ON THE DIRICHLET PROBLEM FOR A CLASS OF ELLIPTIC OPERATOR PENCILS

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1. Introduction

In this paper we consider a partial differential operator depending polynomially on the parameter λ and acting on a manifold M with boundary Γ. The operator is assumed to have the form

$$A(x,D,\lambda) = A_{2m}(x,D) + \lambda A_{2m-1}(x,D) + \dots + \lambda^{2m-2\mu} A_{2\mu}(x,D)\,. \tag{1-1}$$

Here m and μ are integer numbers with $m > \mu \geq 0$, the number λ is a complex parameter and $A_{2\mu}, \dots, A_{2m}$ are partial differential operators of the form

$$A_j(x,D) = \sum_{|\alpha|\leq j} a_{\alpha j}(x) D^\alpha \quad (j = 2\mu,\, 2\mu+1, \dots, 2m) \tag{1-2}$$

with scalar coefficients $a_{\alpha j}(x)$. Here we used the standard multi-index notation. We assume that the manifold, its boundary and the coefficients of A are infinitely smooth.

In typical examples, a pencil of the form (1-1) will not satisfy the condition of ellipticity with parameter due to Agmon [1] and Agranovich–Vishik [3]. We will therefore consider a general notion of ellipticity with parameter which is connected with the Newton polygon and which will turn out to be suitable for pencils of the form (1-1). We define the principal symbol of A_j as

$$A_j^{(0)}(x,\xi) := \sum_{|\alpha|=j} a_{\alpha j}(x)\xi^\alpha \quad (j = 2\mu, \dots, 2m) \tag{1-3}$$

and set

$$A^{(0)}(x,\xi,\lambda) := A_{2m}^{(0)}(x,\xi) + \lambda A_{2m-1}^{(0)}(x,\xi) + \ldots + \lambda^{2m-2\mu} A_{2\mu}^{(0)}(x,\xi)\,. \tag{1-4}$$

An operator pencil of the form (1-1) and acting in $\mathbb{R}^n$ is called N-elliptic with parameter in $[0,\infty)$ if the estimate

$$|A^{(0)}(x,\xi,\lambda)| \geq C\,|\xi|^{2\mu}\,(\lambda+|\xi|)^{2m-2\mu} \quad (x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n,\ \lambda\in[0,\infty)) \tag{1-5}$$

holds with a constant C independent of x, ξ and λ (cf. [4], [7]). For a pencil acting on a manifold the inequality (1-5) has to be fulfilled locally in each coordinate system, cf. Section 4 below.

In the case $\mu = 0$ this is the definition of ellipticity with parameter introduced by Agmon–Agranovich–Vishik, so from now on we will consider the case $\mu > 0$. The main difficulty in the investigation of N-elliptic pencils of the form (1-1) lies in the fact that the principal symbol vanishes for $\xi = 0$ even for positive λ.

Operator pencils depending polynomially on the parameter λ and satisfying a condition of N-ellipticity with parameter arise, for instance, as the determinant of a Douglis–Nirenberg system (mixed order system) of the form $A(x, D) - \lambda I$ which are elliptic with parameter in the sense of [4] (cf. also [8]). There is also a close connection between pencils which are N-elliptic with parameter and parabolic problems. In particular, the composition of two operators which are $2b_j$-parabolic in the sense of Petrovskii ($j = 1, 2$) is in general no longer parabolic in this sense but belongs to the class of N-parabolic operators. See [7] as a reference for N-parabolicity on manifolds without boundary.

The main tool in the theory of pencils of the form (1-1) is given by the Newton polygon approach, cf. [4] and [7]. The idea of this method is, roughly speaking, to assign to the parameter λ various weights and to obtain for every weight a different principal part of $A(x, D, \lambda)$. On manifolds without boundary the operator $A(x, D, \lambda)$ can be considered as a small perturbation of the corresponding principal part in some subregion of all (ξ, λ) which depends on the weight of λ. These considerations lead to existence and uniqueness results as well as to a priori estimates; see [4] for Douglis–Nirenberg systems and [7] for parabolic problems.

On manifolds with boundary, however, the situation is more complicated. Here the pencil $A(x, D, \lambda)$ has to be considered as a singular perturbation of the corresponding principal part, and we have to deal with additional boundary conditions. Replacing λ by ε^{-1} in (1-1) we see the close connection to the theory of singular perturbations (cf. also [6], [9], [10]).

In the following, we will derive an a priori estimate for the Dirichlet problem corresponding to the pencil (1-1). A more detailed exposition can be found in [5]. First we will introduce Sobolev spaces which correspond to Newton polygons. The a priori estimate will take place in these spaces, and one fundamental result is the description of the trace spaces of these Sobolev spaces which can be found in Section 2. The proof of the a priori estimate (Theorem 4.4) will use an estimate on the solutions of an ordinary differential equation on the half-line which can be found in Theorem 3.2.

2. Sobolev spaces corresponding to the Newton polygon

Let us consider a polynomial in the variables $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ of the form

$$P(\xi,\lambda) = \sum_{\alpha,k} a_{\alpha k}\xi^\alpha \lambda^k \tag{2-1}$$

with complex coefficients $a_{\alpha k}$. Denote by $\nu(P)$ the set of all integer points (i,k) such that an α exists with $|\alpha| = i$ and $a_{\alpha k} \neq 0$. Then the Newton polygon $N(P)$ is defined as the convex hull of all points in $\nu(P)$, their projections on the coordinate axes and the origin. To define and investigate Sobolev spaces corresponding to the Newton polygon $N(P)$, we first introduce some simple geometric notions. For a detailed discussion of the Newton polygon, we refer the reader to [7], Chapters 1 and 2.

Let $\Gamma_1, \dots, \Gamma_S$ be the sides of the Newton polygon not lying on the coordinate axes and indexed in the clockwise direction. Suppose that

$$(0,0),\ (a_1,b_1), \dots, (a_{S+1}, b_{S+1})\,, \quad a_1 = 0, \quad b_{S+1} = 0\,,$$

are the vertices of the polygon $N(P)$. Then the side Γ_s is given by

$$\Gamma_s = \{(a,b) \in \mathbb{R}^2 : 1\cdot a + r_s \cdot b = d_s\} \quad (s = 1,\dots,S) \tag{2-2}$$

where $r_s = (a_{s+1} - a_s)/(b_s - b_{s+1})$. The vector $(1, r_s)$ is an exterior normal to the side Γ_s, where we admit $r_1 = \infty$ if Γ_1 is horizontal. Further we have $r_S = 0$ in the case that Γ_S is vertical. In what follows we will suppose that Γ_S is not vertical. Since $N(P)$ is convex, we have

$$\infty \geq r_1 > \ldots > r_S > 0\,.$$

To the Newton polygon $N(P)$ we assign its weight function

$$\Xi_P(\xi,\lambda) := \sum_{(i,k)\in N(P)} |\xi|^i\,|\lambda|^k\,, \tag{2-3}$$

where the summation on the right-hand side is extended over all integer points of $N(P)$. It is easily seen that we have

$$\Xi_P(\xi,\lambda) \approx \prod_{s=1}^{S} \left(|\xi|^2 + |\lambda|^{\frac{2}{r_s}}\right)^{m_s}\,, \tag{2-4}$$

where $m_1, \dots, m_S$ are nonnegative numbers depending on the geometry of the Newton polygon. We restrict ourselves to the case that all m_j are integers.

The Sobolev space $H^{\Xi_P}(\mathbb{R}^n)$ is defined as the space of all distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\Xi_P(\cdot,\lambda)Fu(\cdot) \in L_2(\mathbb{R}^n)$. Here $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the Schwartz space and its dual, respectively, and Fu stands for the Fourier transform of u. The space $H^{\Xi_P}(\mathbb{R}^n)$ is endowed with the norm

$$\|u\|_{\Xi_P,\mathbb{R}^n} := \Big(\int_{\mathbb{R}^n} \Xi_P^2(\xi,\lambda)|Fu(\xi)|^2\,d\xi\Big)^{1/2}. \tag{2-5}$$

For the theory of boundary value problems it is essential to know the trace spaces $\{u(\cdot,0) | u \in H^{\Xi_P}(\mathbb{R}^n)\}$ of all traces $u(\cdot,0)$ defined in $\mathbb{R}^{n-1} = \{(x',0) | x' \in \mathbb{R}^{n-1}\}$. We denote by $\Xi_P^{(-l)}(\xi,\lambda)$ the function corresponding to the Newton polygon which is constructed from $N(P)$ by a shift of length l to the left parallel to the abscissa. We preserve the notation $H^{\Xi_P^{(-l)}}(\mathbb{R}^{n-1})$ for the spaces in $\mathbb{R}^{n-1}$ corresponding to the weight functions $\Xi_P^{(-l)}(\xi',\lambda) := \Xi_P^{(-l)}(\xi',0,\lambda)$ $(\xi' \in \mathbb{R}^{n-1})$.

Theorem 2.1. *For every $\lambda_0 > 0$ there exists a constant $C > 0$, independent of u and λ, such that*

$$\|D_n^l u(x',0)\|_{\Xi_P^{(-l-\frac{1}{2})},\mathbb{R}^{n-1}} \le C\|u\|_{\Xi_P,\mathbb{R}^n} \quad (l = 0,\dots,2m_1+\dots+2m_S-1) \tag{2-6}$$

holds for $u \in H^{\Xi_P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$ with $|\lambda| \ge \lambda_0$.

The proof of this theorem uses the equivalence (2-4), general results on trace spaces as contained in [11] and complicated but elementary estimates on one-dimensional integrals. See [5], Section 2, for details.

For general weight functions $\mu(\xi)$ instead of $\Xi_P(\xi,\lambda)$, the definition of the space $H^\mu(\mathbb{R}^n_+)$ can be found, e.g., in [11]. The norm in these spaces is given as a quotient norm. For $H^{\Xi_P}(\mathbb{R}^n_+)$ it can be seen that an equivalent norm may be defined by

$$\Big(\sum_{l=0}^{M}\int_0^\infty \|(D_n^l u)(\cdot,x_n)\|^2_{\Xi_P^{(-l)},\mathbb{R}^{n-1}}\,dx_n\Big)^{1/2}, \tag{2-7}$$

where $M = 2m_1+\dots+2m_S$ (cf. (2-4)).

3. The zeros of the symbol and an ODE estimate

Now we again consider an operator pencil of the form (1-1) and its model problem in $\mathbb{R}^n_+$ with constant coefficients and without lower order terms. Let $A(\xi,\lambda)$ be a polynomial in $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ of the form

$$A(\xi,\lambda) = A_{2m}(\xi) + \lambda A_{2m-1}(\xi) + \dots + \lambda^{2m-2\mu}A_{2\mu}(\xi), \tag{3-1}$$

where $A_j(\xi)$ is a homogeneous polynomial in ξ of degree j. If $A(\xi,\lambda)$ is N-elliptic with parameter in $[0,\infty)$ in the sense of (1-5) then standard arguments show that the number of roots of $A(\xi',\tau,\lambda)$ with $(\xi',\lambda)\in\mathbb{R}^{n-1}\times\mathbb{C}$, considered as a polynomial in τ, is independent of (ξ',λ) and (for $n>2$) equal to m. Similarly to the theory of singular perturbations (cf. [10]) we consider the auxiliary polynomial of degree $2m-2\mu$ given by

$$Q(\tau) := \tau^{-2\mu} A(0,\tau,1)\,. \tag{3-2}$$

This polynomial has no real roots, and we say that $A(\xi',\tau,\lambda)$ degenerates regularly for $\lambda\to\infty$ if the polynomial $Q(\tau)$ defined in (3-2) has exactly $m-\mu$ roots with positive imaginary part (counted according to their multiplicities).

In order to investigate the behaviour of the fundamental solutions of $A(\xi',D_t,\lambda)$ the main step is to split the roots $\tau_1(\xi',\lambda),\dots,\tau_m(\xi',\lambda)$ of $A(\xi',\cdot,\lambda)$ with positive imaginary part into two groups. This is possible due to the following lemma which can be proved using the theory of Puisseux series (see [5], Section 3).

Lemma 3.1. *Let the polynomial $A(\xi,\lambda)$ in* (3-1) *be N-elliptic with parameter in $[0,\infty)$ and assume that A degenerates regularly for $\lambda\to\infty$. Then, with a suitable numbering of the roots $\tau_j(\xi',\lambda)$ of $A(\xi',\tau,\lambda)$ with positive imaginary part, we have:*

(i) *Let $S(\xi')=\{\tau_1^0(\xi'),\dots,\tau_\mu^0(\xi')\}$ be the set of all zeros of $A_{2\mu}(\xi',\tau)$ with positive imaginary part. Then for all $r>0$ there exists a $\lambda_0>0$ such that the distance between the sets $\{\tau_1(\xi',\lambda),\dots,\tau_\mu(\xi',\lambda)\}$ and $S(\xi')$ is less than r for all ξ' with $|\xi'|=1$ and all $\lambda\geq\lambda_0$.*

(ii) *Let $\tau_{\mu+1}^1,\dots,\tau_m^1$ be the roots of the polynomial $Q(\tau)$ (cf.* (3-2)*) with positive imaginary part. Then*

$$\tau_j(\xi',\lambda)=\lambda\tau_j^1+\tilde{\tau}_j^1(\xi',\lambda)\quad(j=\mu+1,\dots,m)\,, \tag{3-3}$$

and there exist constants K_j and λ_1, independent of ξ' and λ, such that for $\lambda\geq\lambda_1$ the inequality

$$|\tilde{\tau}_j^1(\xi',\lambda)|\leq K_j|\xi'|^{\frac{1}{k_1}}\,\lambda^{1-\frac{1}{k_1}}\quad(|\xi'|\leq\lambda) \tag{3-4}$$

holds, where k_1 is the maximal multiplicity of the roots of $Q(\tau)$.

Now we consider for fixed $\xi'\in\mathbb{R}^{n-1}$, $\lambda\in[0,\infty)$ and $j=1,\dots,m$ the ordinary differential equation on the half-line

$$A(\xi',D_t,\lambda)\,w_j(t)=0\qquad(t>0)\,, \tag{3-5}$$

$$D_t^{k-1}\,w_j(t)|_{t=0}=\delta_{jk}\quad(k=1,\dots,m)\,, \tag{3-6}$$

$$w_j(t)\to 0\qquad(t\to+\infty)\,.$$

Here D_t stands for $-i\frac{\partial}{\partial t}$. The following result will be the main step in the proof of the a priori estimate (Theorem 4.4).

Theorem 3.2. *For every* $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ *and* $\lambda \in [0,\infty)$ *the ordinary differential equation* (3-5)–(3-6) *has a unique solution* $w_j(\xi',t,\lambda)$, *and the estimate*

$$\|D_t^l w_j(\xi',\cdot,\lambda)\|_{L_2(\mathbb{R}_+)} \le C \begin{cases} |\xi'|^{l-j+\frac12}, & j\le\mu,\quad l\le\mu,\\ |\xi'|^{1+\mu-j}(\lambda+|\xi'|)^{l-\mu-\frac12}, & j\le\mu,\quad l>\mu,\\ |\xi'|^{l-\mu}(\lambda+|\xi'|)^{\mu-j+\frac12}, & j>\mu,\quad l\le\mu,\\ (\lambda+|\xi'|)^{l-j+\frac12}, & j>\mu,\quad l>\mu, \end{cases} \tag{3-7}$$

holds with a constant C *not depending on* ξ' *and* λ.

Sketch of proof. In the standard way (cf., e.g., [2], Section 2) we can write w_j in the form

$$w_j(\xi',t,\lambda) = \frac{1}{2\pi i}\int_{\gamma(\xi',\lambda)} \frac{M_j(\xi',\tau,\lambda)}{A_+(\xi',\tau,\lambda)} e^{it\tau}\,d\tau \tag{3-8}$$

where $M_j(\xi',\tau,\lambda)$ is a polynomial in τ of degree $m-j$, and

$$A_+(\xi',\tau,\lambda) := \prod_{k=1}^m (\tau-\tau_j(\xi',\lambda))\,.$$

In (3-8) we have to integrate along a closed contour $\gamma=\gamma(\xi',\lambda)$ which encloses all roots $\tau_1,\dots,\tau_m$. Expressing the coefficients of M in terms of τ_j, we can apply Lemma 3.1 to find estimates for the integrand. Splitting the integral for large λ in the form $\int_\gamma \cdots = \int_{\gamma_1}\cdots + \int_{\gamma_2}\cdots$ according to the splitting of the roots $\{\tau_1,\dots,\tau_m\}$ into two groups (cf. Lemma 3.1), wo obtain for each of the two integrals different estimates. A comparison of the right-hand sides of these estimates finally leads to the inequality (3-7).

4. The main results

To prove the a priori estimate for the Dirichlet boundary value problem corresponding to the operator pencil (1-1), we first consider model problems in $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Let A be a polynomial of the form (3-1). In connection with this polynomial we consider the Newton polygon $N_{m,\mu}$ which is defined as the convex hull of the points

$$(0,0),\ (0,m-\mu),\ (\mu,m-\mu),\ (m,0)\,.$$

For this Newton polygon the weight function $\Xi(\xi,\lambda)$ is equivalent to $(1+|\xi|)^\mu(\lambda+|\xi|)^{m-\mu}$, cf. (2-4). We remark that this weight function differs from $\Xi_A(\xi,\lambda)$ as defined in Section 2 and corresponds to the "energy space" conncected with

$A(D,\lambda)$. The spaces $H^{\Xi}(\mathbb{R}^n)$ and $H^{1/\Xi}(\mathbb{R}^n)$ are defined in analogy to the definition of $H^{\Xi_P}(\mathbb{R}^n)$.

The operator $A(D,\lambda)$ acts continuously from $H^{\Xi}(\mathbb{R}^n)$ to $H^{\frac{1}{\Xi}}(\mathbb{R}^n)$. From Theorem 2.1 we see that the operator

$$D_n^{j-1} : u \mapsto \left[\left(\frac{\partial}{\partial x_n}\right)^{j-1} u\right]\Bigg|_{\mathbb{R}^{n-1}} \qquad (j=1,\dots,m)$$

acts continuously from $H^{\Xi}(\mathbb{R}^n)$ to $H^{\Xi^{(-j+\frac{1}{2})}}(\mathbb{R}^{n-1})$.

The following proposition is proved by elementary calculations.

Proposition 4.1. (A priori estimate in $\mathbb{R}^n$.) *Let $A(\xi,\lambda)$ be N-elliptic with parameter in $[0,\infty)$. Then for every $\lambda_0>0$ the inequality*

$$\|u\|_{\Xi,\mathbb{R}^n} \le C\Big(\|A(D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big) \tag{4-1}$$

holds for all $\lambda \ge \lambda_0$ with a constant $C = C(\lambda_0)$ independent of u and λ.

Theorem 4.2. (A priori estimate in $\mathbb{R}^n_+$.) *Let $A(\xi,\lambda)$ be N-elliptic with parameter in $[0,\infty)$ and degenerate regularly for $\lambda\to\infty$. Then for every $\lambda_0>0$ there exists a constant $C=C(\lambda_0)$ such that for all $\lambda\ge\lambda_0$ and all $u\in H^{\Xi}(\mathbb{R}^n_+)$ the estimate*

$$\begin{aligned}\|u\|_{\Xi,\mathbb{R}^n_+} \le C\Big(&\|A(D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n_+} \\ &+\sum_{j=1}^m \|D_n^{j-1}u\|_{\Xi^{(-j+\frac{1}{2})},\mathbb{R}^{n-1}} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n_+)}\Big)\end{aligned} \tag{4-2}$$

holds.

The proof of this estimate is quite elaborated and can be found in [5]. By standard arguments we may restrict ourselves to the case $f := A(D,\lambda)u = 0$. The main step in the proof is to find an estimate for the solution v of

$$A(D,\lambda)\,v = 0\,, \tag{4-3}$$

$$D_n^{j-1}v(x)|_{x_n=0} = h_j(x')\,, \tag{4-4}$$

of the form

$$\|v\|_{\Xi,\mathbb{R}^n_+} \le \text{ const }\Big(\sum_{j=1}^m \|h_j\|_{\Xi^{(-j+1/2)},\mathbb{R}^{n-1}} + \lambda^{m-\mu}\|v\|_{L_2(\mathbb{R}^n)}\Big)\,. \tag{4-5}$$

Here $h_j\in H^{\Xi^{(-j+1/2)}}(\mathbb{R}^{n-1})$. The proof of (4-5) uses the norm (2-7) with $M=m$, Theorem 3.2 and a comparison of the right-hand side of (3-7) with the weight function $\Xi(\xi',\lambda)$.

Now let $A(x, D, \lambda)$ be an operator pencil of the form (1-1) acting on the compact manifold M with boundary Γ. The Sobolev spaces $H^{\Xi}(M)$ and $H^{\Xi^{(-j+1/2)}}(\Gamma)$ are defined in the usual way, using a partition of unity.

We choose a finite number of coordinate systems. In each of these systems the operator is of the form (1-1). The principal part of the operator is invariantly defined at each of these systems and at every fixed point $x^0 \in M$ it is of the form

$$A^{(0)}(x^0, D, \lambda) = A^{(0)}_{2m}(x^0, D) + \ldots + \lambda^{2m-2\mu} A^{(0)}_{2\mu}(x^0, D) \tag{4-6}$$

(here $A_j^{(0)}$ denotes the principal part of A_j). We suppose that for each $x^0 \in \overline{M}$ our operator is N-elliptic with parameter. From the reason of continuity and compactness the constant C in inequality (1-5) can be chosen independent of x^0.

Now we fix a point $x^0 \in \Gamma$ and a coordinate system in the neighbourhood of x^0 such that in this system locally the boundary Γ is given by the equation $x_n = 0$. In this case we can define an analog of the polynomial (3-2):

$$Q(x^0, \tau) = \tau^{-2\mu} A^{(0)}(x^0, 0, \tau, 1) \tag{4-7}$$

Suppose that at a point $x^0 \in \Gamma$ and in a fixed coordinate system this polynomial has $m - \mu$ roots in the upper half-plane of the complex plane. It easily follows from this fact that every polynomial (4-7) corresponding to an arbitrary $x^0 \in \Gamma$ has the same property. In this case we say that the operator $A(x, D, \lambda)$ degenerates regularly at the boundary Γ.

We consider the Dirichlet boundary value problem connected with the pencil $A(x, D, \lambda)$, i.e. the operator

$$(A(x, D, \lambda), D_\Gamma) \quad \text{where } D_\Gamma u = \left(u|_\Gamma, \frac{\partial}{\partial \nu} u \bigg|_\Gamma, \ldots, \left(\frac{\partial}{\partial \nu}\right)^{m-1} u \bigg|_\Gamma \right).$$

Here $\frac{\partial}{\partial \nu}$ stands for the normal derivative.

We obviously have

Lemma 4.3. *The operator*

$$(A(x, D, \lambda), D_\Gamma) : \; H^{\Xi}(M) \longrightarrow H^{\frac{1}{\Xi}}(M) \times \prod_{j=1}^{m} H^{\Xi^{(-j+\frac{1}{2})}}(\Gamma)$$

is continuous with norm bounded by a constant independent of λ.

One of the main results of this paper states is the following a priori estimate in terms of the Sobolev spaces corresponding to the Newton polygon. In particular, this shows that the definition of these spaces is appropriate for such pencils.

Theorem 4.4. *Let $A(x,D,\lambda)$ be an operator pencil of the form* (1-1)*, acting on the manifold M with boundary Γ. Let A be N-elliptic with parameter in* $[0,\infty)$ *and assume that A degenerates regularly at the boundary Γ. Then for $\lambda \geq \lambda_0$ there exists a constant $C = C(\lambda_0)$, independent of u and λ, such that*

$$\|u\|_{\Xi,M} \leq C\Big(\|A(x,D,\lambda)u\|_{\frac{1}{\Xi},M} + \sum_{j=1}^{m}\Big\|\Big(\frac{\partial}{\partial\nu}\Big)^{j-1}u\Big\|_{\Xi^{(-j+\frac{1}{2})},\Gamma} + \lambda^{m-\mu}\|u\|_{L_2(M)}\Big). \tag{4-8}$$

The proof of this theorem uses the technique of localization ("freezing the coefficients") in order to reduce the proof of the a priori estimate to the corresponding results for the model problems in $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Starting from the representation (3-8) of the fundamental solutions w_j of (3-5)–(3-6) it is possible to construct a (rough) right parametrix for (A, D_Γ); see [5] for details (cf. also [12] for the construction of the parametrix in the case of ellipticity with parameter in the sense of Agmon–Agranovich–Vishik).

Theorem 4.5. *The operator (A, D_Γ) has a (right) parametrix in the sense that*

$$(A, D_\Gamma)B = I + T\,,$$

where I denotes the identity operator in $H^{\frac{1}{\Xi}}(M) \times \prod_{j=1}^m H^{\Xi^{(-j+1/2)}}(\Gamma)$ and

$$T : H^{\frac{1}{\Xi}}(M) \times \prod_{j=1}^{m} H^{\Xi^{(-j+1/2)}}(\Gamma) \to H^{\Theta}(M) \times \prod_{j=1}^{m} H^{\Xi^{(-j+3/2)}}(\Gamma)$$

is continuous with norm bounded by a constant independent of λ. Here we posed $\Theta(\xi,\lambda) = (1+|\xi|)/\Xi(\xi,\lambda)$.

Final remarks. The application of the Newton polygon method to operator pencils of the form (1-1) is not restricted to Dirichlet boundary conditions. It is also possible to prove an a priori estimate for general boundary value problems. This will be done in a forthcoming paper of the authors.

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