The Application of the Determinantal Method to the Finite Hill's Equation

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Important stability properties of the finite Hill's equation

$$(1) \quad y''(x) + \Lambda \, y(x) + 2 \cdot \sum_{n=1}^{b} \cos(2nx) \, G_n \, y(x) = 0 \quad (x \in \mathbb{R})$$

$(\Lambda, G_n \in \mathrm{M}(k \times k, \mathbb{C}))$ can be expressed by the characteristic exponents of (1), i.e. the complex numbers ν where a solution $y(x)$ of (1) exists with $y(x + \pi) = e^{i\pi\nu} y(x)$. According to the determinantal method the characteristic exponents can be calculated by evaluating the determinant of some infinite block-matrix $A(\nu)$ for k different values of ν (see [1],[2]). As for the finite Hill's equation $A(\nu)$ is a bandmatrix with bandwidth b, we investigate the evaluation of the determinants of bandmatrices.

First we consider one-sided infinite matrices. Let $A = (A_{ij})_{i,j=0}^{\infty}$ $(A_{ij} \in \mathrm{M}(k \times k, \mathbb{C}))$ be an infinite bandmatrix with bandwidth b, i.e. $A_{ij} = 0$ for $|i - j| > b$. The infinite determinant is defined as the limit of the finite subdeterminants $\det A_n$ with $A_n := (A_{ij})_{i,j=0}^{n}$ $(n \in \mathbb{N}_0)$. For numerical purposes it is useful to accelerate the convergence of the sequence $(\det A_n)_{n=0}^{\infty}$. Whereas this acceleration has been studied for $k = 1$ by R. Mennicken and E. Wagenführer ([3],[4],[5]), the results in the case $k > 1$ seem to be new.

Similarly to [4] we define (I_k denotes the unit matrix):

Definition: *A sequence $(\gamma_n)_{n=0}^{\infty}$ of complex numbers is called convergent to zero with order m as regards A if there exists a natural number n_0 with*

$$|\gamma_n| \leq \; const \; \cdot \left(\sum_{i,j=n-n_0}^{n} |A_{ij} - \delta_{ij} I_k|^2 \right)^{\frac{m}{2}} \quad (n \; large).$$

To describe the asymptotic behaviour of the sequence $(\det A_n)_{n=0}^{\infty}$ we write A_n for some $l \in \mathbb{N}$ in the form

$$A_n = \begin{pmatrix} & & 0 & 0 \\ A_{n-lb-1} & & R_n^{(l)} & 0 \\ 0 & L_n^{(l)} & D_n^{(l)} & s_n^{(l)} \\ 0 & \underbrace{0}_{b} & \underbrace{z_n^{(l)}}_{lb} & \underbrace{A_{n,n}}_{1} \end{pmatrix} .$$

Theorem 1: *The sequence*

$$\det A_n - \det(A_{nn} - z_n^{(l)}(D_n^{(l)})^{-1} s_n^{(l)}) \cdot \det A_{n-1}$$

converges to zero with order $2l + 2$ as regards A.

Theorem 1 can be used to accelerate the convergence of $(\det A_n)_{n=0}^{\infty}$ by splitting up infinite products:

Theorem 2: *Let $B_n \in \mathrm{M}(k \times k, \mathbb{C})$, $\det B_n \neq 0$ $(n \in \mathbb{N}_0)$, $2 \leq m \leq 2l + 2$. If*

$$|B_n - (A_{nn} - z_n^{(l)}(D_n^{(l)})^{-1} s_n^{(l)})|$$

converges to zero with order m as regards A, then we have for $\tilde{A} := (B_i^{-1} A_{ij})_{i,j=0}^{\infty}$:

a) $\det A = \left(\prod\limits_{n=0}^{\infty} \det B_n \right) \cdot \det \tilde{A}$.

b) $\det \tilde{A}_n - \det \tilde{A}_{n-1}$ converges to zero with order m as regards A.

As part a) of this theorem shows, it is necessary to know the value of the infinite product $\prod\limits_{n=0}^{\infty} \det B_n$ in order to calculate $\det A$.

In the application of Theorem 2 to the finite Hill's equation (1) we have to calculate the determinant of the two-sided infinite matrix $A(\nu) := (A_{ij}(\nu))_{i,j=-\infty}^{\infty}$ $(A_{ij}(\nu) \in \mathrm{M}(k \times k, \mathbb{C}))$ with

$$A_{ij}(\nu) := \delta_{ij} I_k + (\Lambda - (2i + \nu)^2 I_k)^{-1} G_{i-j} \qquad (i, j \in \mathbb{Z}, \; \nu \in H)$$

where we set $H := \{z \in \mathbb{C} : \det(\Lambda - (2n+z)^2 I_k) \neq 0 \text{ for all } n \in \mathbb{Z}\}$, $G_n := G_{-n}$ $(1 \leq n \leq b)$ and $G_n := 0$ $(n = 0 \text{ or } |n| > b)$.

We first consider the values $\nu = 0$ and $\nu = 1$:

Theorem 3: $\det A(0)$ *and* $\det A(1)$ *can be written as the product of two one-sided infinite determinants.*

For $\nu = 0$, say, we set $C(0) = (C_{ij}(0))_{i,j=0}^{\infty}$ and $S(0) = (S_{ij}(0))_{i,j=1}^{\infty}$ with

$$C_{ij}(0) \quad := \quad \begin{cases} \delta_{ij} I_k + \Lambda^{-1} G_{-j} & (i = 0, \, j \in \mathbb{N}_0), \\ \delta_{ij} I_k + (\Lambda - (2i)^2 I_k)^{-1}(G_{i-j} + G_{i+j}) & (i \in \mathbb{N}, \, j \in \mathbb{N}_0), \end{cases}$$

$$S_{ij}(0) \quad := \quad \delta_{ij} I_k + (\Lambda - (2i)^2 I_k)^{-1}(G_{i-j} - G_{i+j}) \quad (i, j \in \mathbb{N}).$$

Then $\det A(0) = \det S(0) \cdot \det C(0)$ holds.

According to Theorem 3 the acceleration of the convergence of $\det A(0)$ and $\det A(1)$ is reduced to that of one-sided infinite determinants as described in Theorem 2. In [1] matrices $(B_n)_{n=0}^{\infty}$ are defined for which the modified determinants $\det \tilde{A}(0)$ and $\det \tilde{A}(1)$ have convergence order $\mathrm{O}(n^{-6})$ and the infinite products can be calculated exactly.

For arbitrary $\nu \in H$ the two-sided infinite determinant $\det A(\nu)$ can be transformed to some one-sided infinite determinant (cf. [3, p. 16]). The following result holds which seems to be new even in the case $k = 1$:

Theorem 4: *For* $\mu \in \{1, \ldots, b\}$ *and* $n \in \mathbb{Z}$ *let*

$$B_{n,\mu} \quad := \quad \begin{cases} I_k, & (0 \leq |n| \leq [\frac{\mu}{2}]), \\ I_k - [(\Lambda - (2n+\nu)(2(n-\mu)+\nu)I_k)^{-1}G_\mu]^2, & (n \geq [\frac{\mu}{2}]+1), \\ I_k - [(\Lambda - (2n+\nu)(2(n+\mu)+\nu)I_k)^{-1}G_\mu]^2, & (n \leq -[\frac{\mu}{2}]-1), \end{cases}$$

$$B_n \quad := \quad \prod_{\mu=1}^{b} B_{n,\mu} \quad (n \in \mathbb{Z}).$$

Then we have for $\tilde{A}(\nu) := (B_i^{-1} A_{ij}(\nu))_{i,j=-\infty}^{\infty}$:

a) $\det \tilde{A}_n(\nu) - \det \tilde{A}_{n-1}(\nu) = \mathrm{O}(n^{-6})$.

b) $\det A(\nu) = \det \tilde{A}(\nu) \cdot \prod_{\mu=1}^{b} \det \left[f(\Lambda,\mu)^{-2} \cdot f(\Lambda + G_\mu, \mu) \cdot f(\Lambda - G_\mu, \mu) \right]$

   where

$$f(\Lambda, \mu) := \begin{cases} (\Lambda + (\mu^2 - \nu^2)I_k)^{-1} \cdot (\cos \pi \sqrt{\Lambda + \mu^2 I_k} - \cos \pi \nu I_k) & (\mu \text{ even}), \\ \cos \pi \sqrt{\Lambda + \mu^2 I_k} + \cos \pi \nu I_k & (\mu \text{ odd}). \end{cases}$$

As numerical examples show, the determinantal method to calculate the characteristic exponents of (1) (combined with the acceleration of convergence as described in Theorem 4) is considerably faster than numerical integration, especially for small values of k and b.

References

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