INFINITE DETERMINANTS CORRESPONDING TO PERIODIC ODE SYSTEMS

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Abstract. The Floquet exponents of a periodic ordinary differential equation system can be characterized as the zeros of a regularized determinant depending on a parameter ν. In this paper we investigate this regularized determinant and show that essentially it is a polynomial in $\exp(\nu)$. The convergence of the infinite determinant can be described and accelerated. So we obtain a new method to compute the Floquet exponents, generalizing the well-known determinantal approach for Hill's equation.

1. Introduction

The aim of this paper is to describe the structure of infinite determinants which correspond to periodic ordinary differential equation systems of the form

$$y'(x) = A(x) \cdot y(x) \tag{1}$$

where $A(\cdot) \in L^\infty(\mathbb{R}, \mathbb{C}^{n\times n})$ with $A(x) = A(x+1)$ almost everywhere. The long history of infinite determinants starts at the end of the 19th century with famous works of Hill [10], Poincaré [17], von Koch [11] and others. Determinants were first defined for infinite matrices but later generalized for trace operators and (in the regularized form) Hilbert–Schmidt operators in abstract Banach spaces (see [6], for instance). For a survey on a general theory of (regularized) determinants in Banach spaces and for infinite matrices, we also refer the reader to [8], [9] and the references therein.

For the special case where (1) is Hill's equation the determinantal concept which was introduced by Hill was also investigated from a numerical point of view, see [14], [15]. Classical results on Hill's equation were generalized to matrix Hill's equations in [1], [2] and [3] where also the question of convergence improvement is discussed. For general systems of the form (1), however, the corresponding infinite determinants have not been investigated in more detail up to now. General results which describe the stability of (1) using regularized determinants are known even for partial differential equations [12] and can be applied here, see Lemma 1.1 below. To obtain a method which can also be used in applications, however, one has to describe the structure of this regularized determinant. This is done in this paper where also some methods of convergence acceleration for the determinants are stated. A large part of the results of this paper is taken from the preprint [5].

Let $Y(x)$ be the fundamental solution of (1), i.e. the matrix valued solution with $Y(0) = I_n$, where I_n denotes the unit matrix. In the following we will deal with the function space $L_p(\mathbb{T}, \mathbb{C}^n)$ where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ stands for the torus and with the

Sobolev space $W_p^1(\mathbb{T},\mathbb{C}^n)$ ($1 \le p \le \infty$) of all absolutely continuous functions whose derivative belongs to $L_p(\mathbb{T},\mathbb{C}^n)$. As usual, we set $H^1(\mathbb{T},\mathbb{C}^n) := W_2^1(\mathbb{T},\mathbb{C}^n)$.

Making use of the isometric isomorphism $L_2(\mathbb{T},\mathbb{C}^n) \cong \ell_2(\mathbb{Z},\mathbb{C}^n) =: H$, we can write the operators in $L_2(\mathbb{T},\mathbb{C}^n)$ as infinite block matrices. For $Z \in L_\infty(\mathbb{T},\mathbb{C}^{n\times n})$ the operator of multiplication by Z is denoted by M_Z. We have $M_Z = (Z_{k-l})_{k,l=-\infty}^{\infty} \in \mathscr{L}(H)$ with

$$Z_k := \int_{\mathbb{T}} Z(t)\, e^{-2\pi i k t}\, dt \in \mathbb{C}^{n\times n} \tag{2}$$

(see [7], p. 566). We will denote by $\mathscr{S}_p(H)$ the Neumann–Schatten class of order p in $\mathscr{L}(H)$. For a Hilbert–Schmidt operator $B \in \mathscr{S}_2(H)$ the regularized determinant $\Delta_2(1-B)$ is defined as the infinite product $\prod_j (1-\lambda_j(B))\exp(\lambda_j(B))$, cf. [6], p. 166. Due to classical Floquet theory, equation (1) is asymptotically stable if and only if all Floquet exponents have negative real part. Here ν is called a Floquet exponent if $\det(Y(1) - \exp(\nu) I_n) = 0$. For the Floquet exponents we have the following known description (see also [12], p. 110).

Lemma 1.1. *For any $\nu \in \mathbb{C}$ the following statements are equivalent:*
(i) *ν is a Floquet exponent of* (1).
(ii) *$-\nu$ is an eigenvalue of the unbounded operator L in $L^2(\mathbb{T},\mathbb{C}^n)$ with domain $H_1(\mathbb{T},\mathbb{C}^n)$ and $Lf := f' - Af$.*
(iii) *The regularized determinant $\Delta_2(1 - B_L(\nu))$ is equal to zero, where $B_L(\nu) \in \mathscr{S}_2(H)$ is defined by*

$$B_L(\nu) := 1 - (L+\nu)F$$

with

$$F := \operatorname{diag}((2\pi i l + \delta_{0l})^{-1} I_n)_{l=-\infty}^{\infty} \in \mathscr{S}_2(H)\,,$$

where δ_{kl} denotes the Kronecker symbol.

In (iii) the operator F acts as a normalization. (Note that the diagonal elements of the infinite matrix L grow to infinity while the diagonal elements of $B_L(\nu)$ tend to I_n.) We will now study the properties of the determinant which appears in (iii).

2. The structure of the regularized determinant

First we describe the behaviour of $\Delta_2(1 - B_L(\nu))$ under similarity transforms.

Lemma 2.1. *Let $Z \in W_\infty^1(\mathbb{T},\mathbb{C}^{n\times n})$ with $\det Z(x) \neq 0$ for all $x \in \mathbb{T}$. Then $\Delta_2(1-B_L(\nu))$ and $\Delta_2(1 - B_{M_Z^{-1}LM_Z}(\nu))$ are equal up to a constant non-vanishing factor which does not depend on ν.*

The proof of this fact is based on the identity

$$1 - B_{M_Z^{-1}LM_Z}(\nu) = M_Z^{-1}(1 - B_L(\nu))F^{-1}M_Z F$$

and on the product formula for regularized determinants ([6], p. 169). We obtain

$$\begin{aligned}\Delta_2(1 - B_{M_Z^{-1}LM_Z}(\nu)) &= \Delta_2(1 - B_L(\nu)) \cdot \Delta_2(F^{-1}M_Z F M_Z^{-1}) \\ &\quad \cdot \exp(-\operatorname{tr}[(1-LF)(1 - F^{-1}M_Z F M_Z^{-1})]) \\ &\quad \cdot \exp(-\operatorname{tr}[\nu(F - M_Z F M_Z^{-1})])\,.\end{aligned} \tag{3}$$

Here tr stands for the trace defined for all trace class operators in $\mathscr{L}(H)$. The only ν-dependent term on the right-hand side of (3) except $\Delta_2(1 - B_L(\nu))$ is $\exp(-\mathrm{tr}\,[\nu(F - M_Z F M_Z^{-1})])$. So for the proof of the lemma we have to show that the trace vanishes. Surprisingly this is not easy (note that F is no trace class operator) and uses estimates of the trace norm of infinite matrices which can be found in [16], p. 239. To apply these estimates, we use the fact that $Z' \in L_\infty$. Moreover, it is easy to see that there are examples $F \in \mathscr{S}_2(H)$ and $B \in \mathscr{L}(H)$ where $\mathrm{tr}\,(F - BFB^{-1})$ does not exist. See [5] for details.

To investigate the determinant appearing in Lemma 1.1 (iii) it is convenient to use the connection of regularized determinants and the determinants of infinite matrices. For an infinite block matrix $M = (M_{kl})_{k,l=-\infty}^{\infty}$ with $M_{kl} \in \mathbb{C}^{n\times n}$ and for $N \in \mathbb{N}_0$ we set

$$\mathrm{Det}_N(M) := \det(M_{kl})_{k,l=-N}^{N} \tag{4}$$

and $\mathrm{Det}\,M := \lim_{N\to\infty} \mathrm{Det}_N(M)$, provided the limit exists. In the case considered here, it is easily seen from the results in [6], p. 169, that the infinite determinant $\mathrm{Det}(1 - B_L(\nu))$ exists, and that we have

$$\mathrm{Det}(1 - B_L(\nu)) = \exp(-n(1-\nu) - \mathrm{tr}\,A_0) \cdot \Delta_2(1 - B_L(\nu))\,. \tag{5}$$

The following lemma is a first example of a modification of the infinite matrix $1 - B_L(\nu)$ which will enable us in Section 3 to improve the convergence of the infinite determinant. Note that the modified matrix appearing in this lemma is normalized to unity on the diagonal. As usual, the trigonometric functions of matrices appearing in this lemma and later are defined using the infinite series of these functions. The hyperbolic trigonometric functions will be denoted by sinh and cosh. For abbreviation, we will write z instead of zI_n.

Lemma 2.2. *For $\nu \in \Lambda := \{z \in \mathbb{C} : \det\sinh\frac{z-A_0}{2} \neq 0\}$ we set*

$$\overline{B}_L(\nu) := \Big((1-\delta_{kl})A_{k-l}(2\pi i l + \nu - A_0)^{-1}\Big)_{k,l=-\infty}^{\infty}.$$

Then $\mathrm{Det}(1 - \overline{B}_L(\nu))$ exists for $\nu \in \Lambda$, and we obtain
a) $\mathrm{Det}(1 - B_L(\nu)) = \mathrm{Det}(1 - \overline{B}_L(\nu)) \cdot \det(2\sinh\frac{\nu - A_0}{2})$ *for $\nu \in \Lambda$,*
b) $\mathrm{Det}(1 - \overline{B}_L(\nu)) \to 1$ *for $|\mathrm{Re}\,\nu| \to \infty$.*

Proof. Comparing the definitions of $B_L(\nu)$ and $\overline{B}_L(\nu)$ we see for $k, l \in \mathbb{Z}$ and $\nu \in \Lambda$

$$(1 - B_L(\nu))_{kl} = (1 - \overline{B}_L(\nu))_{kl} \cdot \frac{2\pi i l + \nu - A_0}{2\pi i l + \delta_{0,l}}\,. \tag{6}$$

Therefore the finite section determinants fulfill

$$\begin{aligned}\mathrm{Det}_N(1 - B_L(\nu)) &= \mathrm{Det}_N(1 - \overline{B}_L(\nu)) \cdot \prod_{l=-N}^{N} \det\left(\frac{2\pi i l + \nu - A_0}{2\pi i l + \delta_{0,l}}\right) \\ &= \mathrm{Det}_N(1 - \overline{B}_L(\nu)) \cdot \det\left[(\nu - A_0)\prod_{l=1}^{N}\left(1 + \left(\frac{\nu - A_0}{2\pi l}\right)^2\right)\right].\end{aligned}$$

For $N \to \infty$ the last determinant converges to $\det(2\sinh\frac{\nu - A_0}{2}) \neq 0$ as we can see from the Weierstraß product formula for the sinh-function applied to matrices. Thus $\mathrm{Det}(1 - \overline{B}_L(\nu))$ exists for $\nu \in \Lambda$ and equality a) holds. To obtain b), we use the estimation

$$\begin{aligned}\|\overline{B}_L(\nu)\|^2_{\mathscr{S}_2} &= \sum_{k,l} |(\overline{B}_L(\nu))_{kl}|^2 \\ &\leq \sum_{k\neq l} |A_{k-l}|^2 \cdot |(2\pi i l + \nu - A_0)^{-1}|^2 \\ &\leq \|A\|^2_{L_2} \cdot \sum_l |(2\pi i l + \nu - A_0)^{-1}|^2\end{aligned}$$

which shows $\overline{B}_L(\nu) \in \mathscr{S}_2(H)$ and $\|\overline{B}_L(\nu)\|_{\mathscr{S}_2} \to 0$ for $|\mathrm{Re}\,\nu| \to \infty$. From the continuity of the regularized determinant we see

$$\mathrm{Det}(1 - \overline{B}_L(\nu)) = \Delta_2(1 - \overline{B}_L(\nu)) \to 1 \text{ for } |\mathrm{Re}\,\nu| \to \infty\,.$$

□

Now we can state the main result of this paper which gives a very simple connection between the infinite determinant $\mathrm{Det}(1 - B_L(\nu))$ and the $n \times n$-determinant $\det(Y(1) - \exp(\nu)I_n)$ which appears in the definition of the Floquet exponents.

Theorem 2.3. *The determinant* $\mathrm{Det}(1 - B_L(\nu))$ *is (up to normalization) a polynomial in* $\exp(\nu)$. *More precisely, the following equality holds for every* $\nu \in \mathbb{C}$:

$$\mathrm{Det}(1 - B_L(\nu)) = (-1)^n \exp\Big(-\frac{1}{2}(n\nu + \mathrm{tr}\,A_0)\Big) \cdot \det(Y(1) - \exp(\nu)I_n)\,. \tag{7}$$

Proof. Due to the theorem of Floquet–Lyapunov there exists a $Z \in W^1_\infty(\mathbb{T}, \mathbb{C}^{n\times n})$ with $\det Z(x) \neq 0$ ($x \in \mathbb{T}$) which transforms (1) to a constant system, i.e. we have

$$(M_Z^{-1} L M_Z)f = f' - Kf \quad (f \in H_1(\mathbb{T}, \mathbb{C}^n))$$

where $K \in \mathbb{C}^{n\times n}$ is a constant matrix with $\exp K = Y(1)$. From Lemma 2.1 we obtain the existence of some constant $c \neq 0$, not depending on ν, with

$$\begin{aligned}\mathrm{Det}(1 - B_L(\nu)) &= \exp(-n(1-\nu) - \mathrm{tr}\,A_0) \cdot \Delta_2(1 - B_L(\nu)) \\ &= c \cdot \exp(-n(1-\nu) - \mathrm{tr}\,A_0) \cdot \Delta_2(1 - B_{M_Z^{-1}LM_Z}(\nu)) \\ &= c \cdot \exp(\mathrm{tr}\,K - \mathrm{tr}\,A_0) \cdot \mathrm{Det}(1 - B_{M_Z^{-1}LM_Z}(\nu))\,.\end{aligned}$$

We calculate the last determinant explicitly ($B_{M_Z^{-1}LM_Z}(\nu)$ is block diagonal). Similarly to the proof of Lemma 2.2 (or using this lemma) we get

$$\begin{aligned}\mathrm{Det}(1 - B_{M_Z^{-1}LM_Z}(\nu)) &= \det\Big(2\sinh\frac{\nu - K}{2}\Big) \\ &= \det\Big[\exp\Big(-\frac{\nu + K}{2}\Big) \cdot (\exp(\nu)I_n - \exp K)\Big] \\ &= (-1)^n \exp\Big(-\frac{1}{2}(n\nu + \mathrm{tr}\,K)\Big) \cdot \det(Y(1) - \exp(\nu)I_n)\,,\end{aligned}$$

and therefore

$$
\begin{aligned}
&\exp\Big(\frac{1}{2}(n\nu + \operatorname{tr} A_0)\Big) \cdot \operatorname{Det}(1 - B_L(\nu)) \\
&\quad = (-1)^n\, c \exp\Big(\frac{1}{2}(\operatorname{tr} K - \operatorname{tr} A_0)\Big) \cdot \det(Y(1) - \exp(\nu) I_n) \\
&\quad = (-1)^n\, \tilde{c} \cdot \det(Y(1) - \exp(\nu) I_n)\,.
\end{aligned} \tag{8}
$$

Here $\tilde{c} := c \exp(\frac{1}{2}(\operatorname{tr} K - \operatorname{tr} A_0))$ is independent of ν.

It remains to compute $\tilde{c}$. The left-hand side of (8) can be written as (cf. Lemma 2.2 a))

$$
\begin{aligned}
\exp\Big(\frac{1}{2}(n\nu + \operatorname{tr} A_0)\Big) \cdot \det\Big(2 \sinh \frac{\nu - A_0}{2}\Big) \cdot \operatorname{Det}(1 - \overline{B}_L(\nu)) \\
= (-1)^n \det(\exp A_0 - \exp(\nu) I_n) \cdot \operatorname{Det}(1 - \overline{B}_L(\nu))\,.
\end{aligned}
$$

Due to Lemma 2.2 b) this expression tends to $(-1)^n \det(\exp A_0)$ for $\operatorname{Re} \nu \to -\infty$. Using the formula of Liouville we get

$$
\det(\exp A_0) = \exp\Big(\operatorname{tr} \int_0^1 A(t)\, dt\Big) = \det Y(1)\,.
$$

Comparing the limits of both sides of (8) for $\operatorname{Re}\nu \to -\infty$ the constant $\tilde{c}$ is seen to be equal to 1 which finishes the proof of the theorem. □

Remark 2.4. a) We want to point out that from this theorem the equivalence of (i) and (iii) of Lemma 1.1 immediately follows. (This equivalence was not used in the proofs.)

b) Equation (7) is the basis for numerical methods using infinite determinants. We only have to compute the values of $\operatorname{Det}(1 - B_L(\nu))$ for $n - 1$ different values of ν to determine the coefficients of the polynomial appearing on the right-hand side of (7). Then the zeros of this polynomial can be found by standard methods.

c) Instead of the proof which was used here, one could also think of different approaches to prove Theorem 2.3. First, one could think of using the theory of entire functions of finite order to show that both sides of (7) are equal at least up to an entire function without zeros. (Due to Lemma 1.1, both sides have the same zeros.) But for this approach one would also have to investigate the multiplicity of the zeros which seems to be not easy in the case of the infinite determinant.

Another approach to prove this theorem would be to use more results about the determinants of infinite matrices to obtain that the left-hand side of (7) is a polynomial in $\exp(\nu)$. For this one has to show first that the left-hand side with ν replaced by $i\nu$ is a periodic function of ν. Then one can use growth estimates to see that the Fourier series of this function is only a finite sum. If we know that both sides of (7) are polynomials in $\exp(\nu)$ we can prove the theorem using again Lemma 1.1, at least if the zeros of both polynomials are simple. This proof which was already used by Hill works in the case of (matrix) Hill's equation, see, e.g. [2], [10], [13]. However, in the case of (matrix) Hill's equation the convergence of the corresponding infinite determinant is much better than in the case considered here. This makes it possible to use classical results on infinite determinants due to Poincaré [17] and von Koch [11] and their generalizations described in [15]. In general, the determinants corresponding to (1) are not absolutely convergent in the sense of [11]. This makes it difficult to apply classical theories on infinite matrices, and thus the use of regularized determinants seems to be more appropriate here.

Remark 2.5. Even if the function $A(x)$ is constant (i.e. in the case of an ODE system with constant coefficients) the result of Theorem 2.3 is not trivial. In this case we receive the infinite product formula for trigonometric functions, applied to matrices. In the special case where $A(x) = A_0 + \cos 2\pi x A_1$ with

$$A_0 = \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}$$

(Mathieu's equation), there are well-known recurrence formulas for the finite section determinants $\mathrm{Det}_N(1 - B_L(\nu))$. Even in this simple case where the infinite matrix $1 - B_L(\nu)$ is tridiagonal is seems to be impossible to prove equation (7) directly.

3. Evaluation of the infinite determinant

Due to Theorem 2.3, the calculation of the Floquet exponents is reduced to the evaluation of $n-1$ infinite determinants $\mathrm{Det}(1 - B_L(\nu))$. In this section we want to investigate and improve the convergence of this determinant. We fix ν and set $\delta_N := \mathrm{Det}_N(1 - B_L(\nu))$.

Remark 3.1. We have $\delta_N - \delta_{N-1} = O(N^{-2})$ $(N \to \infty)$ and in general the exponent cannot be replaced by a smaller number (see [5]). A similar result is valid for the infinite matrix $1 - \overline{B}_L(\nu)$ defined in Lemma 2.2. Therefore the convergence has to be improved in order to obtain a method which is useful from a numerical point of view. There are several possibilities to improve the order of convergence. As the proofs of such convergence accelerations are of quite technical nature, we restrict ourselves to state two examples.

Theorem 3.2 (see [5]). *Assume $A_k = 0$ for $|k| > b$. Let $f_0(\mathrm{tr}\,(\nu - A_0)^2) \neq 0$ and $f_p(2\,\mathrm{tr}\,(A_p A_{-p})) \neq 0$ for $p = 1, \dots, b$ where the auxiliary functions f_p are defined by*

$$f_p(z) := \begin{cases} \left(\frac{2}{\sqrt{z}}\right) \cdot \sinh\left(\frac{\sqrt{z}}{2}\right) & \text{if } p \text{ even,} \\ \cosh\left(\frac{\sqrt{z}}{2}\right) & \text{if } p \text{ odd.} \end{cases}$$

Let the modified sequence $(\tilde{\delta}_N)_N$ be given by

$$\tilde{\delta}_N := \delta_N \cdot \prod_{m=1}^{N} \left[\left(1 + \frac{\mathrm{tr}\,(\nu - A_0)^2}{(2\pi m)^2}\right) \prod_{\substack{p=1 \\ p<2m}}^{b} \left(1 + \frac{2\,\mathrm{tr}\,(A_p A_{-p})}{\pi^2 (2m-p)^2}\right) \right]^{-1}.$$

Then $\tilde{\delta}_N - \tilde{\delta}_{N-1} = O(N^{-4})$ and

$$\mathrm{Det}(1 - B_L(\nu)) = \left(\lim_{N \to \infty} \tilde{\delta}_N\right) \cdot f_0(\mathrm{tr}\,(\nu - A_0)^2) \prod_{p=1}^{b} f_p(2\,\mathrm{tr}\,(A_p A_{-p})).$$

Theorem 3.3 (see [3]). *Consider a matrix Hill equation of the form*

$$y''(x) + A(x)\, y(x) = 0\,, \tag{9}$$

where $A(x) = \sum_{k=-b}^{b} \exp(2\pi i k x) A_k,\ A_k = A_{-k} \in \mathbb{R}^{n\times n}$.

a) *Define the modified sequence* $(\tilde{\delta}_N)_N$ *by*

$$\tilde{\delta}_N := \delta_N \cdot \Big(\prod_{m=1}^{\infty} \gamma_m \Big)^{-1}$$

with

$$\gamma_m := \Big(\prod_{p=0}^{b} \gamma_{m,p}(\nu) \cdot \gamma_{m,p}(-\nu) \Big) \cdot \Big(\prod_{p,q=1}^{b} \gamma_{m,p,q} \Big) \tag{10}$$

where we define for $m \in \mathbb{N}$ *and* $p, q = 1, \dots, b$*:*

$$\gamma_{m,0}(\nu) := \det \frac{(2\pi m + \nu)^2 - A_0}{(2\pi m)^2}\,,$$

$$\gamma_{m,p}(\nu) := \begin{cases} \det \Big(I_n - [((2\pi m+\nu)(2\pi(m-p)+\nu) - A_0)^{-1} A_p]^2 \Big) & \text{if } m > \frac{p}{2}\,, \\ 1 & \text{else,} \end{cases}$$

$$\gamma_{m,p,q} := \begin{cases} 1 - \dfrac{2\mathrm{tr}\,(A_p A_{q-p} A_q)}{(2\pi(m-(p+q)/3))^6} & \text{if } p \neq q \text{ and } m > \frac{p+q}{3}\,, \\ 1 & \text{else.} \end{cases}$$

Then we have $\tilde{\delta}_N - \tilde{\delta}_{N-1} = O(N^{-8})$

b) *The infinite determinant* $\mathrm{Det}(1 - B_L(\nu))$ *is given by*

$$\mathrm{Det}(1 - B_L(\nu)) = \Big(\lim_{N\to\infty} \tilde{\delta}_N \Big) \cdot \prod_{m=1}^{\infty} \gamma_m\,, \tag{11}$$

and the infinite product on the right-hand side of (11) *can be expressed explicitly as a combination of trigonometric functions of the matrices* $A_0, \dots, A_b$.

For the explicit value of the infinite product in part b) we refer the reader to [3].

4. Numerical examples and final remarks

Starting from the abstract theory of regularized determinants of Hilbert–Schmidt operators, we described in detail the structure of this determinants in the case considered here and reduced the problem of stability of (1) to the evaluation of $n-1$ infinite matrix determinants. The convergence of these matrix determinants can be improved using the methods described in Section 3. So we obtain a new algorithm to compute the Floquet exponents. The aim of this section is to show that this new method is useful also from a numerical point of view and to discuss some further developments.

The standard method to compute the Floquet exponents is to solve the initial value problem (1) with initial value $Y(0) = I_n$ directly and to use the definition of the Floquet exponents as the zeros of $\det(Y(1) - \exp(\nu)I_n)$. We compare the CPU times needed by the two methods to achieve a given accuracy for the Floquet exponents. In the case of the numerical integration of (1) we use the minimum of the times needed by the Runge–Kutta–Fehlberg method and by a variable-order variable-step Adams method. All computations were done on a SUN–Sparc workstation in Fortran 77, using the NAG library. The following examples are model problems which show a typical behaviour. See also [3] and [18] for more details on the implementation.

First we investigate the effect of convergence acceleration as proposed in Theorem 3.2. The following table shows the relative error of the estimates for the Floquet exponents using the finite section determinants δ_N (left) and their modified values $\tilde{\delta}_N$ (right). We can see the improvement due to convergence acceleration. Moreover, the computing time for the modification is neglectible compared with the time needed to the evaluation of the finite section determinants. In Table 1 the computing times are given in CPU seconds.

	Using (δ_N)		Using $(\tilde{\delta}_N)$	
N	Error	Time	Error	Time
5	$1.6 \cdot 10^{-1}$	0.03	$2.2 \cdot 10^{-3}$	0.03
10	$8.7 \cdot 10^{-2}$	0.05	$2.9 \cdot 10^{-4}$	0.05
20	$4.5 \cdot 10^{-2}$	0.10	$3.7 \cdot 10^{-5}$	0.11
40	$2.3 \cdot 10^{-2}$	0.27	$4.6 \cdot 10^{-6}$	0.27

Table 1. Convergence acceleration using Theorem 3.2

Now we want to compare the computing times of the determinantal method and of the standard approach. We consider a model problem of the form (9) with $n = 2$ and different values of b. In Table 2 the CPU time needed for numerical integration is set to 1.

Error	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$
10^{-6}	0.09	0.20	0.27	0.44	0.58	0.77
10^{-7}	0.09	0.18	0.25	0.32	0.47	0.61
10^{-8}	0.08	0.18	0.27	0.33	0.44	0.52
10^{-9}	0.08	0.16	0.23	0.34	0.42	0.58
10^{-10}	0.08	0.16	0.25	0.33	0.41	0.51

Table 2. CPU-time of the determinantal method using Theorem 3.3 compared to the time of numerical integration which is set to 1

We can see from this example that the determinantal approach is significantly faster (at the same level of accuracy) than the numerical solution of the initial value problem, in particular for small values of b. For further examples and discussions we refer to [3], [18], [19].

Finally, we want to remark that it is also possible to use the eigenvalues of the finite sections of the infinite matrix L (cf. Lemma 1.1 (ii)) as estimates for the Floquet exponents. In [4] a combination of determinantal and eigenvalue approach is investigated which is up to 10–50 times faster than the standard method. In all these examples one can see that the beautiful abstract concept of infinite determinants (finally) leads to a very fast and precise numerical method. It seems that some questions could be of interest from a numerical point of view, for instance, the behaviour of this method in the case of stiff equations.

References

1. R. Denk, *On the Floquet exponents of Hill's equation systems*, Math. Nachr. **172** (1995), 87–94.
2. R. Denk, *Hill's equation systems and infinite determinants*, Math. Nachr. **175** (1995), 47–60.
3. R. Denk, *Convergence improvement for the infinite determinants of Hill systems*, Z. angew. Math. Mech. **75** (1995), 463–470.
4. R. Denk, *The determinantal method for Hill systems*, Z. angew. Math. Mech. **76** (1996), 509–510.
5. R. Denk, *On Hilbert–Schmidt operators and determinants corresponding to periodic ODE systems*, Oper. Theory Adv. Appl. (to appear).
6. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Transl. Math. Monogr. **18**, Amer. Math. Soc., Providence, R. I., 1969.
7. I. Gohberg, S. Goldberg, and M. A. Kasshoek, *Classes of linear operators.*, Vol. II, Birkhäuser Verlag, Basel etc., 1993.
8. I. Gohberg, S. Goldberg, and N. Krupnik, *Traces and determinants of linear operators*, Integral Equations Operator Theory **26** (1996), 136–187.
9. I. Gohberg, S. Goldberg, and N. Krupnik, *Hilbert–Carleman and regularized determinants for linear operators*, Integral Equations Operator Theory **27** (1997), 10–47.
10. G. W. Hill, *On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon*, Acta Math. **8** (1886), 1–36.
11. H. von Koch, *Sur les déterminants infinis et les équations différentielles lineáres*, Acta Math. **16** (1892), 217–295.
12. P. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser Verlag, Basel, 1993.
13. J. Meixner and F. W. Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen*, Springer-Verlag, Berlin etc., 1954.
14. R. Mennicken, *On the convergence of infinite Hill-type determinants*, Arch. Rational Mech. Anal. **30** (1968), 12–37.
15. R. Mennicken and E. Wagenführer, *Über die Konvergenz verallgemeinerter Hillscher Determinanten*, Math. Nachr. **72** (1976), 21–49.
16. A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, Cambridge, 1987.
17. H. Poincaré, *Sur les déterminants d'ordre infini*, Bull. Soc. Math. France **14** (1886), 77–90.
18. E. Wagenführer, *Ein Verfahren höherer Konvergenzordnung zur Berechnung des charakteristischen Exponenten der Mathieuschen Differentialgleichung*, Numer. Math. **27** (1976), 53–65.
19. E. Wagenführer, *Die Determinantenmethode zur Berechnung des charakteristischen Exponenten der endlichen Hillschen Differentialgleichung*, Numer. Math. **35** (1980), 405–420.

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