Robust hedging with proportional transaction costs. Finance and Stochastics 18, 327–347.

[9] Y. Dolinsky and H.M. Soner (2014). Martingale optimal transport and robust hedging in continuous time. Prob. Theory Rel. Fields 160, 391–427.

[10] R. M. Dudley (2002). Real Analysis and Probability. Vol. 74 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge.

[11] H. Föllmer and A. Schied (2011). Stochastic Finance: An Introduction in Discrete Time. Walter de Gruyter, 3rd Edition.

[12] A. Galichon, P. Henry-Labordère and N. Touzi (2014). A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. Ann. Appl. Prob., 24(1), 312–336.

[13] W. Gangbo (2004). An Introduction to the Mass Transportation Theory and its Applications. Lecture Notes. Georgia Institute of Technology.

[14] X. Guo, X. Tan and N. Touzi. Tightness and duality of martingale transport on the Skorokhod space. arXiv Preprint.

[15] J. Jacod and A. Shiryaev (1998). Local martingales and the fundamental asset pricing theorems in the discrete-time case. Fin. Stoch. 2, 259–273.

[16] L. V. Kantorovich (1942). On the translocation of masses. Dokl. Akad. Nauk. USSR 37, 199–201. English translation in J. Math. Sci. 133, 4 (2006), 1381–1382.

[17] H. G. Kellerer (1984). Duality theorems for marginal problems. Z. Wahrsch. Verw. Gebiete 67, 4, 399–432.

[18] S.T Rachev and L. Rüschendorf (1998). Mass Transportation Problems Vol I & II. Probability and its Applications. Springer-Verlag, New York.

[19] C. Villani (2009). Optimal Transport: Old and New, Volume 338 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag Berlin Heidelberg.