

Equilibrium Pricing in Incomplete Markets under Translation Invariant Preferences*

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Abstract

We propose a general discrete-time framework for deriving equilibrium prices of financial securities. It allows for heterogeneous agents, unspanned random endowments and convex trading constraints. We give a dual characterization of equilibria and provide general results on their existence and uniqueness. In the special case where all agents have preferences of the same type and in equilibrium, all random endowments are replicable by trading in the financial market, we show that a one-fund theorem holds and give an explicit expression for the equilibrium pricing kernel.

Keywords: Competitive equilibrium, incomplete markets, translation invariant preferences, heterogenous agents, trading constraints, one-fund theorem.

1 Introduction

We consider a discrete-time model of an incomplete financial markets with finitely many agents. Our main interest is in equilibrium prices of derivatives or structured products with maturities T that are short compared to the time horizon of a typical life-time consumption-investment problem. So the risk of fluctuating interest rates does not play a big role, and we assume them to be exogenously given. We suppose our agents invest in the financial market with the goal of optimizing the utility of their wealth at time T . Our setup can accommodate heterogeneous agents, unspanned random endowments and general convex trading constraints. We try to find equilibria of plans, prices and price expectations in the spirit of Radner (1972). In dynamic models with general preferences, several consumption goods, incomplete security markets and no a priori bounds on trading strategies such an equilibrium does not always exist, and if there is one, it is typically not unique; see Hart (1975), the review articles by Geanakoplos (1990) and Magill and Shafer (1991) or the textbook by Magill and Quinzii (1996).

In this paper the agents are concerned with the level of their wealth at time T , and they are all assumed to have translation invariant preferences. This allows us to prove existence and uniqueness of an equilibrium under general assumptions by backward induction. Our proofs are based on convex duality arguments and lead to recursive algorithms for computing the equilibrium. Typical examples of translation invariant preferences are those induced by expected exponential utility, the monotone mean-variance preferences of Maccheroni et al. (2009), mean-risk type preferences where risk is measured with a convex risk measure, optimized certainty equivalents à la Ben-Tal and Teboulle (1986, 1987) or the divergence utilities of Cherny and Kupper (2009). The assumption of translation invariant preferences is justified if, for instance, agents are understood as professional traders or entire financial institutions who make investment decisions on the basis of expected return and risk.

We assume there exist two kinds of assets. The first type of assets are liquidly traded in large volumes and their prices are not affected by the actions of our agents. Their dynamics will be exogenously given. Assets of the second kind entitle their holders to an uncertain payoff at time T . We think of them as derivatives or structured products

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which can also depend on non-financial risk such as temperature, rain or political events. They exist in fixed supply and are only traded by our agents. The goal is to price them by matching demand and supply. The situation where there are no exogenous instruments and all assets are priced in equilibrium is a special case.

The standard way to price derivatives is to compute the expectation of their discounted payoffs under an equivalent martingale measure \mathbb{Q} . Binomial tree models and the standard Black–Scholes model are complete, and there is exactly one equivalent martingale measure. But extensions such as trinomial tree, GARCH-type, stochastic volatility or jump-diffusion models are incomplete and admit infinitely many equivalent martingale measures. The question then is, which one should be used for pricing. In practice, models are often built directly under \mathbb{Q} , then calibrated to market prices of liquidly traded options and used to value more exotic ones; see for instance, Lipton (2002) for an overview of popular stochastic volatility models. Some pricing measures that have been discussed in the literature are the minimal martingale measure of Föllmer and Schweizer (1991), the Esscher transformed measure proposed by Gerber and Shiu (1994), the variance-optimal martingale measure studied in Schweizer (1995, 1996) and Delbaen and Schachermayer (1996) or the minimal entropy martingale measure of Frittelli (2000). More recently, several authors have applied utility indifference arguments to the valuation of complex financial products; see for instance, Henderson and Hobson (2009) for an overview. But utility indifference prices are personal, reflecting the preferences of a single agent. Our approach provides a way of deriving the pricing rule from equilibrium considerations. We show that if at least one agent has differentiable preferences and open trading constraints, our pricing method is given by a probability measure \mathbb{Q} that is absolutely continuous with respect to the reference measure \mathbb{P} . If in addition, the agent’s preference functional is strictly monotone, \mathbb{Q} is equivalent to \mathbb{P} . On the other hand, if agents have closed trading constraints, equilibrium prices in our model are not necessarily given by a linear functional; see also Hugonnier (2012) and the references therein for equilibrium models with constraints and rational asset pricing bubbles, or Avellaneda and Lipkin (2009) for a dynamic model of stock and option prices under short-selling constraints.

Our method of proving existence of an equilibrium is to recursively construct one-time-step representative agents with preferences over the space of financial gains realizable by investing in the financial market. In every step we take a Negishi approach and distribute resources in a socially optimal manner. But due to market incompleteness and trading constraints, optimal allocations have to be found in suitably restricted subsets. If it can be shown that in every step optimal allocations exist, equilibrium prices can be constructed with the help of conditional subgradients. That the approach yields a multi-period equilibrium is due to the assumption of translation invariant preferences. The consumption sets in our framework can be unbounded from below. To guarantee the existence of an equilibrium we assume that each agent either is sensitive to large losses or has conditionally compact trading constraints. Sensitivity to large losses, defined in Section 2.3, means that a position which will be negative in at least some states of the world is becoming undesirable if it is multiplied with a sufficiently large number. It is a sufficient condition that is easy to check and is different from the ones in e.g. Werner (1987), Cheng (1991), Brown and Werner (1995), or Dana et al. (1997, 1999). It also differs from the dual conditions used by e.g. Filipović and Kupper (2008), Dana and Le Van (2010), or Anthropelos and Žitković (2010) to prove the existence of static equilibria for convex risk measures in different models with incomplete markets.

If an equilibrium exists and in addition, at least one agent has differentiable preferences and open constraints, we show that equilibrium prices are unique. To show uniqueness of the agents’ optimal trading gains one needs strict convexity assumptions on the preferences. The latter is in line with the uniqueness result of Anthropelos and Žitković (2010), which in the static case, gives uniqueness of prices and strategies under assumptions of strict convexity and non-degeneracy. Duffie (1987) has shown the existence of an equilibrium in a model with complete spot markets and an incomplete market of purely financial securities. The proof is based on a fixed point argument and in general, the equilibrium is not unique. In Cuoco and He (2001) a static representative agent is constructed in an economy with incomplete securities markets. But in that paper an equilibrium does not always exist, and the construction of the representative agent involves an aggregation of the single agents with stochastic weights.

In the special case where all agents have preferences of the same type, like for instance, expected exponential utility with different risk aversions, and at the same time, all random endowments are spanned by attainable trading gains, we show that a one-fund theorem holds. If in addition, the preferences are differentiable, the equilibrium pricing kernel can be given in explicit form. If there are exogenous assets, the pricing kernel contains optimal trading gains from investing in them. Otherwise, similar to the standard CAPM, it just consists of the gradient of the base preference functional at the aggregate endowment. As an example we study the effects of stochastic volatility, demand pressure and short-selling constraints on prices of options on single stocks and indexes.

The remainder of the paper is organized as follows. In Section 2 we introduce our model and the notation. In Section 3 we give a dual characterization of equilibrium. Then we show that if every agent either is sensitive to large losses or has conditionally compact trading constraints, an equilibrium exists. In Section 4 we prove uniqueness of equilibrium prices if preferences are differentiable and uniqueness of optimal wealth dynamics if preferences satisfy a strict convexity property. Section 5 provides a one-fund theorem for the special case where agents have preferences of

the same type and random endowments are replicable by trading in the financial market. As an application we discuss the effects of stochastic volatility, demand pressure and short-selling constraints on option prices. All proofs are given in the appendix.

2 Notation and setup

We consider a group of finitely many agents \mathbb{A} who trade financial assets. Time is discrete and runs through the set $\{0, 1, \dots, T\}$. Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the flow of information by a filtration $(\mathcal{F}_t)_{t=0}^T$. We assume that at time t , all agents have access to the information represented by \mathcal{F}_t , and all events in \mathcal{F}_0 have probability 0 or 1. \mathbb{P} is a reference probability measure. It does not necessarily reflect the beliefs of our agents, but we assume they all agree that an event $A \in \mathcal{F}$ is impossible if $\mathbb{P}[A] = 0$. $L^0(\mathcal{F}_t)$ denotes the set of all \mathcal{F}_t -measurable random variables and $L^\infty(\mathcal{F}_t)$ the set of essentially bounded random variables, where random variables are identified if they are equal \mathbb{P} -almost surely. Accordingly, all equalities and inequalities between random variables will be understood in the \mathbb{P} -almost sure sense. Expectation with respect to \mathbb{P} is denoted by \mathbb{E} . Notation for expectations with respect to other probability measures will be introduced where it is needed. In the special case where the sample space Ω is finite, all random variables are bounded and the filtration (\mathcal{F}_t) can be thought of as an event tree.

2.1 The financial market and endowments

All agents can lend funds to and borrow from a money market account at the same exogenously given interest rate and invest in a financial market consisting of $J + K$ assets. We use the money market account as numeraire, that is, all prices will be expressed in terms of the value of one dollar invested in the money market at time 0. The prices of the first J assets are exogenously given by a J -dimensional bounded adapted process $(R_t)_{t=0}^T$. Our agents can buy and sell arbitrary quantities of them without influencing their prices. The other K assets yield time- T payoffs of $S^k \in L^\infty(\mathcal{F}_T)$, $k = 1, \dots, K$, per share. Our goal is to find equilibrium price processes $(S_t^k)_{t=0}^T$ satisfying the terminal conditions $S_T^k = S^k$ together with optimal investment strategies for all agents $a \in \mathbb{A}$. In the special case $J = 0$, the price evolutions of all assets, except the money market account, are determined by supply and demand. The possibility to include exogenously given assets in the model is helpful for the study of derivatives and structured products. For instance, a weather derivative might only be traded by an insurance company and a few end users. They can also invest in large cap stocks. But while their demands will determine the price of the weather derivative, they are too small to influence the stock prices.

We assume that agent $a \in \mathbb{A}$ is initially endowed with $g^{a,R,j} \in \mathbb{R}$ shares of R^j , $g^{a,S,k} \in \mathbb{R}$ shares of S^k and an uncertain payoff $G^a \in L^0(\mathcal{F}_T)$ that is bounded from below. Then

$$H^a := \sum_{j=1}^J g^{a,R,j} R_T^j + \sum_{k=1}^K g^{a,S,k} S^k + G^a$$

is an element of $L^0(\mathcal{F}_T)$ that is again bounded from below.

In addition to the assets initially held by our agents, there is an external supply of $n^k \in \mathbb{R}$ shares of S^k . In the case $n^k > 0$, this means that at time 0, someone not included in \mathbb{A} is selling n^k shares of S^k to our agents at a price such that their aggregate excess demand is exactly n^k . Similarly, if $n^k < 0$, $|n^k|$ shares of S^k are bought from our agents at a price that clears the market among them. By $n \in \mathbb{R}^K$ we denote the vector with components n^k , and (S_t) will be the K -dimensional process with components (S_t^k) . A situation with negative n^k could arise for example, if \mathbb{A} consists of different option dealers that sell put options to end users and dynamically trade them among each other, while the end users buy the options at time 0 at the price at which they are offered by the dealers and hold them until time T .

A *trading strategy* for agent $a \in \mathbb{A}$ is given by an \mathbb{R}^{J+K} -valued predictable stochastic process $(\vartheta_t^a)_{t=1}^T$ describing the deviations of agent a 's investments from the initial endowments $g^{a,R}$ and $g^{a,S}$. By $\vartheta_t^{a,R}$ we denote the first J components of ϑ_t^a and by $\vartheta_t^{a,S}$ the remaining K ones. They are all assumed to be measurable with respect to \mathcal{F}_{t-1} . $g^{a,R,j} + \vartheta_t^{a,R,j}$ is the number of shares of R^j agent a is holding from time $t-1$ to t , and $g^{a,S,k} + \vartheta_t^{a,S,k}$ the number of shares of S^k . We assume that there is no consumption or infusion of funds at intermediate times. If agents want to buy more of an asset, they have to finance it by selling others. Since all prices are expressed in discounted terms, investments in the money market do not change their value, and investor a 's time T wealth resulting from a trading strategy $(\vartheta_t^a)_{t=1}^T$ can be written as

$$H^a + \sum_{t=1}^T \vartheta_t^{a,R} \cdot \Delta R_t + \vartheta_t^{a,S} \cdot \Delta S_t := H^a + \sum_{t=1}^T \left(\sum_{j=1}^J \vartheta_t^{a,R,j} \Delta R_t^j + \sum_{k=1}^K \vartheta_t^{a,S,k} \Delta S_t^k \right),$$

where we denote $\Delta R_t^j := R_t^j - R_{t-1}^j$ and $\Delta S_t^k := S_t^k - S_{t-1}^k$.

We assume that the R -assets satisfy the following no-arbitrage condition:

(NA) No arbitrage in the R -assets and money market account:

For every predictable trading strategy $(\vartheta_t)_{t=1}^T$ in the R -assets one has that

$$\sum_{t=1}^T \vartheta_t \cdot \Delta R_t \geq 0 \quad \text{implies} \quad \sum_{t=1}^T \vartheta_t \cdot \Delta R_t = 0.$$

By the Dalang–Morton–Willinger theorem (see Dalang et al., 1990) this is equivalent to the existence of a probability measure \mathbb{Q} equivalent to \mathbb{P} such that $R_t^j = \mathbb{E}_{\mathbb{Q}} \left[R_T^j \mid \mathcal{F}_t \right]$ for all j and t .

2.2 Trading constraints

We suppose that our agents face trading constraints described by sets C_{t+1}^a of admissible one-step strategies $\vartheta_{t+1} \in L^0(\mathcal{F}_t)^{J+K}$ satisfying the following two conditions:

(C1) there exist strategies $\bar{\vartheta}_{t+1}^a \in C_{t+1}^a \cap L^\infty(\mathcal{F}_t)^{J+K}$ such that $\sum_{a \in \mathbb{A}} \bar{\vartheta}_{t+1}^{a,S} = n$

(C2) $\lambda \vartheta_{t+1} + (1 - \lambda) \vartheta'_{t+1} \in C_{t+1}^a$ for all $\vartheta_{t+1}, \vartheta'_{t+1} \in C_{t+1}^a$ and $\lambda \in L^0(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$

Condition (C1) guarantees that there exists at least one admissible trading strategy for each agent such that aggregate demand is equal to supply. For example, condition (C1) is fulfilled if the external supply of the S -assets is zero and for all agents it is admissible to just keep their funds in the money market account. (C2) is a conditional convexity condition which will be needed in our proof that an equilibrium exists. In the case $C_{t+1}^a = L^0(\mathcal{F}_t)^{J+K}$, we say that agent a is unconstrained at time t . Stochastic constraints can, for instance, be introduced by specifying C_{t+1}^a as

$$C_{t+1}^a = \left\{ \vartheta_{t+1} \in L^0(\mathcal{F}_t)^{J+K} : \underline{d}_{t+1}^j \leq \vartheta_{t+1}^{R,j} \leq \bar{d}_{t+1}^j, \underline{e}_{t+1}^k \leq \vartheta_{t+1}^{S,k} \leq \bar{e}_{t+1}^k \right\}, \quad (2.1)$$

where $\underline{d}_{t+1}^j, \bar{d}_{t+1}^j, \underline{e}_{t+1}^k, \bar{e}_{t+1}^k$ are \mathcal{F}_t -measurable bounds on the number of shares of the R - and S -assets that can be held or sold short from time t to $t + 1$. It is easy to see that sets of the form (2.1) fulfill the convexity condition (C2). To satisfy (C1), the bounds $\underline{d}_{t+1}^j, \bar{d}_{t+1}^j, \underline{e}_{t+1}^k, \bar{e}_{t+1}^k$ have to be chosen appropriately.

For $x \in L^0(\mathcal{F}_t)^{J+K}$, we set

$$\|x\|_{\mathcal{F}_t} := \left(\sum_{i=1}^{J+K} (x^i)^2 \right)^{1/2}$$

and say C_{t+1}^a is \mathcal{F}_t -bounded if there exists an \mathcal{F}_t -measurable random variable Y such that $\|x\|_{\mathcal{F}_t} \leq Y$ for all $x \in C_{t+1}^a$. Similarly, we call C_{t+1}^a \mathcal{F}_t -open if for every $x \in C_{t+1}^a$ there exists an \mathcal{F}_t -measurable random variable $\varepsilon > 0$ such that $x' \in C_{t+1}^a$ for all $x' \in L^0(\mathcal{F}_t)^{J+K}$ satisfying $\|x' - x\|_{\mathcal{F}_t} \leq \varepsilon$. We say C_{t+1}^a is sequentially closed if it contains every $x \in (L^0)^{J+K}$ that is an almost sure limit of a sequence of elements in C_{t+1}^a .¹

2.3 Translation invariant preferences

Agent a 's goal at time $t \in \{0, \dots, T\}$ is to invest in the financial market so as to optimize a preference functional

$$U_t^a : L^0(\mathcal{F}_T) \rightarrow \underline{L}(\mathcal{F}_t),$$

where $\underline{L}(\mathcal{F}_t)$ denotes the set of all \mathcal{F}_t -measurable random variables with values in $\mathbb{R} \cup \{-\infty\}$. Usually, preference functionals take values in \mathbb{R} . But our agents update their preferences as they learn about information contained in \mathcal{F}_t . So their utilities at time t are \mathcal{F}_t -measurable, and allowing U_t^a to take values in $\underline{L}(\mathcal{F}_t)$ instead of $L^0(\mathcal{F}_t)$ allows for more general examples. Also, since we work with discounted units of money, the agents' preferences are over discounted payoffs. That is, they measure the performance of investments relative to the riskless asset. However, if interest rates are assumed to be deterministic, the discount rate can be absorbed by adjusting the preference functional. So in this case, assessing discounted payoffs with a given preference functional is equivalent to assessing undiscounted payoffs with an appropriately distorted preference functional.

We will also need the larger sets $L(\mathcal{F}_t)$ of \mathcal{F}_t -measurable random variables taking values in $\mathbb{R} \cup \{\pm\infty\}$. We assume that U_t^a has the following properties:

¹Note that sequentially closed sets are not complements of \mathcal{F}_t -open sets.

(N) Normalization: $U_t^a(0) = 0$

(M) Monotonicity: $U_t^a(X) \geq U_t^a(Y)$ for all $X, Y \in L^0(\mathcal{F}_T)$ such that $X \geq Y$

(C) \mathcal{F}_t -Concavity: $U_t^a(\lambda X + (1 - \lambda)Y) \geq \lambda U_t^a(X) + (1 - \lambda)U_t^a(Y)$ for all $X, Y \in L^0(\mathcal{F}_T)$ and $\lambda \in L^0(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$, where $0(-\infty)$ is understood to be 0

(T) Translation property: $U_t^a(X + Y) = U_t^a(X) + Y$ for all $X \in L^0(\mathcal{F}_T)$ and $Y \in L^0(\mathcal{F}_t)$

Every preference functional $U_t : L^0(\mathcal{F}_T) \rightarrow \underline{L}(\mathcal{F}_t)$ satisfying $U_t(0) \in L^0(\mathcal{F}_t)$ can be normalized without changing the preference order by passing to $U_t(X) - U_t(0)$. So one can assume (N) without loss of generality as soon as $U_t^a(0) \in L^0(\mathcal{F}_t)$. The monotonicity assumption (M) is standard. It just means that more is preferred to less. Condition (C) is an extension of ordinary concavity to a situation where agents make decisions based on the information contained in \mathcal{F}_t .² Condition (T) means that our preference orders are invariant under a shift of random payoffs by a known amount of cash. This restricts the class of preferences that fall into our framework. But it is for instance, satisfied by the certainty equivalent of expected exponential utility or mean-risk type preferences, and it covers the case of professional investors which maximize expected values under constraints on the amount of risk they are allowed to take; some specific cases of preference functionals with the translation property (T) are discussed in Example 2.1 below.

It is a direct consequence of condition (C) that U_t^a has the following local property:

$$1_A U_t^a(X) = 1_A U_t^a(Y) \quad \text{for all } X, Y \in L^0(\mathcal{F}_T) \text{ and } A \in \mathcal{F}_t \text{ such that } 1_A X = 1_A Y. \quad (2.2)$$

That is, in the event A , the utility $U_t(X)$ only depends on values X can attain in states of the world contained in A . A short proof of (2.2) is given in the appendix. As a consequence, one obtains that for $\vartheta \in L^0(\mathcal{F}_t)$ and $X \in L^\infty(\mathcal{F}_T)$, one has

$$U_t^a(\vartheta X) = \sum_{m \geq 1} 1_{\{m-1 \leq |\vartheta| < m\}} U_t^a(1_{\{m-1 \leq |\vartheta| < m\}} \vartheta X) \in L^0(\mathcal{F}_t). \quad (2.3)$$

In addition to (N), (M), (C) and (T), we also assume that the preferences are time-consistent in the following sense:

(TC) Time-consistency: For all $X, Y \in L^0(\mathcal{F}_T)$ and $t = 0, \dots, T - 1$,

$$U_{t+1}^a(X) \geq U_{t+1}^a(Y) \quad \text{implies} \quad U_t^a(X) \geq U_t^a(Y). \quad (2.4)$$

By (N) and (T) one has $U_{t+1}^a(U_{t+1}^a(X)) = U_{t+1}^a(X)$ for all random variables X belonging to the set

$$\text{dom } U_{t+1}^a := \{X \in L^0(\mathcal{F}_T) : U_{t+1}^a(X) \in L^0(\mathcal{F}_{t+1})\}.$$

Applying (2.4) to the random variable $Y = U_{t+1}^a(X)$ shows that time-consistency implies the following recursive structure of the preference functionals:

$$U_t^a(X) = U_t^a(U_{t+1}^a(X)) \quad \text{for all } t = 0, \dots, T - 1 \text{ and } X \in \text{dom } U_{t+1}^a. \quad (2.5)$$

For some of the results in this paper we will also need the preferences to satisfy one or more of the following conditions:

(SL) Sensitivity to large losses: $\lim_{\lambda \rightarrow \infty} U_0^a(\lambda X) = -\infty$ for all $X \in L^0(\mathcal{F}_T)$ with the property $\mathbb{P}[X < 0] > 0$.

(SM) Strict monotonicity: $U_0^a(X) > U_0^a(Y)$ for all $X, Y \in \text{dom } U_0^a$ such that $X \geq Y$ and $\mathbb{P}[X > Y] > 0$.

(SC) Strict concavity modulo translation: $U_0^a(\lambda X + (1 - \lambda)Y) > \lambda U_0^a(X) + (1 - \lambda)U_0^a(Y)$ for all $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$ and $X, Y \in \text{dom } U_0^a$ such that $X - Y$ is not constant.

Note that since the functionals U_0^a have the translation property (T), they cannot be strictly concave under translation by constants. But property (SC) will be sufficient for our purposes. Equivalent conditions are given in Cheridito and Li (2008).

²Convex preferences correspond to quasi-concave preference functionals. However, quasi-concavity and the translation property (T) imply concavity; see Lemma 2.1 in Cheridito and Kupper (2009).

Examples 2.1

1. Entropic preference functionals

The standard example of a family of preference functionals satisfying (N), (M), (C), (T), (TC) is given by the conditional certainty equivalents of expected exponential utility, also called entropic preference functionals:

$$U_t^a(X) = -\frac{1}{\gamma} \log \mathbb{E}[\exp(-\gamma X) \mid \mathcal{F}_t] \quad \text{for a constant } \gamma > 0 \quad (2.6)$$

(for general $X \in L^0(\mathcal{F}_T)$, we understand the conditional expectation as $\lim_{m \rightarrow \infty} \mathbb{E}[m \wedge \exp(-\gamma X) \mid \mathcal{F}_t]$). They induce the same preferences as the conditional expected exponential utilities $\mathbb{E}[-\exp(-\gamma X) \mid \mathcal{F}_t]$. But only in the form (2.6) do they have the translation property (T). U_0^a also satisfies (SL), (SM) and (SC). (SL) and (SM) are obvious. (SC) follows from Theorem 5.3 in Cheridito and Li (2008).

2. Pasting together one-step preference functionals

A general method of constructing time-consistent preference functionals in discrete time is by pasting together one-step preference functionals; see Cheridito and Kupper (2011). Assume, for instance, that

$$v_t : L^\infty(\mathcal{F}_{t+1}) \rightarrow L^\infty(\mathcal{F}_t), \quad t = 0, \dots, T-1,$$

are mappings satisfying (N), (M), (C) and (T) such that the extensions

$$V_t(X) = \lim_{m \rightarrow \infty} \lim_{l \rightarrow -\infty} v_t((X \wedge m) \vee l) \text{ map } \underline{L}(\mathcal{F}_{t+1}) \text{ to } \underline{L}(\mathcal{F}_t). \quad (2.7)$$

Then the compositions

$$U_t(X) = V_t \circ \dots \circ V_{T-1}(X), \quad X \in L^0(\mathcal{F}_T) \quad (2.8)$$

inherit (N), (M), (C), (T) and are automatically time-consistent.

It is convenient to define conditional preference functionals with the translation property (T) (or conditional risk measures) for bounded random variables. Then the outcomes are automatically bounded as well. The double limit in (2.7) provides a procedure for extending them to unbounded and extended random variables. But to be able to define U_t through the recursion (2.8), one needs that for $X \in \underline{L}(\mathcal{F}_{t+1})$, $V_t(X)$ belongs to $\underline{L}(\mathcal{F}_t)$. General conditions for this to hold are given in Cheridito et al. (2006). In Example 5 below, we provide a wide class of functionals for which it can be shown directly.

In the sequel we give some specific examples of one-step preference functionals $v_t : L^\infty(\mathcal{F}_{t+1}) \rightarrow L^\infty(\mathcal{F}_t)$.

3. Monotone mean-variance preferences

Standard conditional mean variance

$$MV_t^\lambda(X) = \mathbb{E}[X \mid \mathcal{F}_t] - \frac{\lambda}{2} \text{Var}(X \mid \mathcal{F}_t)$$

fulfills (N), (C), (T) but not the monotonicity property (M); see for instance, Maccheroni et al. (2009). This can be corrected by slightly modifying its dual representation. For $X \in L^\infty(\mathcal{F}_1)$, $MV_0^\lambda(X)$ has a dual representation of the form

$$MV_0^\lambda(X) = \inf_{\xi \in \mathcal{E}_1} \mathbb{E}[X\xi + G^\lambda(\xi) \mid \mathcal{F}_0],$$

where

$$\mathcal{E}_1 = \{\xi \in L^1(\mathcal{F}_1) : \mathbb{E}[\xi] = 1\} \quad \text{and} \quad G^\lambda(x) = \frac{1}{2\lambda}(x-1)^2.$$

This extends to

$$MV_t^\lambda(X) = \text{ess inf}_{\xi \in \mathcal{E}_{t+1}} \mathbb{E}[X\xi + G^\lambda(\xi) \mid \mathcal{F}_t], \quad X \in L^\infty(\mathcal{F}_{t+1}), \quad (2.9)$$

where

$$\mathcal{E}_{t+1} = \{\xi \in L^1(\mathcal{F}_{t+1}) : \mathbb{E}[\xi \mid \mathcal{F}_t] = 1\}$$

and ess inf denotes the largest lower bound of a family of random variables with respect to the \mathbb{P} -almost sure order; see for instance, Proposition VI.1.1 of Neveu (1975). If one modifies (2.9) to

$$v_t(X) = \text{ess inf}_{\xi \in \mathcal{D}_{t+1}} \mathbb{E}[X\xi + G^\lambda(\xi) \mid \mathcal{F}_t], \quad (2.10)$$

for

$$\mathcal{D}_{t+1} = \{\xi \in L^1(\mathcal{F}_{t+1}) : \xi \geq 0, \mathbb{E}[\xi | \mathcal{F}_t] = 1\},$$

one obtains one-step preference functionals satisfying (N), (M), (C), (T). They belong to the class of divergence utilities, which are shown to satisfy condition (2.7) in the appendix.

4. Mean-risk preferences

Instead of modifying mean-variance as in (2.10), one can also replace the variance term by a conditional convex risk measure and set

$$v_t(X) = \lambda \mathbb{E}[X | \mathcal{F}_t] - (1 - \lambda) \rho_t(X), \quad (2.11)$$

where λ is a number in $(0, 1)$ and

$$\rho_t : L^\infty(\mathcal{F}_{t+1}) \rightarrow L^\infty(\mathcal{F}_t)$$

a normalized conditional convex risk measure, that is, $-\rho_t$ satisfies (N), (M), (C) and (T); see e.g. Föllmer and Schied (2004) for an introduction to convex risk measures in a static framework and Cheridito and Kupper (2011) for dynamic risk measures. Whether condition (2.7) holds, depends on ρ .

5. Divergence utilities

The monotone mean-variance preference functional (2.10) can be generalized by replacing the function G^λ with a more general measurable function $G : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\text{ess inf}_{\xi \in \mathcal{D}_{t+1}} \mathbb{E}[G(\xi) | \mathcal{F}_t] = 0$. Then

$$v_t(X) = \text{ess inf}_{\xi \in \mathcal{D}_{t+1}} \mathbb{E}[X\xi + G(\xi) | \mathcal{F}_t], \quad (2.12)$$

has all the properties (N), (M), (C), (T). For suitable functions $H : \mathbb{R} \rightarrow \mathbb{R}$, conditional optimized certainty equivalents

$$\text{ess sup}_{s \in \mathbb{R}} \{s - \mathbb{E}[H(s - X) | \mathcal{F}_t]\} \quad (2.13)$$

are of this form (ess sup denotes the least upper bound of a family of random variables in the \mathbb{P} -almost sure order). For instance, if H is increasing and convex such that $\max_{x \in \mathbb{R}} (x - H(x)) = 0$, then (2.13) is of the form (2.12) with

$$G(y) = H^*(y) = \sup_{x \in \mathbb{R}} \{xy - H(x)\}.$$

$\mathbb{E}[G(\xi)]$ is an f-divergence after Csiszar (1967). In the special case $G(x) = \frac{1}{\lambda} x \log(x)$ it is relative entropy and v_t becomes the conditional entropic preference functional of Example 2.1.1. Unconditional functionals of the form (2.12) and (2.13) have, for instance, been studied by Ben-Tal and Teboulle (1987), Cheridito and Li (2008), Cheridito and Li (2009), Cherny and Kupper (2009). It is shown in the appendix that the extensions

$$V_t(X) = \lim_{m \rightarrow \infty} \lim_{l \rightarrow -\infty} v_t((X \wedge m) \vee l) \quad \text{satisfy} \quad (2.7) \quad (2.14)$$

and that

$$U_0 = V_0 \circ \dots \circ V_{T-1} \text{ is sensitive to large losses.} \quad (2.15)$$

2.4 Definition of equilibrium

We work with the concept of an equilibrium of plans, prices and price expectations as introduced by Radner (1972) in a slightly different setup. At every time $t = 0, 1, \dots, T-1$ the goal of each agent $a \in \mathbb{A}$ is to invest in such a way that the utility of final wealth becomes maximal. It is assumed that at time t , all agents trade the assets at the same current prices and agree on what the possible scenarios for future price evolutions are. Having invested according to some trading strategy $\vartheta_1^a, \dots, \vartheta_t^a$ up to time t , agent a 's optimization problem is given by

$$\text{ess sup}_{\vartheta_s^a \in C_s^a, s \geq t+1} U_t^a \left(H^a + \sum_{s=1}^T \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right). \quad (2.16)$$

But since U_t^a has the translation property (T), $U_t^a \left(H^a + \sum_{s=1}^T \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right)$ can be written as

$$\sum_{s=1}^t (\vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s) + U_t^a \left(H^a + \sum_{s=t+1}^T \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right).$$

So (2.16) is equivalent to

$$\operatorname{ess\,sup}_{\vartheta_s^a \in C_s^a, s \geq t+1} U_t^a \left(H^a + \sum_{s=t+1}^T \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right). \quad (2.17)$$

Definition 2.2 An equilibrium consists of a bounded, \mathbb{R}^K -valued, adapted process $(S_t)_{t=0}^T$ satisfying the terminal condition $S_T = S$ together with admissible trading strategies $(\vartheta_t^a)_{t=1}^T$ for all agents $a \in \mathbb{A}$, such that the following two conditions hold:

(i) **Individual optimality**

$$U_t^a \left(H^a + \sum_{s=t+1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \geq U_t^a \left(H^a + \sum_{s=t+1}^T \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right)$$

for all $t = 0, \dots, T-1$ and admissible continuation strategies $(\vartheta_s^a)_{s=t+1}^T$.

(ii) **Market clearing** $\sum_{a \in \mathbb{A}} \hat{\vartheta}_t^{a,S} = n$ for all $t = 1, \dots, T$.

Remark 2.3 If U_0^a is strictly monotone, then individual optimality at all times t follows from the time 0 optimality condition

$$U_0^a \left(H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \geq U_0^a \left(H^a + \sum_{s=1}^T \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right) \quad (2.18)$$

for all admissible strategies $(\vartheta_s^a)_{s=1}^T$. Indeed, let us assume to the contrary that (2.18) holds but there exist $t \geq 1$ and an admissible continuation strategy $(\vartheta_s^a)_{s=t+1}^T$ such that

$$U_t^a \left(H^a + \sum_{s=t+1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) < U_t^a \left(H^a + \sum_{s=t+1}^T \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s \right)$$

on an \mathcal{F}_t -measurable set A with $\mathbb{P}[A] > 0$. Then by time-consistency, one has

$$\begin{aligned} & U_0^a \left(H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \\ &= U_0^a \left(\sum_{s=1}^t \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s + U_t^a \left(H^a + \sum_{s=t+1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \right) \\ &< U_0^a \left(\sum_{s=1}^t \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s + U_t^a \left(H^a + \sum_{s=t+1}^T \tilde{\vartheta}_s^{a,R} \cdot \Delta R_s + \tilde{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \right) \\ &= U_0^a \left(H^a + \sum_{s=1}^t \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s + \sum_{s=t+1}^T \tilde{\vartheta}_s^{a,R} \cdot \Delta R_s + \tilde{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \end{aligned}$$

for the admissible strategies $\tilde{\vartheta}_s^a = 1_{A^c} \hat{\vartheta}_s^a + 1_A \vartheta_s^a$, $s = t+1, \dots, T$. But this contradicts (2.18).

3 Dual characterization and existence of equilibrium

A standard way of proving the existence of a competitive equilibrium in an economy with complete markets, going back to Negishi (1960), is to construct a suitable representative agent. But the form of the representative agent typically depends on the equilibrium to be supported, and as a consequence, the proof involves a fixed-point argument. For translation invariant preferences the situation turns out to be simpler. Since optimal behavior in the future does not depend on past decisions, an equilibrium can be constructed recursively by forming one-step representative agents even if markets are incomplete. Moreover, in every step, preferences can be aggregated without using fixed-point arguments.

Assume that equilibrium prices S_{t+1}, \dots, S_T and admissible trading strategies $\hat{\vartheta}_{t+2}^a, \dots, \hat{\vartheta}_T^a$ for all agents $a \in \mathbb{A}$ have already been determined such that the components of S_{t+1}, \dots, S_T are bounded. We then define the continuation value of agent $a \in \mathbb{A}$ at time $t+1$ by

$$H_{t+1}^a = U_{t+1}^a \left(H^a + \sum_{s=t+2}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right), \quad \text{where} \quad H_T^a = U_T^a(H^a) = H^a. \quad (3.1)$$

Since H^a is assumed to be bounded from below and there exist bounded admissible one-step strategies $\bar{\vartheta}_s^a$, $s = t+2, \dots, T$, there exists a constant $c \in \mathbb{R}$ such that

$$H_{t+1}^a = U_{t+1}^a \left(H^a + \sum_{s=t+2}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \geq U_{t+1}^a \left(H^a + \sum_{s=t+2}^T \bar{\vartheta}_s^{a,R} \cdot \Delta R_s + \bar{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \geq c.$$

In particular, H_{t+1}^a belongs to $L^0(\mathcal{F}_{t+1})$ and the following recursive relation holds:

$$\begin{aligned} H_{t+1}^a &= U_{t+1}^a \left(U_{t+2}^a \left(H^a + \sum_{s=t+3}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) + \hat{\vartheta}_{t+2}^{a,R} \cdot \Delta R_{t+2} + \hat{\vartheta}_{t+2}^{a,S} \cdot \Delta S_{t+2} \right) \\ &= U_{t+1}^a \left(H_{t+2}^a + \hat{\vartheta}_{t+2}^{a,R} \cdot \Delta R_{t+2} + \hat{\vartheta}_{t+2}^{a,S} \cdot \Delta S_{t+2} \right). \end{aligned}$$

The usual approach of defining a representative agent in a complete market framework would be to pool all available resources and redistribute them in a socially optimal manner. But in our model the agents cannot pool and redistribute resources arbitrarily. They can only exchange their risk exposures by trading the financial assets. In addition, they face trading constraints. To account for that we build a one-step representative agent at time t with preferences over one-step gains that can be realized by taking admissible positions in the financial market. This will allow us to construct equilibrium prices S_t , continuation values H_t^a and optimal strategies $\hat{\vartheta}_{t+1}^a$ recursively from S_{t+1} and H_{t+1}^a . Observe that agent a 's time- t utility from investing according to an admissible one-step trading strategy $\vartheta_{t+1}^a \in C_{t+1}^a$ is

$$U_t^a(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot \Delta S_{t+1}).$$

We want to extend this to any $\vartheta_{t+1}^a \in L^0(\mathcal{F}_t)^{J+K}$ by setting it equal to $-\infty$ for those states of the world $\omega \in \Omega$ in which the trading constraints are violated. This can be done in an \mathcal{F}_t -measurable way by defining

$$\hat{u}_t^a(\vartheta_{t+1}^a) := \begin{cases} U_t^a(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot \Delta S_{t+1}) & \text{on the set } \{c_t^a(\vartheta_{t+1}^a) = 1\} \\ -\infty & \text{on the set } \{c_t^a(\vartheta_{t+1}^a) = 0\} \end{cases}, \quad (3.2)$$

where $c_t^a(\vartheta_{t+1}^a)$ is the \mathcal{F}_t -measurable function $c_t^a(\vartheta_{t+1}^a) = \text{ess sup}_{\vartheta_{t+1}^a \in C_{t+1}^a} 1_{\{\vartheta_{t+1}^a = \vartheta_{t+1}^a\}}$. However, since we go backwards in time, S_t is not known yet. Therefore, we replace the increment ΔS_{t+1} in (3.2) by S_{t+1} and define the mappings $u_t^a : L^0(\mathcal{F}_t)^{J+K} \rightarrow \underline{L}(\mathcal{F}_t)$ by

$$u_t^a(\vartheta_{t+1}^a) := \begin{cases} U_t^a(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot S_{t+1}) & \text{on the set } \{c_t^a(\vartheta_{t+1}^a) = 1\} \\ -\infty & \text{on the set } \{c_t^a(\vartheta_{t+1}^a) = 0\} \end{cases}. \quad (3.3)$$

The role of the one-step representative agent at time t will be played by the conditional sup-convolution

$$\hat{u}_t(x) = \text{ess sup}_{\substack{\vartheta_{t+1}^a \in L^0(\mathcal{F}_t)^{J+K} \\ \sum_{a \in \mathbb{A}} \vartheta_{t+1}^{a,S} = x}} \sum_{a \in \mathbb{A}} u_t^a(\vartheta_{t+1}^a), \quad x \in L^0(\mathcal{F}_t)^K. \quad (3.4)$$

There are two main differences between this construction and the representative agent that is usually built in a model with complete markets. First, \hat{u}_t is defined on a space of trading strategies and not cash-flows. As a consequence, conditioned on \mathcal{F}_t , it becomes a function on the finite-dimensional space \mathbb{R}^K as opposed to an infinite-dimensional space of random variables. Second, for more general preferences, the Negishi construction involves a weighted average of the individual utility functions, and the proof that the weights can be chosen in such a way that the method yields an equilibrium is based on a fixed point argument. But here, all the weights can be chosen equal to one because the preference functionals U_t^a have the translation property (T) and therefore, react in the same way to the addition of one unit of cash. In particular, our approach does not rely on abstract fixed point results.

If $+\infty - \infty$ is understood as $-\infty$, the mapping $\hat{u}_t : L^0(\mathcal{F}_t)^K \rightarrow L(\mathcal{F}_t)$ is \mathcal{F}_t -concave. Moreover, since the preference functionals of all agents have the local property (2.2), one has

$$1_A \hat{u}_t(x) = 1_A \hat{u}_t(y) \quad \text{for all } x, y \in L^0(\mathcal{F}_t)^K \text{ and } A \in \mathcal{F}_t \text{ such that } 1_A x = 1_A y.$$

We define the *conditional concave conjugate* $\hat{u}_t^* : L^0(\mathcal{F}_t)^K \rightarrow \underline{L}(\mathcal{F}_t)$ by

$$\hat{u}_t^*(y) = \operatorname{ess\,inf}_{x \in L^0(\mathcal{F}_t)^K} \{x \cdot y - \hat{u}_t(x)\}$$

and call a random vector $y \in L^0(\mathcal{F}_t)^K$ a *conditional supergradient* of \hat{u}_t at $x \in L^0(\mathcal{F}_t)^K$ if

$$\hat{u}_t(x) \in L^0(\mathcal{F}_t) \quad \text{and} \quad \hat{u}_t(x+z) - \hat{u}_t(x) \leq z \cdot y \quad \text{for all } z \in L^0(\mathcal{F}_t)^K.$$

The *conditional superdifferential* $\partial \hat{u}_t(x)$ is the set of all conditional supergradients of \hat{u}_t at x . As in standard convex analysis, one has

$$\hat{u}_t(x) + \hat{u}_t^*(y) \leq x \cdot y \quad \text{for all } x, y \in L^0(\mathcal{F}_t)^K,$$

with equality if and only if $y \in \partial \hat{u}_t(x)$. With this notation we now are ready to give a dual characterization of equilibrium:

Theorem 3.1 *A bounded \mathbb{R}^K -valued adapted process $(S_t)_{t=0}^T$ satisfying $S_T = S$ together with admissible trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$ for all agents $a \in \mathbb{A}$ form an equilibrium if and only if for all times $t = 0, \dots, T-1$ the following three conditions hold:*

- (i) $S_t \in \partial \hat{u}_t(n)$
- (ii) $\sum_{a \in \mathbb{A}} U_t^a(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1}) = \hat{u}_t(n)$
- (iii) $\sum_{a \in \mathbb{A}} \hat{\vartheta}_{t+1}^{a,S} = n$.

In particular, if (i)–(iii) hold and $\partial \hat{u}_t(n) = \{S_t\}$ for all $t = 0, \dots, T-1$, then $(S_t)_{t=0}^T$ is the unique equilibrium price process.

The following proposition shows that there exists an equilibrium pricing measure if at least one of the agents has strictly monotone preferences and open trading constraints.

Proposition 3.2 *If the market is in equilibrium and there exists at least one agent $a \in \mathbb{A}$ such that U_0^a is strictly monotone and C_{t+1}^a is \mathcal{F}_t -open for all $t \leq T-1$, then there exists a probability measure \mathbb{Q} on (Ω, \mathcal{F}) equivalent to \mathbb{P} such that*

$$R_t = \mathbb{E}_{\mathbb{Q}}[R_T | \mathcal{F}_t] \quad \text{and} \quad S_t = \mathbb{E}_{\mathbb{Q}}[S_T | \mathcal{F}_t] \quad \text{for all } t = 0, \dots, T. \quad (3.5)$$

To ensure existence of an equilibrium one needs assumptions on the preferences and trading constraints which guarantee that at every time t , the one-step representative agent's utility is finite and attained. To motivate these assumptions, we give a simple example where an equilibrium does not exist.

Example 3.3 Assume the probability space contains only finitely many elements $\{\omega_1, \dots, \omega_N\}$, the time horizon is 1 and the preferences of the agents are given by expectations $U_0^a(\cdot) = \mathbb{E}^a[\cdot]$ corresponding to probability measures \mathbb{P}^a , $a \in \mathbb{A}$. If there exist agents $a, b \in \mathbb{A}$ with no trading constraints and a payoff S^k such that $\mathbb{E}^a[S^k] \neq \mathbb{E}^b[S^k]$, an equilibrium price for this payoff cannot exist. Indeed, no matter how one chooses the initial price $S_0^k \in \mathbb{R}$, at least one of the expectations $\mathbb{E}^a[S^k]$, $\mathbb{E}^b[S^k]$ is different from S_0^k . If for instance, $\mathbb{E}^a[S^k] \neq S_0^k$, then

$$\sup_{\vartheta^a \in \mathbb{R}^{J+K}} U_0^a(H^a + \vartheta^{a,R} \cdot \Delta R_1 + \vartheta^{a,S} \cdot \Delta S_1) \geq \mathbb{E}^a[H^a] + \sup_{\vartheta \in \mathbb{R}} \mathbb{E}^a[\vartheta \Delta S_1^k] = +\infty,$$

and there exists no optimal trading strategy for agent a .

Of course, if in Example 3.3, all agents have preferences given by $\mathbb{E}_{\mathbb{Q}}[\cdot]$ for the same probability measure \mathbb{Q} and $R_0^j = \mathbb{E}_{\mathbb{Q}}[R_1^j]$ for all $j = 1, \dots, J$, then $S_0^k = \mathbb{E}_{\mathbb{Q}}[S^k]$, $k = 1, \dots, K$, are equilibrium prices for the S -assets, and one has

$$\sup_{\vartheta^a \in \mathbb{R}^{J+K}} \mathbb{E}_{\mathbb{Q}}[H^a + \vartheta^{a,R} \cdot \Delta R_1 + \vartheta^{a,S} \cdot \Delta S_1] = \mathbb{E}_{\mathbb{Q}}[H^a + \bar{\vartheta}^{a,R} \cdot \Delta R_1 + \bar{\vartheta}^{a,S} \cdot \Delta S_1] = \mathbb{E}_{\mathbb{Q}}[H^a]$$

for all $a \in \mathbb{A}$. That is, every trading strategy yields the same utility. So all of them are optimal.

Another extreme case is when the agents have general preferences but for all of them, there exists only one admissible trading strategy $(\hat{\vartheta}_t^a)_{t=1}^T$. Then any process $(S_t)_{t=0}^T$ with bounded components together with $(\hat{\vartheta}_t^a)_{t=1}^T = (\tilde{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$, forms an equilibrium.

In the following theorem we give a general existence result. We say that the trading constraints C_{t+1}^a factorize if they are of the form

$$C_{t+1}^a = D_{t+1}^{a,1} \times \cdots \times D_{t+1}^{a,J} \times E_{t+1}^{a,1} \times \cdots \times E_{t+1}^{a,K}$$

for non-empty \mathcal{F}_t -convex subsets $D_{t+1}^{a,1}, \dots, D_{t+1}^{a,J}, E_{t+1}^{a,1}, \dots, E_{t+1}^{a,K}$ of $L^0(\mathcal{F}_t)$.

Theorem 3.4 *Assume that for all $a \in \mathbb{A}$ and $t \leq T-1$, C_{t+1}^a factorizes and is sequentially closed. If there exists a (possibly empty) subset \mathbb{A}' of \mathbb{A} such that*

- (i) U_0^a is sensitive to large losses for all $a \in \mathbb{A}'$ and
- (ii) C_{t+1}^a is \mathcal{F}_t -bounded for all $a \in \mathbb{A} \setminus \mathbb{A}'$ and $t \leq T-1$,

then an equilibrium exists.

4 Differentiable preferences and uniqueness of equilibrium

In this section we introduce a differentiability condition on the preferences and introduce assumptions that guarantee uniqueness of equilibrium prices and optimal wealth dynamics. Condition (D) in the following definition is a conditional version of Gâteaux-differentiability.

Definition 4.1 *We say a preference functional $U_t : L^0(\mathcal{F}_T) \rightarrow \underline{L}(\mathcal{F}_t)$ satisfies the differentiability condition (D) if for all $X \in \text{dom } U_t$ there exists a random variable $Z \in L^1(\mathcal{F}_T)$ such that*

$$\lim_{m \rightarrow \infty} \frac{U_t(X + Y/m) - U_t(X)}{1/m} = \mathbb{E}[YZ \mid \mathcal{F}_t] \quad \text{for all } Y \in L^\infty(\mathcal{F}_T). \quad (4.1)$$

If such a random variable Z exists, it is unique and we denote it by $\nabla U_t(X)$. If for $X \in \text{dom } U_t \cap L^0(\mathcal{F}_{t+1})$, there exists a $Z \in L^1(\mathcal{F}_{t+1})$ such that (4.1) holds for all $Y \in L^\infty(\mathcal{F}_{t+1})$, it is also unique and we denote it by $\tilde{\nabla} U_t(X)$.

If for a random variable $X \in L^0(\mathcal{F}_{t+1})$ the gradient $\nabla U_t(X)$ exists, then so does $\tilde{\nabla} U_t(X)$, and

$$\tilde{\nabla} U_t(X) = \mathbb{E}[\nabla U_t(X) \mid \mathcal{F}_{t+1}].$$

On the other hand, if U_0, \dots, U_{T-1} is a time-consistent family of preference functionals and there is an $X \in L^0(\mathcal{F}_T)$ such that $\tilde{\nabla} U_t(U_{t+1}(X))$ exists for all $t \leq T-1$, then $\nabla U_t(X)$ exists too and can be written as

$$\nabla U_t(X) = \prod_{s=t}^{T-1} \tilde{\nabla} U_s(U_{s+1}(X)). \quad (4.2)$$

Proposition 4.2 *Assume there exists an $a \in \mathbb{A}$ such that for every $t \leq T-1$, the preference functional U_t^a satisfies the differentiability condition (D) and the set C_{t+1}^a is \mathcal{F}_t -open. Then there exists at most one equilibrium price process $(S_t)_{t=0}^T$. Moreover, if the market is in equilibrium,*

$$\frac{d\mathbb{Q}_t^a}{d\mathbb{P}} = \nabla U_t^a \left(H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \quad (4.3)$$

defines a probability measure \mathbb{Q}_t^a on (Ω, \mathcal{F}) such that $\mathbb{Q}_t^a \mid_{\mathcal{F}_t} = \mathbb{P} \mid_{\mathcal{F}_t}$ and

$$R_t = \mathbb{E}_{\mathbb{Q}_t^a} [R_T \mid \mathcal{F}_t], \quad S_t = \mathbb{E}_{\mathbb{Q}_t^a} [S_T \mid \mathcal{F}_t]. \quad (4.4)$$

If in addition, U_0^a is strictly monotone, then $\mathbb{Q}^a := \mathbb{Q}_0^a$ is equivalent to \mathbb{P} and one has,

$$R_t = \mathbb{E}_{\mathbb{Q}^a} [R_T \mid \mathcal{F}_t] \quad \text{and} \quad S_t = \mathbb{E}_{\mathbb{Q}^a} [S_T \mid \mathcal{F}_t] \quad \text{for all } t. \quad (4.5)$$

Remark 4.3 If under the assumptions of Proposition 4.2 an equilibrium exists but the preference functional U_0^a is not strictly monotone, one still has

$$R_t = \mathbb{E}_{\mathbb{Q}^a} [R_T | \mathcal{F}_t] \quad \text{and} \quad S_t = \mathbb{E}_{\mathbb{Q}^a} [S_T | \mathcal{F}_t] \quad \mathbb{Q}^a\text{-almost surely.} \quad (4.6)$$

But if \mathbb{Q}^a is not equivalent to \mathbb{P} on \mathcal{F}_t , (4.6) is a weaker statement than (4.4) since for $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ and $\mathbb{Q}[A] = 0$, (4.6) does not give any information about R_t and S_t in the event A . On the other hand, if \mathbb{Q}^a is equivalent to \mathbb{P} , then (4.6) contains the same amount of information as (4.4).

Since we have made no assumptions on non-redundancy of the assets, we cannot say anything about the uniqueness of optimal trading strategies ($\hat{\vartheta}_t^a$). If for instance, $R_t^1 = R_t^2$ for all t , then any investment in R^1 can arbitrarily be replaced by one in R^2 . However, if equilibrium prices are unique and U_0^a is strictly concave modulo translation, it can be shown that the optimal trading gains of the corresponding agent are unique.

Proposition 4.4 *If there exists a unique equilibrium price process $(S_t)_{t=0}^T$ and U_0^a is strictly concave modulo translation for some agent $a \in \mathbb{A}$, then the optimal one-step trading gains*

$$\hat{\vartheta}_t^{a,R} \cdot \Delta R_t + \hat{\vartheta}_t^{a,S} \cdot \Delta S_t, \quad t = 1, \dots, T,$$

are unique.

5 Base preferences and attainable initial endowments

In this section we consider the case where all agents have preferences of the same type and all endowments can be attained by trading in the financial market. Then, after hedging the endowment, every agent invests in the same portfolio. If preferences are differentiable, the equilibrium pricing kernel can be given in explicit form. In Subsection 5.1 we show a one-fund theorem. In Subsection 5.2 we discuss option prices under stochastic volatility, demand pressure and short-selling constraints.

5.1 One-fund theorem

Note that mean-variance preferences of the form $MV^\gamma(X) = \mathbb{E}[X] - \gamma \text{Var}(X)$ for a parameter $\gamma > 0$ can be written as $MV^\gamma(X) = \gamma^{-1} MV^1(\gamma X)$. If the preferences of all our agents are related in the same way to a base preference functional satisfying our assumptions, and all initial endowments are attainable by trading, the following one fund result holds.

Theorem 5.1 (One fund theorem)

Assume there exists an equilibrium such that $(S_t)_{t=0}^T$ as well as all optimal one-step trading gains

$$\hat{\vartheta}_t^{a,R} \cdot \Delta R_t + \hat{\vartheta}_t^{a,S} \cdot \Delta S_t$$

are unique and the initial endowments are of the form

$$H^a = c^a + \sum_{t=1}^T \eta_t^{a,R} \cdot \Delta R_t + \eta_t^{a,S} \cdot \Delta S_t,$$

for constants $c^a \in \mathbb{R}$ and trading strategies $(\eta_t^a)_{t=1}^T$, $a \in \mathbb{A}$. Moreover, suppose there exists a sequence of base preference functionals

$$U_t : L^0(\mathcal{F}_T) \rightarrow \underline{L}(\mathcal{F}_t), \quad \text{and non-empty subsets } C_{t+1} \in L^0(\mathcal{F}_t)^{J+K}, \quad t = 0, \dots, T-1,$$

such that the preferences and trading constraints of agent $a \in \mathbb{A}$ are given by

$$U_t^a(X) = \frac{1}{\gamma^a} U_t(\gamma^a X) \quad \text{and} \quad C_{t+1}^a = \frac{1}{\gamma^a} C_{t+1} - \eta_{t+1}^a \quad (5.1)$$

for parameters $\gamma^a > 0$, $a \in \mathbb{A}$. Denote

$$\gamma = \left(\sum_{a \in \mathbb{A}} \frac{1}{\gamma^a} \right)^{-1} \quad \text{and} \quad \eta_t^S = \sum_{a \in \mathbb{A}} \eta_t^{a,S}.$$

Then there exists a J -dimensional trading strategy $(\hat{\vartheta}_t^R)_{t=1}^T$ which for every $t \leq T-1$, maximizes

$$U_t \left(\sum_{s=t+1}^T \vartheta_s^R \cdot \Delta R_s + \gamma(n + \eta_s^S) \cdot \Delta S_s \right) \quad (5.2)$$

over all $(\vartheta_s^R)_{s=t+1}^T$ satisfying $(\vartheta_s^R, \gamma(n + \eta_s^S)) \in C_s$, $s = t+1, \dots, T$, and agent a 's optimal one-step trading gains are of the form

$$\left(\frac{1}{\gamma^a} \hat{\vartheta}_t^R - \eta_t^{a,R} \right) \cdot \Delta R_t + \left(\frac{\gamma}{\gamma^a} (n + \eta_t^S) - \eta_t^{a,S} \right) \cdot \Delta S_t, \quad t = 1, \dots, T. \quad (5.3)$$

If, in addition, U_t satisfies the differentiability condition (D) and C_{t+1} is \mathcal{F}_t -open for all $t \leq T-1$, then

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}} = \nabla U_t \left(\sum_{s=1}^T \hat{\vartheta}_s^R \cdot \Delta R_s + \gamma(n + \eta_s^S) \cdot \Delta S_s \right)$$

defines probability measures satisfying $\mathbb{Q}_t |_{\mathcal{F}_t} = \mathbb{P} |_{\mathcal{F}_t}$ such that

$$R_t = \mathbb{E}_{\mathbb{Q}_t} [R_T | \mathcal{F}_t] \quad \text{and} \quad S_t = \mathbb{E}_{\mathbb{Q}_t} [S_T | \mathcal{F}_t] \quad \text{for all } t \leq T-1.$$

If moreover, U_0 is strictly monotone, then $\mathbb{Q} := \mathbb{Q}_0$ is equivalent to \mathbb{P} , and one has

$$R_t = \mathbb{E}_{\mathbb{Q}} [R_T | \mathcal{F}_t] \quad \text{as well as} \quad S_t = \mathbb{E}_{\mathbb{Q}} [S_T | \mathcal{F}_t] \quad \text{for all } t.$$

Remark 5.2 If under the assumptions of Theorem 5.1, there are no R -assets ($J = 0$) and the endowments are of the form $H^a = c^a + g^{a,S} \cdot S_T$ for constants $c^a \in \mathbb{R}$ and deterministic vectors $g^{a,S} \in \mathbb{R}^K$, then in equilibrium, H^a can be written as

$$H^a = c^a + g^{a,S} \cdot S_0 + \sum_{t=1}^T g^{a,S} \cdot \Delta S_t.$$

So it follows from Theorem 5.1 that agent a 's optimal one-step trading gains are of the form

$$\left(\frac{\gamma}{\gamma^a} (n + g^S) - g^{a,S} \right) \cdot \Delta S_t, \quad t = 1, \dots, T, \quad \text{where } g^S = \sum_{a \in \mathbb{A}} g^{a,S}.$$

That is, after hedging the endowment, every agent, like in a one-time-step CAPM, takes a static position in the market portfolio. Moreover, if U_0 is strictly monotone and has the differentiability property (D), the equilibrium pricing kernel simplifies to

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \nabla U_0 (\gamma(n + g^S) \cdot (S_T - S_0)) = \nabla U_0 (\gamma(n + g^S) \cdot S_T).$$

In particular, the equilibrium pricing measure only depends on aggregate endowment and external supply and not on the distribution of wealth among the agents. Moreover, the introduction of new securities in zero net supply does not change existing asset prices. Of course, the situation is different if the agents' preferences and constraints are not related to each other as in (5.1) or if endowments are unspanned in equilibrium.

Example 5.3 If the agents have entropic utility functionals

$$U_t^a(X) = -\frac{1}{\gamma^a} \log \mathbb{E} [\exp(-\gamma^a X) | \mathcal{F}_t] \quad \text{for constants } \gamma^a > 0, \quad a \in \mathbb{A},$$

one can write $U_t^a(X) = U_t(\gamma^a X) / \gamma^a$ for the base preference functionals

$$U_t(X) = -\log \mathbb{E} [\exp(-X) | \mathcal{F}_t], \quad t = 0, \dots, T.$$

We know from Example 2.1 that they have the properties (M), (T), (C), (TC) and U_0 satisfies (SL), (SM), (SC). Moreover, U_t satisfies the differentiability condition (D) with

$$\nabla U_t(X) = \frac{\exp(-X)}{\mathbb{E} [\exp(-X) | \mathcal{F}_t]}.$$

So if all agents are unconstrained and all H^a are bounded from below, one obtains from Theorem 3.4 that an equilibrium exists. By Propositions 4.2 and 4.4, the equilibrium prices and optimal one-step trading gains are unique. If in equilibrium the endowments can be written as

$$H^a = c^a + \sum_{t=1}^T \eta_t^{a,R} \cdot \Delta R_t + \eta_t^{a,S} \cdot \Delta S_t,$$

one obtains from Theorem 5.1 that for $\gamma = (\sum_{a \in \mathbb{A}} (\gamma^a)^{-1})^{-1}$ and $\eta_t^S = \sum_{a \in \mathbb{A}} \eta_t^{a,S}$, there exists a J -dimensional trading strategy $(\hat{\vartheta}_t^R)_{t=1}^T$ such that (5.2)–(5.3) hold. Moreover,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \nabla U_0 \left(\sum_{s=1}^T \hat{\vartheta}_s^R \cdot \Delta R_s + \gamma(n + \eta_s^S) \cdot \Delta S_s \right) = \frac{\exp \left(- \sum_{s=1}^T \left\{ \hat{\vartheta}_s^R \cdot \Delta R_s + \gamma(n + \eta_s^S) \cdot \Delta S_s \right\} \right)}{\mathbb{E} \left[\exp \left(- \sum_{s=1}^T \left\{ \hat{\vartheta}_s^R \cdot \Delta R_s + \gamma(n + \eta_s^S) \cdot \Delta S_s \right\} \right) \right]} \quad (5.4)$$

defines a probability measure \mathbb{Q} equivalent to \mathbb{P} for which

$$R_t = \mathbb{E}_{\mathbb{Q}} [R_T | \mathcal{F}_t] \quad \text{and} \quad S_t = \mathbb{E}_{\mathbb{Q}} [S_T | \mathcal{F}_t] \quad \text{for all } t = 0, \dots, T.$$

In the special case where there are no R -assets ($J = 0$) and endowments are of the form $H^a = c^a + g^{a,S} \cdot S$ for deterministic vectors $g^{a,S} \in \mathbb{R}^K$, the pricing kernel simplifies to

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\exp \left(-\gamma(n + g^S) \cdot S_T \right)}{\mathbb{E} \left[\exp \left(-\gamma(n + g^S) \cdot S_T \right) \right]} \quad \text{for } g^S = \sum_{a \in \mathbb{A}} g^{a,S}.$$

5.2 Simulation of option prices in a discrete Heston model

As an application of Theorem 5.1 we calculate equilibrium prices of equity options and study the effects of stochastic volatility, demand pressure and short-selling constraints. It has been observed that implied volatility smiles of index options and options on single stocks look differently even though the underlyings are distributed similarly. Typically, index options appear to be more expensive, and their smiles are steeper. A possible explanation for this difference is that there usually is positive aggregate demand for out-of-the-money put index options by end users. If option dealers sell these options to end users and cannot fully hedge themselves, they expose themselves to the risk of a decline of the index. To compensate for that they are asking higher prices; see Bakshi et al. (2003), Bollen and Whaley (2004), Gârleanu et al. (2009) and the references therein.

We here propose an incomplete market equilibrium model to account for this phenomenon. It is similar to the one in Gârleanu et al. (2009). But it starts with individual agents and not a representative one. We assume our agents \mathbb{A} are option dealers with expected exponential utility preferences with absolute risk aversions $\gamma_a > 0$, $a \in \mathbb{A}$. They have no endowments and trade in the underlying and the options. On the other side there are end users such as for instance, pension funds who buy put options to insure their investment portfolios. We assume that end users demand a fixed portfolio of put options and pay the price at which it is offered by the dealers. For our simulations we suppose they demand $m \geq 0$ put options with discounted strike $K_0 = 92$ and maturity T . From the dealers' point of view the external supply is $n = -m \leq 0$. We assume that the dealers do not influence the price of the underlying R but determine the option prices through demand and supply among them. Suppose the underlying moves according to a discretized Heston model

$$\begin{aligned} R_{t+h} &= |R_t + \mu R_t h + \sqrt{v_t} R_t \Delta b_{t+h}^1|, & R_0 &= 100 \\ v_{t+h} &= |v_t + \alpha(m - v_t)h + \beta \sqrt{v_t} \Delta b_{t+h}^2|, & v_0 &= 0.04. \end{aligned}$$

The absolute values are here to guarantee that R_t and v_t stay above zero. We choose maturity $T = 0.5$ years and make 100 steps of size $h = 0.005$. The other parameters are $\mu = 0.1$, $\alpha = 0.2$, $m = 0.04$, $\beta = 0.3$. $(b_{nh}^1)_{n=0}^{100}$ and $(b_{nh}^2)_{n=0}^{100}$ are two Bernoulli random walks with independent increments that have distribution $\mathbb{P}[\Delta b_t^i = \pm \sqrt{h}] = 1/2$ and correlation $\mathbb{E}[\Delta b_t^1 \Delta b_t^2] / h = -0.3$. We are interested in the prices of put options on R . The discounted time- T payoff corresponding to discounted strike K and maturity T is $S = (K - R_T)^+$. While for the simulation of (R_t) we make steps of size $h = 0.005$, we assume the trading dates to be a subset \mathbb{T} of $\mathcal{T} = \{0, h, \dots, T\}$. If \mathbb{T} is coarse, option dealers can rebalance their portfolios less frequently, and the model becomes more incomplete. We think of situations where transaction costs are high or there are trading constraints. Denote by $\Theta_{\mathbb{T}}^R$ the set of all investment strategies

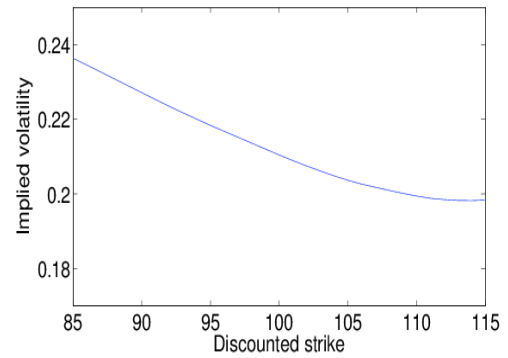
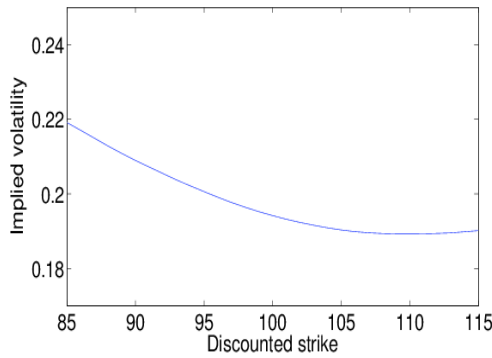
in the underlying that are constant on the intervals $[t_{i-1}, t_i)$, where $\mathbb{T} = \{t_0 = 0, t_1, \dots, T\}$. By Formula (5.4), the equilibrium pricing kernel takes the form

$$\frac{\exp\left(-\sum \hat{\vartheta}_t^R \Delta R_t + \gamma m P\right)}{\mathbb{E}\left[\exp\left(-\sum_{t \in \mathbb{T}} \hat{\vartheta}_t^R \Delta R_t + \gamma m P\right)\right]},$$

where $\gamma = (\sum_{a \in \mathbb{A}} (\gamma^a)^{-1})^{-1}$, $P = (K_0 - R_T)^+$ and $(\hat{\vartheta}_t^R)$ is the maximizer of the expected utility

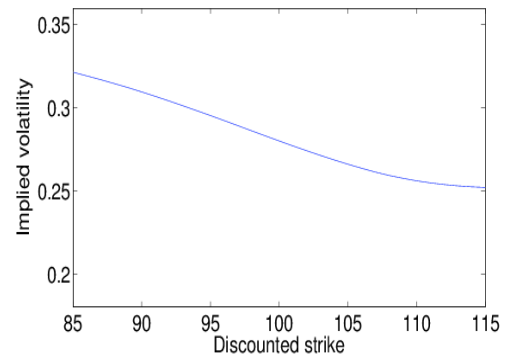
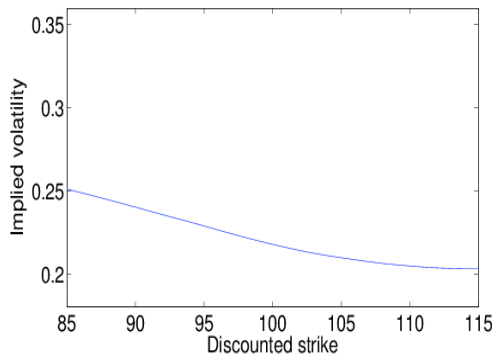
$$-\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\sum \vartheta_t^R \Delta R_t + \gamma m P\right)\right] \quad \text{over the set } \Theta_{\mathbb{T}}^R.$$

In the following we calculate implied volatilities of put options with discounted strikes between 85 and 115 for different choices of m and \mathbb{T} . We first assume $m = 0$ (no demand pressure) and think of R as the price of a single stock. The first of the two figures below shows implied volatilities for the case $m = 0$ and $\mathbb{T} = \mathcal{T}$ (option dealers rebalance their portfolios frequently). The second figure shows the situation for $m = 0$ and $\mathbb{T} = \{0, T\}$ (option dealers have to form their portfolios at time 0 and keep them constant until T).



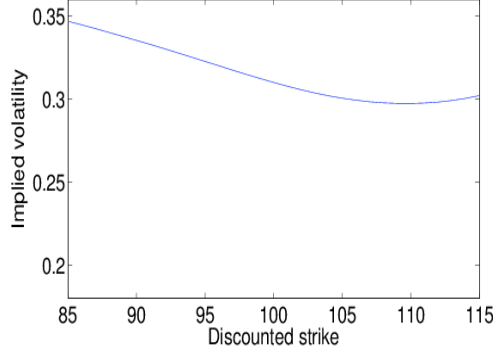
For $m = 0$, trading restrictions increase implied volatilities, and therefore option prices, only slightly because option dealers do not have to hedge the options. The only difference between frequent and less frequent trading is the quality of the dealers' investment strategy in the underlying R .

Now assume that net demand by end users for put options with discounted strike K_0 is positive, as is typical for index options. The first of the following two figures shows implied volatilities for the case $m > 0$ (positive demand) and $\mathbb{T} = \mathcal{T}$ (dealers rebalance frequently). The second one is for $m > 0$ (positive demand) and $\mathbb{T} = \{0, T\}$ (dealers have to invest statically).



It can be seen that net demand for put options with discounted strike $K_0 = 92$ increases prices of put options of all strikes, but especially those corresponding to low strikes. Also, trading restrictions have more of an influence on prices than in the case $m = 0$.

As a limit case, the next figure shows results for $m > 0$ and $\mathbb{T} = \emptyset$. That is, there is positive demand by end users for put options with discounted strike K_0 . But option dealers are not allowed to trade the underlying. This can be interpreted as a short-selling constraint. If dealers are short in put options, they would like to hedge by shorting the underlying. But under short-selling constraints, the best they can do is to have a zero position in the underlying. This increases prices of put options further compared to the case of demand pressure and few trading dates.



See also Avellaneda and Lipkin (2009) for a continuous-time model for hard-to-borrow stocks and the valuation of options on them.

A Proofs of Section 2

Proof of (2.2)

Since U_t^a satisfies (C), one has $1_A U_t^a(X) = 1_A U_t^a(1_A Y + 1_{A^c} X) \geq 1_A U_t^a(Y)$ and by symmetry, $1_A U_t^a(X) \leq 1_A U_t^a(Y)$. This gives $1_A U_t^a(X) = 1_A U_t^a(Y)$. \square

Proof of (2.14)

Let $X \in \underline{L}(\mathcal{F}_{t+1})$ and introduce the \mathcal{F}_t -measurable sets

$$A_0 = \{\mathbb{P}[X \leq 0 \mid \mathcal{F}_t] > 0\}, \quad A_i = \{\mathbb{P}[X \leq i \mid \mathcal{F}_t] > 0 \text{ and } \mathbb{P}[X \leq i-1 \mid \mathcal{F}_t] = 0\} \quad \text{for } i \geq 1.$$

Since Ω is the union of the mutually disjoint sets A_0, A_1, \dots , the random variable

$$\xi = \sum_{i \geq 0} 1_{A_i} \frac{1_{\{X \leq i\}}}{\mathbb{P}[X \leq i \mid \mathcal{F}_t]}$$

is in \mathcal{D}_{t+1} , and one has

$$\mathbb{E}[(X \wedge m] \vee l) \xi + G(\xi) \mid \mathcal{F}_t] \leq \sum_{i \geq 0} 1_{A_i} (i + \mathbb{E}[G(\xi) \mid \mathcal{F}_t]) < +\infty$$

for all $m \geq 0$ and $l \leq 0$. This shows (2.14). \square

Proof of (2.15)

Let $X \in \underline{L}(\mathcal{F}_{t+1})$ such that $\mathbb{P}[X < 0] > 0$. We first show that for every $m \in \mathbb{N}$, there exists a constant $\lambda_t \geq 1$ such that

$$\mathbb{P}[V_t(\lambda_t X) \leq -m] > 0. \tag{A.1}$$

To do that, we introduce the \mathcal{F}_t -measurable set $A = \{\mathbb{P}[X < 0 \mid \mathcal{F}_t] > 0\}$ and the conditional density

$$\xi = 1_A \frac{1_{\{X < 0\}}}{\mathbb{P}[X < 0 \mid \mathcal{F}_t]} + 1_{A^c} \in \mathcal{D}_{t+1}.$$

The claim (A.1) now follows from the fact that

$$1_A V_t(\lambda_t X) \leq 1_A (\mathbb{E}[\lambda_t X \xi + G(\xi) \mid \mathcal{F}_t]) \rightarrow -\infty 1_A \quad \text{as } \lambda_t \rightarrow \infty.$$

So one obtains that for every $m \in \mathbb{N}$ there exist constants $\lambda_{t-1}, \lambda_t \geq 1$ such that

$$\mathbb{P}[V_{t-1}(\lambda_{t-1} V_t(\lambda_t X)) \leq -m] > 0.$$

Since $V_t(0) = 0$, one obtains from concavity that

$$V_t(\lambda_{t-1} \lambda_t X) \leq \lambda_{t-1} V_t(\lambda_t X),$$

and it follows that

$$\mathbb{P}[V_{t-1}(\lambda_{t-1} \lambda_t X) \leq -m] \geq \mathbb{P}[V_{t-1}(\lambda_{t-1} V_t(\lambda_t X)) \leq -m] > 0$$

for all λ_{t-1} and λ_t sufficiently large. Iterating this argument yields (2.15). \square

B Proofs of Section 3

Proof of Theorem 3.1

Let us first assume that the bounded adapted process $(S_t)_{t=0}^T$ and the strategies $(\hat{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$, form an equilibrium. Then condition (iii) of Theorem 3.1 holds by definition. Moreover, the continuation value processes (H_t^a) are bounded from below, and one obtains

$$\begin{aligned}
-\hat{u}_t^*(S_t) &= \operatorname{ess\,sup}_{x \in L^0(\mathcal{F}_t)^K} \{ \hat{u}_t(x) - x \cdot S_t \} \\
&= \operatorname{ess\,sup}_{\vartheta^a \in L^0(\mathcal{F}_t)^{J+K}} \sum_{a \in \mathbb{A}} \{ u_t^a(\vartheta^a) - \vartheta^{a,S} \cdot S_t \} \\
&= \operatorname{ess\,sup}_{\vartheta^a \in C_{t+1}^a} \sum_{a \in \mathbb{A}} U_t^a (H_{t+1}^a + \vartheta^{a,R} \cdot \Delta R_{t+1} + \vartheta^{a,S} \cdot \Delta S_{t+1}) \\
&= \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1} \right) \\
&= \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1} \right) - n \cdot S_t \\
&\leq \hat{u}_t(n) - n \cdot S_t.
\end{aligned} \tag{B.1}$$

Since one always has $\hat{u}_t(n) + \hat{u}_t^*(S_t) \leq n \cdot S_t$, the inequality in (B.1) must be an equality, and it follows that

$$S_t \in \partial \hat{u}_t(n) \quad \text{as well as} \quad \sum_a U_t^a \left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1} \right) = \hat{u}_t(n).$$

This shows that the conditions (i) and (ii) hold.

For the reverse implication, assume that (i)–(iii) are satisfied. Then the market clearing condition holds. Moreover, one has for all admissible trading strategies $(\vartheta_t^a)_{t=1}^T$, $a \in \mathbb{A}$,

$$\begin{aligned}
&\sum_{a \in \mathbb{A}} U_t^a (H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot \Delta S_{t+1}) \\
&= \sum_{a \in \mathbb{A}} U_t^a (H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot S_{t+1}) - \vartheta_{t+1}^{a,S} \cdot S_t \\
&\leq \hat{u}_t \left(\sum_{a \in \mathbb{A}} \vartheta_{t+1}^{a,S} \right) - \sum_{a \in \mathbb{A}} \vartheta_{t+1}^{a,S} \cdot S_t \\
&\leq -\hat{u}_t^*(S_t) = \hat{u}_t(n) - n \cdot S_t \\
&= \sum_{a \in \mathbb{A}} U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}).
\end{aligned}$$

From here it follows by backwards induction that $(\hat{\vartheta}_t^a)_{t=1}^T$ is an optimal strategy for agent $a \in \mathbb{A}$. \square

Proof of Proposition 3.2

Suppose there exists no probability measure \mathbb{Q} equivalent to \mathbb{P} satisfying (3.5). Then it follows from the Dalang–Morton–Willinger theorem (Dalang et al. 1990) that there exists a $t \leq T - 1$ and a one-step trading strategy $\vartheta_{t+1} \in L^0(\mathcal{F}_t)^{J+K}$ such that $\vartheta_{t+1}^R \cdot \Delta R_{t+1} + \vartheta_{t+1}^S \cdot \Delta S_{t+1}$ is non-negative and strictly positive with positive \mathbb{P} -probability. The same is true for $\varepsilon_t(\vartheta_{t+1}^R \cdot \Delta R_{t+1} + \vartheta_{t+1}^S \cdot \Delta S_{t+1})$ for arbitrary \mathcal{F}_t -measurable $\varepsilon_t > 0$. But this means that there can exist no optimal trading strategy for the agents with strictly monotone preference functionals and open trading constraints, a contradiction to the assumption that the market is in equilibrium. \square

Proof of Theorem 3.4

Set $S_T = S$ and $H_T^a = H^a$, $a \in \mathbb{A}$. Then the existence of an equilibrium follows from Theorem 3.1 if we can show that for every $t \leq T - 1$, $S_{t+1} \in L^\infty(\mathcal{F}_{t+1})^K$ and bounded from below $H_{t+1}^a \in L^0(\mathcal{F}_{t+1})$, $a \in \mathbb{A}$, the following hold:

(a) there exist one-step strategies $\hat{\vartheta}_{t+1}^a \in C_{t+1}^a$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{R}} \hat{\vartheta}_{t+1}^{a,S} = n \quad \text{and} \quad \sum_{a \in \mathbb{A}} U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1}) = \hat{u}_t(n);$$

(b) there exists a random vector $S_t \in \partial \hat{u}_t(n) \cap L^\infty(\mathcal{F}_t)^K$.

(a) follows from Lemma B.1 below and (b) from Lemma B.2. To prove the two lemmas, we need the following concepts from Cheridito et al. (2014): Let C be a subset of $L^0(\mathcal{F})^d$.

- We call C \mathcal{F} -stable if $1_A x + 1_{A^c} y \in C$ for all $x, y \in L^0(\mathcal{F})^d$ and $A \in \mathcal{F}$;
- We call C \mathcal{F} -linear if $\lambda x + y \in C$ for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$;
- We call C \mathcal{F} -convex if $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$ such that $0 \leq \lambda \leq 1$;
- We say C is an \mathcal{F} -convex polyhedral set if it is of the form

$$C = \{x \in L^0(\mathcal{F})^d : x \cdot a_i \leq b_i, i = 1, \dots, I\}$$

for some integer $I \in \mathbb{N}$, $a_1, \dots, a_I \in L^0(\mathcal{F})^d$ and $b_1, \dots, b_I \in L^0(\mathcal{F})$;

- By C^\perp we denote the conditional orthogonal complement $\{x \in L^0(\mathcal{F})^d : x \cdot y = 0 \text{ for all } y \in C\}$.
- A mapping $f : L^0(\mathcal{F})^d \rightarrow L^0(\mathcal{F})^m$ is \mathcal{F} -linear if $f(\lambda x + y) = \lambda f(x) + f(y)$ for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$;
- A mapping $f : L^0(\mathcal{F})^d \rightarrow L^0(\mathcal{F})$ is \mathcal{F} -concave if $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$ for all $x, y \in L^0(\mathcal{F})^d$ and $\lambda \in L^0(\mathcal{F})$ satisfying $0 \leq \lambda \leq 1$;
- By $\mathbb{N}(\mathcal{F})$ we denote the set of all \mathcal{F} -measurable random variables taking values in $\mathbb{N} = \{1, 2, \dots\}$. For a sequence $(x_m)_{m \in \mathbb{N}}$ in $L^0(\mathcal{F})^d$ and $M \in \mathbb{N}(\mathcal{F})$, we define $x_M := \sum_{m \in \mathbb{N}} 1_{\{M=m\}} x_m$.

Lemma B.1 Fix $t \leq T - 1$ and $S_{t+1} \in L^\infty(\mathcal{F}_{t+1})^K$. Let H_{t+1}^a , $a \in \mathbb{A}$, be random variables in $L^0(\mathcal{F}_{t+1})$ that are bounded from below. Assume the sets C_{t+1}^a , $a \in \mathbb{A}$, factorize and are sequentially closed. If there exists a (possibly empty) subset \mathbb{A}' of \mathbb{A} such that U_0^a is sensitive to large losses for all $a \in \mathbb{A}'$ and C_{t+1}^a is \mathcal{F}_t -bounded for all $a \in \mathbb{A} \setminus \mathbb{A}'$, then $\hat{u}_t(x) < +\infty$ for all $x \in L^0(\mathcal{F}_t)^K$, $\Theta := \{x \in L^0(\mathcal{F}_t)^K : \hat{u}_t(x) \in L^0(\mathcal{F}_t)\}$ is an \mathcal{F}_t -convex polyhedral set, and for all $x \in \Theta$, there exist one-step trading strategies $\vartheta_{t+1}^a \in C_{t+1}^a$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{R}} \vartheta_{t+1}^{a,S} = x \quad \text{and} \quad \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot S_{t+1} \right) = \hat{u}_t(x). \quad (\text{B.2})$$

In particular, $\hat{u}_t(n) \in L^0(\mathcal{F}_t)$, and there exist one-step trading strategies $\hat{\vartheta}_{t+1}^a \in C_{t+1}^a$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{R}} \hat{\vartheta}_{t+1}^{a,S} = n \quad \text{and} \quad \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1} \right) = \hat{u}_t(n).$$

Proof. Let $f : L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)} \rightarrow L^0(\mathcal{F}_t)^K$ be the \mathcal{F}_t -linear mapping given by $f((\vartheta^a)_{a \in \mathbb{A}}) := \sum_{a \in \mathbb{A}} \vartheta^a \cdot S$. Since ΔR_{t+1} and S_{t+1} are bounded, one obtains from (2.3) that $f(\prod_{a \in \mathbb{A}} C_{t+1}^a) \subseteq \tilde{\Theta} := \{x \in L^0(\mathcal{F}_t)^K : \hat{u}_t(x) > -\infty\}$. On the other hand, it follows from the definition of \hat{u}_t that for every $x \in \tilde{\Theta}$, there exists a sequence (A_m) of \mathcal{F}_t -measurable events together with one-step strategies $\vartheta_m^a \in C_{t+1}^a$ such that $A_m \uparrow \Omega$ almost surely and $\sum_{a \in \mathbb{A}} \vartheta_m^{a,S} = x$ on the set $A_m \setminus A_{m-1}$, where $A_0 = \emptyset$. Since the sets C_{t+1}^a are \mathcal{F}_t -convex and sequentially closed, $\vartheta^a = \sum_m 1_{A_m \setminus A_{m-1}} \vartheta_m^a$ belongs to C_{t+1}^a for all $a \in \mathbb{A}$, and one has $\sum_{a \in \mathbb{A}} \vartheta^{a,S} = x$. This shows that $f(\prod_{a \in \mathbb{A}} C_{t+1}^a) = \tilde{\Theta}$. Clearly, $\prod_{a \in \mathbb{A}} C_{t+1}^a$ is an \mathcal{F}_t -convex polyhedral set, and by Theorem 4.5 of Wu (2013), a non-empty \mathcal{F}_t -convex subset of $L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)}$ is polyhedral if and only if it can be represented as

$$C = \left\{ \sum_{i=1}^m \lambda_i c_i : \lambda_i \in L^0_+, \sum_{i=1}^l \lambda_i = 1 \right\} \quad (\text{B.3})$$

for integers $0 \leq l \leq m \in \mathbb{N}$ and random vectors $c_1, \dots, c_m \in L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)}$. Since f is \mathcal{F}_t -linear, $f(\prod_{a \in \mathbb{A}} C_{t+1}^a) = \tilde{\Theta}$ is a subset of $L^0(\mathcal{F}_t)^K$ with a representation of the form (B.3), and therefore, an \mathcal{F}_t -convex polyhedral set. n belongs to $\tilde{\Theta}$ because the sets C_{t+1}^a were assumed to fulfill (C1). So if we can show (B.2) for all $x \in \tilde{\Theta}$, one obtains $\tilde{\Theta} = \Theta$, and the lemma follows. To do this, we fix $x \in \tilde{\Theta}$. Since the price process $(R_t)_{t=0}^T$ satisfies (NA), one obtains from the Dalang–Morton–Willinger theorem (Dalang et al. 1990) a probability measure \mathbb{Q} equivalent to \mathbb{P} such that $R_t^j = \mathbb{E}_{\mathbb{Q}} \left[R_{t+1}^j \mid \mathcal{F}_t \right]$ for all j . Set $W^k = S_{t+1}^k - \mathbb{E}_{\mathbb{Q}} \left[S_{t+1}^k \mid \mathcal{F}_t \right]$ for all k . Due to (2.3),

$$g(\eta) = \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \eta^{a,R} \cdot \Delta R_{t+1} + \eta^{a,S} \cdot W \right)$$

defines an \mathcal{F}_t -concave mapping $g : L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)} \rightarrow L^0(\mathcal{F}_t)$, and there exist one-step strategies $\vartheta_{t+1}^a \in C_{t+1}^a$, $a \in \mathbb{A}$, satisfying (B.2) if and only if the conditional optimization problem $\text{ess sup}_{\eta \in B} g(\eta)$ has an optimal solution, where B is the \mathcal{F}_t -convex polyhedral set

$$B := \left\{ \eta = (\eta^a)_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} C_{t+1}^a : \sum_{a \in \mathbb{A}} \eta^{a,S} = x \right\}.$$

Consider the sequentially closed \mathcal{F}_t -linear set

$$E := \left\{ \theta \in L^0(\mathcal{F}_t)^{(J+K)} : \theta^R \cdot \Delta R_{t+1} + \theta^S \cdot W = 0 \right\}.$$

It follows from Corollary 2.12 of Cheridito et al. (2014) that every $\eta \in L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)}$ has a unique decomposition $\eta = \eta' + \eta''$ with $\eta' \in (E^\perp)^{|\mathbb{A}'|(J+K)} \times L^0(\mathcal{F}_t)^{|\mathbb{A} \setminus \mathbb{A}'|(J+K)}$ and $\eta'' \in E^{|\mathbb{A}'|(J+K)} \times \{0\}^{|\mathbb{A} \setminus \mathbb{A}'|(J+K)}$. Denote by Π the projection from $L^0(\mathcal{F}_t)^{|\mathbb{A}|(J+K)}$ to $(E^\perp)^{|\mathbb{A}'|(J+K)} \times L^0(\mathcal{F}_t)^{|\mathbb{A} \setminus \mathbb{A}'|(J+K)}$ given by $\Pi(\eta) := \eta'$. It follows as above that $\Pi(B)$ is an \mathcal{F}_t -convex polyhedral set. In particular, it is sequentially closed and \mathcal{F}_t -stable. Choose $\bar{\eta} \in \prod_{a \in \mathbb{A}} C_{t+1}^a$ such that $\sum_{a \in \mathbb{A}} \bar{\eta}^{a,S} = x$. g has a maximizer in B if and only if it has one in the set

$$C := \{ \eta \in \Pi(B) : g(\eta) \geq g(\bar{\eta}) \}.$$

By Lemma 4.3 and Theorems 4.4 and 7.2 of Cheridito et al. (2014), g has a maximizer in C if C is \mathcal{F}_t -bounded. But this follows from Theorem 3.13 of Cheridito et al. (2014) if it can be shown that for every $\eta \in C \setminus \{\bar{\eta}\}$ there exists an $m \in \mathbb{N}$ such that $m(\eta - \bar{\eta}) + \bar{\eta} \notin C$. If $\eta^a \neq \bar{\eta}^a = 0$ for some $a \in \mathbb{A} \setminus \mathbb{A}'$, this is a direct consequence of the assumption that C_{t+1}^a is \mathcal{F}_t -bounded for all $a \in \mathbb{A} \setminus \mathbb{A}'$. On the other hand, if $\eta^a = \bar{\eta}^a$ for all $a \in \mathbb{A} \setminus \mathbb{A}'$ and $\eta^a \neq \bar{\eta}^a$ for some $a \in \mathbb{A}'$, there exists a set $A \in \mathcal{F}_t$ with $\mathbb{P}[A] > 0$ and a non-empty subset \mathbb{A}'' of \mathbb{A}' such that

$$\mathbb{P}[(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + (\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W \neq 0 \mid \mathcal{F}_t] > 0 \quad \text{on } A \quad \text{for all } a \in \mathbb{A}''$$

and

$$(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + (\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W = 0 \quad \text{on } A \quad \text{for all } a \in \mathbb{A} \setminus \mathbb{A}''.$$

Since ΔR_{t+1} and W admit no arbitrage, $\mathbb{P}[(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + (\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W < 0 \mid \mathcal{F}_t]$ must be strictly positive on A for all $a \in \mathbb{A}''$. So it follows from the sensitivity to large losses of the functionals U_0^a that for $\eta^{m,a} := m(\eta^a - \bar{\eta}^a) + \bar{\eta}^a$,

$$\lim_{m \rightarrow \infty} U_t^a (H_{t+1}^a + (\eta^{m,a,R} \cdot \Delta R_{t+1} + \eta^{m,a,S} \cdot W)) \rightarrow -\infty \quad \text{almost surely on } A \quad \text{for all } a \in \mathbb{A}'' \quad (\text{B.4})$$

Indeed, assume to the contrary that there exist $a \in \mathbb{A}''$, $A' \in \mathcal{F}_t$ with $A' \subset A$ and $\mathbb{P}[A'] > 0$ such that

$$\limsup_{l \rightarrow \infty} \sup_{m \geq l} U_t^a (H_{t+1}^a + (\eta^{m,a,R} \cdot \Delta R_{t+1} + \eta^{m,a,S} \cdot W)) > -\infty \quad \text{almost surely on } A'.$$

Then there exist $c \in \mathbb{R}$ and $A'' \in \mathcal{F}_t$ with $A'' \subset A'$ and $\mathbb{P}[A''] > 0$ such that

$$\limsup_{l \rightarrow \infty} \sup_{m \geq l} U_t^a (H_{t+1}^a + (\eta^{m,a,R} \cdot \Delta R_{t+1} + \eta^{m,a,S} \cdot W)) \geq c \quad \text{almost surely on } A''.$$

It follows that there exists a sequence $(M_l)_{l \in \mathbb{N}}$ in $\mathbb{N}(\mathcal{F}_t)$ such that $M_{l+1} \geq M_l \geq l$ for all $l \in \mathbb{N}$ and

$$U_t^a (H_{t+1}^a + \bar{\eta}^{a,R} \cdot \Delta R_{t+1} + \bar{\eta}^{a,S} \cdot W + M_l(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + M_l(\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W) \geq c - 1 \quad \text{on } A''.$$

But since U_t^a is \mathcal{F}_t -concave, this implies that

$$U_t^a (H_{t+1}^a + \bar{\eta}^{a,R} \cdot \Delta R_{t+1} + \bar{\eta}^{a,S} \cdot W + l(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + l(\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W) \geq c - 1 \quad (\text{B.5})$$

on $A'' \cap \{l \geq M_1\}$ for all $l \in \mathbb{N}$. Choose $l_0 \in \mathbb{N}$ such that $\mathbb{P}[A'''] > 0$ for $A''' = A'' \cap \{l_0 \geq M_1\}$. For $m \in \mathbb{N}$ large enough, one has

$$\mathbb{P} [1_{A'''} \{ (H_{t+1}^a + \bar{\eta}^{a,R} \cdot \Delta R_{t+1} + \bar{\eta}^{a,S} \cdot W)^+ + m(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + m(\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W \} < 0] > 0.$$

Since U_0^a is sensitive to large losses, one obtains

$$\begin{aligned} & U_0^a (1_{A'''} \{ H_{t+1}^a + \bar{\eta}^{a,R} \cdot \Delta R_{t+1} + \bar{\eta}^{a,S} \cdot W + l(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + l(\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W \}) \\ & \leq U_0^a \left(\frac{l}{m} 1_{A'''} \{ (H_{t+1}^a + \bar{\eta}^{a,R} \cdot \Delta R_{t+1} + \bar{\eta}^{a,S} \cdot W)^+ + m(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + m(\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W \} \right) \rightarrow -\infty \end{aligned}$$

for $l \rightarrow \infty$. In particular,

$$U_0^a \left(1_{A^c} U_t^a \left(H_{t+1}^a + \bar{\eta}^{a,R} \cdot \Delta R_{t+1} + \bar{\eta}^{a,S} \cdot W + l(\eta^{a,R} - \bar{\eta}^{a,R}) \cdot \Delta R_{t+1} + l(\eta^{a,S} - \bar{\eta}^{a,S}) \cdot W \right) \right) \rightarrow -\infty \quad \text{for } l \rightarrow \infty.$$

But this contradicts (B.5). So (B.4) must be true, and the proof is complete. \square

Lemma B.2 *Under the assumptions of Lemma B.1 there exists a random vector S_t in $\partial \hat{u}_t(n) \cap L^\infty(\mathcal{F}_t)^K$.*

Proof. It follows from Lemma B.1 that $\hat{u}_t(n) \in L^0(\mathcal{F}_t)$. If we can show that there exists a constant $M \in \mathbb{R}_+$ such that

$$\hat{u}_t(n+x) \leq \hat{u}_t(n) + M \|x\|_{\mathcal{F}_t} \quad \text{for all } x \in L^0(\mathcal{F}_t)^K, \quad (\text{B.6})$$

the lemma follows from Theorem 7.10 of Cheridito et al. (2014). To prove (B.6), we choose $x \in L^0(\mathcal{F}_t)^K$ such that $\hat{u}_t(n+x) > -\infty$ (inequality (B.6) holds trivially on the event $\{\hat{u}_t(n+x) = -\infty\}$). By Lemma B.1, there exist one-step strategies $\vartheta_{t+1}^a \in C_{t+1}^a$, $a \in \mathbb{A}$, such that

$$\sum_{a \in \mathbb{A}} \vartheta_{t+1}^{a,S} = n+x \quad \text{and} \quad \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot S_{t+1} \right) = \hat{u}_t(n+x).$$

Since the sets C_{t+1}^a factorize and there exist one-step strategies $\hat{\vartheta}_{t+1}^a \in C_{t+1}^a$, $a \in \mathbb{A}$, satisfying

$$\sum_{a \in \mathbb{A}} \hat{\vartheta}_{t+1}^{a,S} = n \quad \text{and} \quad \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1} \right) = \hat{u}_t(n),$$

there exist one-step strategies $\eta_{t+1}^a \in C_{t+1}^a$ such that for all $a \in \mathbb{A}$,

$$\vartheta_{t+1}^{a,R} = \eta_{t+1}^{a,R}, \quad \text{sign}(\vartheta_{t+1}^{a,S,k} - \eta_{t+1}^{a,S,k}) = \text{sign}(x^k) \quad \text{for every } k = 1, \dots, K, \quad \text{and} \quad \sum_{a \in \mathbb{A}} \eta_{t+1}^{a,S} = n.$$

In particular, $|\vartheta_{t+1}^{a,S} - \eta_{t+1}^{a,S}| \cdot S_{t+1} \leq \sum_{k=1}^K |x^k| \|S_{t+1}^k\|_\infty$. So it follows from (M) and (T) that

$$\begin{aligned} \hat{u}_t(n) &\geq \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \eta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \eta_{t+1}^{a,S} \cdot S_{t+1} \right) \\ &\geq \sum_{a \in \mathbb{A}} U_t^a \left(H_{t+1}^a + \vartheta_{t+1}^{a,R} \cdot \Delta R_{t+1} + \vartheta_{t+1}^{a,S} \cdot S_{t+1} \right) - \sum_{k=1}^K |x^k| \|S_{t+1}^k\|_\infty \\ &\geq \hat{u}_t(n+x) - \|x\|_{\mathcal{F}_t} \left(\sum_{k=1}^K \|S_{t+1}^k\|_\infty^2 \right)^{1/2}, \end{aligned}$$

which implies (B.6). \square

C Proofs of Section 4

Proof of Proposition 4.2

Suppose there exists an equilibrium price process $(S_t)_{t=0}^T$ and equilibrium trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$, $a \in \mathbb{A}$. By Theorem 3.1, one has for all $t = 0, \dots, T-1$, $S_t \in \partial \hat{u}_t(n)$, $\sum_{a \in \mathbb{A}} U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1}) = \hat{u}_t(n)$ and $\sum_{a \in \mathbb{A}} \hat{\vartheta}_{t+1}^{a,S} = n$. By assumption, there exists an $a \in \mathbb{A}$ such for every $t \leq T-1$, U_t^a satisfies (D) and C_{t+1}^a is \mathcal{F}_t -open. It follows that for all $t \leq T-1$ and $x \in L^0(\mathcal{F}_t)^K$,

$$\begin{aligned} &\lim_{m \rightarrow \infty} \frac{\hat{u}_t(n+x/m) - \hat{u}_t(n)}{1/m} \\ &\geq \lim_{m \rightarrow \infty} \frac{U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + (\hat{\vartheta}_{t+1}^{a,S} + x/m) \cdot S_{t+1}) - U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1})}{1/m} \\ &= x \cdot y_t \end{aligned}$$

for

$$y_t^k = \mathbb{E} \left[S_{t+1}^k \tilde{\nabla} U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot S_{t+1}) \mid \mathcal{F}_t \right], \quad k = 1, \dots, K.$$

But since $\hat{u}_t : L^0(\mathcal{F}_t)^d \rightarrow \underline{L}(\mathcal{F}_t)$ is \mathcal{F}_t -concave, one must have

$$\lim_{m \rightarrow \infty} \frac{\hat{u}_t(n + x/m) - \hat{u}_t(n)}{1/m} = x \cdot y_t,$$

and it follows that $\partial \hat{u}_t(n) = \{y_t\}$. So one obtains from Theorem 3.1 that $S_t = y_t$. In particular, the process $(S_t)_{t=0}^T$ is unique.

Furthermore, $\hat{\vartheta}_{t+1}^a$ maximizes

$$U_t^a (H_{t+1}^a + \vartheta_{t+1}^R \cdot \Delta R_{t+1} + \vartheta_{t+1}^S \cdot \Delta S_{t+1})$$

over all $\vartheta_{t+1} \in C_{t+1}^a$. So one has for each $j = 1, \dots, J$,

$$\begin{aligned} & \mathbb{E} \left[\Delta R_{t+1}^j \tilde{\nabla} U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}) \mid \mathcal{F}_t \right] \\ &= \lim_{m \rightarrow \infty} \frac{U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \Delta R_{t+1}^j / m + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}) - U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1})}{1/m} \\ &= 0. \end{aligned}$$

This shows that

$$R_t^j = \mathbb{E} \left[R_{t+1}^j \tilde{\nabla} U_t^a (H_{t+1}^a + \hat{\vartheta}_{t+1}^{a,R} \cdot \Delta R_{t+1} + \hat{\vartheta}_{t+1}^{a,S} \cdot \Delta S_{t+1}) \mid \mathcal{F}_t \right],$$

and one obtains by backwards induction

$$\begin{aligned} R_t^j &= \mathbb{E} \left[R_T^j \prod_{s=t}^{T-1} \tilde{\nabla} U_s^a (H_{s+1}^a + \hat{\vartheta}_{s+1}^{a,R} \cdot \Delta R_{s+1} + \hat{\vartheta}_{s+1}^{a,S} \cdot \Delta S_{s+1}) \mid \mathcal{F}_t \right] \\ &= \mathbb{E} \left[R_T^j \prod_{s=t}^{T-1} \tilde{\nabla} U_s^a \left(U_{s+1}^a \left(H^a + \sum_{r=s+1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r \right) \right) \mid \mathcal{F}_t \right] \\ &= \mathbb{E} \left[R_T^j \prod_{s=t}^{T-1} \tilde{\nabla} U_s^a \left(U_{s+1}^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r \right) \right) \mid \mathcal{F}_t \right] \\ &= \mathbb{E} \left[R_T^j \nabla U_t^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r \right) \mid \mathcal{F}_t \right]. \end{aligned}$$

The second equality is a consequence of the definition of the process $(H_t^a)_{t=0}^T$, the third holds because U_s^a and U_{s+1}^a have the translation property (T), and the fourth one follows from Formula (4.2). Analogously, one gets

$$S_t^k = \mathbb{E} \left[S_T^k \nabla U_t^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r \right) \mid \mathcal{F}_t \right] \quad \text{for all } k = 1, \dots, K.$$

That

$$\frac{d\mathbb{Q}_t^a}{d\mathbb{P}} = \nabla U_t^a \left(H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right)$$

defines a probability measure \mathbb{Q}^a follows from the fact that U_t^a has the properties (M) and (T). If U_0^a is strictly monotone, one has

$$\nabla U_0^a \left(H^a + \sum_{r=1}^T \hat{\vartheta}_r^{a,R} \cdot \Delta R_r + \hat{\vartheta}_r^{a,S} \cdot \Delta S_r \right) > 0.$$

So $\mathbb{Q}^a := \mathbb{Q}_0^a$ is equivalent to \mathbb{P} , and one obtains (4.5). \square

Proof of Proposition 4.4

Assume there exist two optimal admissible trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$ and $(\vartheta_t^a)_{t=1}^T$ for agent a and a time s such that

$$\hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \neq \vartheta_s^{a,R} \cdot \Delta R_s + \vartheta_s^{a,S} \cdot \Delta S_s. \quad (\text{C.1})$$

Then it follows by backwards induction that the strategy $(\tilde{\vartheta}_t^a)_{t=1}^T$ given by

$$\tilde{\vartheta}_t^a = \begin{cases} \hat{\vartheta}_t^a & \text{if } t \neq s \\ \vartheta_t^a & \text{if } t = s \end{cases}$$

is again optimal. The two sides of (C.1) cannot differ by a non-zero constant, for otherwise one of the two strategies would dominate the other by the same constant. Therefore, strict concavity modulo translation implies

$$U_0^a \left(H^a + \sum_{t=1}^T \frac{\hat{\vartheta}_t^{a,R} + \tilde{\vartheta}_t^{a,R}}{2} \cdot \Delta R_t + \frac{\hat{\vartheta}_t^{a,R} + \tilde{\vartheta}_t^{a,S}}{2} \cdot \Delta S_t \right) > U_0^a \left(H^a + \sum_{t=1}^T \hat{\vartheta}_t^{a,R} \cdot \Delta R_t + \hat{\vartheta}_t^{a,S} \cdot \Delta S_t \right),$$

a contradiction to the optimality of $(\hat{\vartheta}_t^a)_{t=1}^T$. \square

D Proof of Theorem 5.1

By assumption, there exist optimal admissible trading strategies $(\hat{\vartheta}_t^a)_{t=1}^T$ for all agents $a \in \mathbb{A}$, and the optimal one-step trading gains

$$\hat{\vartheta}_t^{a,R} \cdot \Delta R_t + \hat{\vartheta}_t^{a,S} \cdot \Delta S_t, \quad t \geq 1,$$

are unique. Since $C_{t+1}^a = C_{t+1}/\gamma^a - \eta_{t+1}^a$, the strategy $\tilde{\vartheta}_{t+1}^a := \gamma^a(\hat{\vartheta}_{t+1}^a + \eta_{t+1}^a)$ is in C_{t+1} for all $a \in \mathbb{A}$ and $t \leq T-1$. Moreover, since $U_t(X) = \gamma^a U_t^a(X/\gamma^a)$, one has

$$U_t \left(\sum_{s=t+1}^T \tilde{\vartheta}_s^{a,R} \cdot \Delta R_s + \tilde{\vartheta}_s^{a,S} \cdot \Delta S_s \right) = \text{ess sup}_{\vartheta_s \in C_s} U_t \left(\sum_{s=t+1}^T \vartheta_s^R \cdot \Delta R_s + \vartheta_s^S \cdot \Delta S_s \right)$$

and

$$\tilde{\vartheta}_t^{a,R} \cdot \Delta R_t + \tilde{\vartheta}_t^{a,S} \cdot \Delta S_t = \tilde{\vartheta}_t^{b,R} \cdot \Delta R_t + \tilde{\vartheta}_t^{b,S} \cdot \Delta S_t \quad \text{for all } a, b \in \mathbb{A} \text{ and } t \geq 1.$$

It follows that the strategy

$$(\hat{\vartheta}_t^R, \hat{\vartheta}_t^S) = \sum_{a \in \mathbb{A}} \frac{\gamma}{\gamma^a} (\tilde{\vartheta}_t^{a,R}, \tilde{\vartheta}_t^{a,S}) = \gamma \left(\sum_{a \in \mathbb{A}} \hat{\vartheta}_t^{a,R} + \eta_t^{a,R}, n + \eta_{t+1}^S \right), \quad t \geq 1,$$

satisfies

$$U_t \left(\sum_{s=t+1}^T \hat{\vartheta}_s^R \cdot \Delta R_s + \hat{\vartheta}_s^S \cdot \Delta S_s \right) = \text{ess sup}_{\vartheta_s \in C_s} U_t \left(\sum_{s=t+1}^T \vartheta_s^R \cdot \Delta R_s + \vartheta_s^S \cdot \Delta S_s \right)$$

for all $t \geq 1$. This shows (5.2) and (5.3) because U_t is of the form $U_t(X) = \gamma^a U_t^a(X/\gamma^a)$.

The rest of the theorem follows from Proposition 4.2 by noting that if U_t satisfies (D) for all $t \leq T-1$, then

$$\begin{aligned} & \nabla U_t \left(\sum_{s=1}^T \hat{\vartheta}_s^R \cdot \Delta R_s + \gamma (n + \eta_t^S) \cdot \Delta S_s \right) \\ &= \nabla U_t^a \left(\sum_{s=1}^T (\hat{\vartheta}_s^{a,R} + \eta_s^a) \cdot \Delta R_s + (\hat{\vartheta}_s^{a,S} + \eta_s^{a,S}) \cdot \Delta S_s \right) \\ &= \nabla U_t^a \left(H^a + \sum_{s=1}^T \hat{\vartheta}_s^{a,R} \cdot \Delta R_s + \hat{\vartheta}_s^{a,S} \cdot \Delta S_s \right) \end{aligned}$$

for all $a \in \mathbb{A}$. \square

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