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"On the Stabilizability of Infinite Dimensional Systems via Receding Horizon Control"

Dissertation

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To my parents

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Abstract

One efficient strategy for dealing with optimal control problems on an infinite time horizon is the receding horizon framework which is also known as Model Predictive Control (MPC). In this approach, an infinite horizon optimal control problem is approximated by a sequence of finite horizon problems in a receding horizon fashion. Stability (convergence to the steady state) is not generally ensured due to the use of a finite prediction horizon. Thus, in order to ensure the asymptotic stability of the controlled system, additional terminal cost functions and/or terminal constraints are often needed to add to the finite horizon problems. In this thesis we are concerned with the stabilization of several classes of infinite-dimensional controlled systems by means of a Receding Horizon Control (RHC) scheme. In this scheme, no terminal costs or terminal constraints are used to ensured the stability. The key assumption is the stabilizability of the underlying system. Based on this condition the suboptimality and stability of RHC are investigated. To justify the applicability of this framework, we consider controlled systems governed by three different types of partial differential equations, including the linear wave equation, the viscous Burgers equation, and the nonlinear KdV equation. For all these cases, the well-posedness of the underlying controlled system, the asymptotic stability of RHC, and the suboptimality of RHC are studied. Moreover, numerical experiments are given to validate the theoretical results.

Keywords: receding horizon control, model predictive control, asymptotic stability, infinite-dimensional systems, pde-constrained optimization

Zusammenfassung

Eine effiziente Strategie um optimale Steuerungsprobleme mit unendlichem Zeithorizont zu behandeln, ist die modellprädiktive Regelung oder das sogenannte Prinzip des zurückweichenden Horizonts. In dieser Strategie wird ein optimales Steuerungsproblem mit unendlichem Zeithorizont durch eine Folge von optimalen Steuerungsproblemen mit endlichem Zeithorizont nach Art eines sogenannten zurückweichenden Horizonts (receding horizon fashion) approximiert. Aufgrund der Verwendung eines endlichen Vorhersagehorizonts ist die Stabilität im Allgemeinen jedoch nicht garantiert. Um die asymptotische Stabilität des kontrollierten Systems sicherzustellen, ist es daher oft notwendig, den Problemen mit endlichem Zeithorizont zusätzliche Terminal-Kostenfunktionen und/oder zusätzliche Endzeitpunkt-Restriktionen hinzuzufügen. In dieser Arbeit beschäftigen wir uns mit der Stabilisierung einer Klasse von unendlichdimensionalen kontrollierten Systemen mit der Hilfe eines RHC-Schemas (Receding Horizon Control). In diesem Schema werden weder Terminal-Kostenfunktionen noch Endzeitpunkt-Restriktionen benötigt, um die Stabilität sicherzustellen. Die Stabilisierbarkeit des zugrunde liegenden Systems ist hier die Schlüsselbedingung. Basierend auf dieser Bedingung werden die Suboptimalität und die Stabilität der RHC-Kontrolle untersucht. Um die Anwendbarkeit dieses Schemas zu rechtfertigen, betrachten wir kontrollierte Systeme für verschiedene Typen partieller Differentialgleichungen. Dazu gehören die lineare Wellengleichung, die viskose Burgers Gleichung, und die nichtlineare KdV-Gleichung. Für diese Fälle werden die Wohldefiniertheit des kontrollierten Systems, die asymptotische Stabilität der RHC-Kontrolle, und die Suboptimalität von der RHC-Kontrolle untersucht. Ferner werden numerische Experimente präsentiert um die theoretischen Ergebnisse zu bestätigen.

Schlagworte: zurückweichende Horizont-kontrolle, modellprädiktive Regelung, asymptotische Stabilität, unendlichdimensionalen Systemen, Optimierung mit Differentialgleichungsnebenbedingungen

Preface

This thesis contains six chapters. The first chapter briefly overviews the main results of the thesis and the last chapter is devoted to the conclusion and future work. The remaining four chapters are besed on material from three papers by the author [10, 11, 12]. Chapter 2 and 4 use the material from the research paper [11], coauthored with Karl Kunisch. Chapter 3 originates mostly from Reference [12], coauthored with Karl Kunisch. Finally, Chapter 5 is based on Reference [10], coauthored with Karl Kunisch and Anne-Céline Boulanger.

Contents

Chapter 1

General Introduction

1.1 Background and Problem Formulation

In this thesis, we consider the optimal control problem which consists of minimizing

$$J_\infty(u, y_0) := \int_0^\infty \ell(y(t), u(t))dt \tag{1.1}$$

subject to

$$\begin{cases} \frac{d}{dt}y(t) = f(y(t)) + Bu(t) & \text{for } t > 0, \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where $f(0) = 0$, $\ell(0, 0) = 0$. The state $y(t)$ and the control $u(t)$ are respectively elements of spatially dependent function spaces H and U, and $B \in \mathcal{L}(U, H)$ is the control operator. Furthermore, the incremental cost function $\ell(\cdot, \cdot)$ is assumed to be uniformly positive definite in both the state and control variables. We denote the dual space of H with H^*.

To deal with problem (1.1)-(1.2) in practice, we can first replace ∞ by a large finite time horizon $T_\infty > 0$ in (1.1). Then by writing the first-order optimality conditions for the resulting optimal control problem defined on the interval $(0, T_\infty)$, we obtain the following two-point boundary-value system of equations

$$\begin{cases} \dfrac{d}{dt}y(t) = \partial_p \mathcal{H}(y(t), u(t), p(t)) & t \in (0, T_\infty), \quad y(0) = y_0, \\ \dfrac{d}{dt}p(t) = -\partial_y \mathcal{H}(y(t), u(t), p(t)) & t \in (0, T_\infty), \quad p(T_\infty) = 0, \\ u(t) = \arg\min_{u \in U} \mathcal{H}(y(t), u, p(t)) & t \in (0, T_\infty), \end{cases} \tag{1.3}$$

where for every $y \in H$, $p \in H^*$, and $u \in U$, the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(y, u, p) := \ell(y, u) + \langle p, f(y) + Bu \rangle_{H^*, H}.$$

Here, numerically solving the coupled system (1.3) is of significant challenge for the case that T_∞ is large. Therefore, a large number of research efforts have been devoted to the numerical treatment of the two-points boundary-value coupled system of the form (1.3). An alternative approach to deal with (1.1)-(1.2) consists in constructing the feedback law based on the Bellman's dynamics programming principle. We need to find the solution $V_\infty : H \to \mathbb{R}_+$ of the following Hamilton-Jacobi-Bellman equation

$$\min_{u\in U} \mathcal{H}(y, u, \partial_y V_\infty) = 0, \tag{HJB}$$

where the positive function V_∞ is the value function and defined by

$$V_\infty(y_0) := \inf_{u\in L^2(0,\infty;U)} \{J_\infty(u, y_0) \text{ subject to (1.2)}\}.$$

But for the case of infinite-dimensional controlled systems, discretization leads to finite-dimensional (HJB) equations of very large dimensions and due to the curse of dimensionality, it is infeasible to numerically solve (HJB).

Another strategy for solving problem (1.1)-(1.2) numerically employs the receding horizon control (RHC), which is also known as model predictive control (MPC). This method consists of obtaining a suboptimal solution of the infinite horizon problem by concatenation of a sequence of finite horizon optimal controls on a sequence of overlapping intervals. Further, within this framework, the process of generating the sequence of intervals and concatenation are carried out in such way that the resulting control has a feedback mechanism and it is defined on the whole of the interval $[0, \infty)$. Indeed, the receding horizon framework bridges to a certain degree the gap between closed-loop control and open-loop control. Since when proceeding in this manner, the exact solution of (1.1)-(1.2) is not obtained, the question of justifying the RHC technique arises. This is typically addressed by analysing whether the RHC control meets the control objective which is formulated within (1.1)-(1.2). Frequently this control objective is given by the stabilization problem. Due to replacing the infinite prediction horizon by a family of finite ones, the asymptotic stability of the receding horizon control scheme is not a priori guaranteed. It can even be demonstrated by a simple linear example that the naive application of a RHC strategy can lead to an unstable controlled system. Thus, often it is necessary to impose additional conditions or add terminal costs or terminal constraints to the finite horizon problems to guarantee the desired system performance.

Here, the control objective formulated within (1.1)-(1.2) is given by the stabilization problem. In other words, within (1.1)-(1.2) we aim to find a control $u \in L^2(0, \infty; U)$ which steers the system to the steady state and, simultaneously, minimizes the infinite horizon running cost (1.1). Usually, the succinct use of the structure of the dynamical system under consideration together with possible terminal costs and/or terminal constraints for the finite horizon problems can ensure the asymptotic stability of RHC under appropriate assumptions.

In the past three decades, numerous results have been published on RHC for finite-dimensional systems, see [1, 2, 41, 61, 66, 76, 77, 106] and the many references therein. Some of these frameworks use a terminal constraint as well as a terminal cost. Among them we can refer to [41] for the continuous-time controlled system and [113] for discrete-time controlled system. In both these approaches, a terminal cost and a terminal inequality constraint are added to every finite time horizon problem, where the terminal cost is a quadratic penalty term with a positive definite matrix. Due to the presence of terminal costs and terminal constraints in the finite horizon problems, these strategies require high computational effort.

Later in [117], a globally stabilizing RHC law was obtained by using a global Control Lyapunov Function (CLF) as the terminal cost functions [7, 129]. CLF is considered as a generalization of the Lyapunov function for the controlled system. It is a uniformly positive and continuously differentiable function whose value decreases along atleast a controlled trajectory from a given initial points. Moreover, due to the Artstein theorem [7] for finite-dimensional controlled systems the existence a stabilizing feedback control is equivalent to existence of a global CLF. This framework is attractive in the sense that the stability of controlled system is ensured without enforcing terminal constraints. On the other hand, finding and constructing a global CLF for the underlying controlled system is often difficult or even impossible. For similar works in this direction, we refer to [78, 117]. In [78], finite-dimensional controlled systems have been considered for which CLF exist just locally (on a neighbourhood of the steady state). Afterwards, in [75] this framework has been extended for the case of infinite-dimensional controlled systems. By considering an appropriate functional analytic setting, they generalized the definition of CLF to infinite-dimensional controlled systems. Here it is more delicate to show the stability of RHC, since the Artstein theorem is not valid any more and the Heine-Borel property fails to hold for the strong topology.

More recently several authors (see, e.g., [61, 62, 66, 76]) have managed to prove the asymptotic stability of RHC even without use of CLF and terminal constraints. In this framework the stability and also suboptimality of RHC are achieved through generating an appropriate sequence of overlapping intervals and applying a suitable concatenation scheme. So far, this framework has been well studied for finite-dimensional dynamical systems [76, 120] and discrete-time dynamical systems [61, 62, 66]. In [62], a general scheme for finite- and infinite-dimensional controlled systems with discrete-time was presented. Relying on controllability properties of the underlying system, the stability and the optimality of RHC was established. Afterwards in [3], this RHC scheme was applied to the controlled system governed by partial differential equations and the performance of RHC was analysed. But as far as we know, for infinite-dimensional controlled systems with *continuous-time* dynamical systems there still does not exist a rigorous theory. In this thesis we make a step in this direction. First we derive an abstract framework for a general infinite-dimensional controlled system. Then, we investigate the suboptimality and stability of RHC based on a few natural assumptions. Among them a stabilizability condition of the controlled system under consideration is the key condition. This framework is inspired by, but different from, [120] since we treat partial rather than ordinary differential equations. As a consequence, Barbalat's lemma which relies on the finite dimensionality of the state space, is not applicable. Moreover, we consider also systems which are locally or semi-globally stabilizable. To justify the applicability of this framework, we apply it to three different types of partial differential equations including second order hyperbolic equations, parabolic equations, and dispersive equations. With respect to the nature of the differential equation under study, we investigate the prerequisite assumptions of the receding horizon framework and derive different types of stability results. Moreover for every controlled system, numerical simulations are presented to validate our theoretical results.

1.2 Outline of the Thesis

The outline and the contributions of the thesis are as follows.

Chapter 2: A General RHC Scheme for Infinite-dimensional Controlled Systems In this chapter, a receding horizon algorithm for infinite-dimensional controlled systems of the form (1.1)-(1.2) is introduced. For this framework we consider an appropriate functional analytic setting. The stabilizability of the controlled system (1.2) and well-posedness of finite horizon open-loop problems are the necessary conditions for this framework. Relying on these conditions and Bellman's principle, we show that there is a sufficiently long horizon length for which RHC is suboptimal. This suboptimality property consists of inequalities which estimate the value of the running cost (1.1) evaluated along RHC in terms of the infinite horizon value function V_∞ from above and below. Then we show that for a sufficiently long horizon length, the finite horizon value function decreases exponentially along the receding horizon trajectory. Throughout the chapter, we consider both of the cases local and global stabilizability. At the end of chapter, we discuss the exponential stability of RHC with regard to the uniformly positiveness of the finite horizon value function.

Chapter 3: On the global Stabilizability of the Wave Equation via RHC In this chapter, we apply the proposed receding horizon algorithm in Chapter 2 for the stabilization of the linear wave equation. Here different control actions, namely, distributed control, Dirichlet boundary control, and Neumann boundary control are considered. For each case, depending on the regularity of the solution, the global stabilizability assumption, well-posedness of the controlled system, well-posedness, and first-order optimality conditions for the finite horizon open-loop problems are investigated. The observability conditions are essential here. We will see that these conditions are equivalent to the stabilizability conditions. In addition, depending on these conditions, first the uniform positiveness of the finite horizon value function is obtained. Then, as a consequence, the global exponential stability of RHC is shown. At the end, we end the chapter with numerical experiments for each RHC actions.

Chapter 4: On the Stabilizability of the Burgers Equation by RHC In this chapter, we apply the proposed receding horizon algorithm in Chapter 2 for the stabilization of the Burgers equation with periodic boundary conditions and homogeneous Neumann boundary conditions. Here RHC is active on a small open subset of the physical domain. With respect to the underlying boundary conditions, the global or local stabilizability assumption for the controlled system is investigated.

Here for the Burgers equation, the uniform positiveness of the finite horizon value function fails to hold. Thus, by using other techniques the asymptotic stability of RHC is demonstrated. Finally we present numerical experiments in which the effect of RHC control with and without terminal control penalty is compared.

Chapter 5: On the semi-global Stabilizability of the KdV Equation via RHC This chapter is devoted to the stabilization of the nonlinear KdV equation via RHC which is active on a small open subset of the domain. First, we study the global well-posedness of the nonlinear KdV equation in the weak sense. Then existence of the finite horizon optimal control is investigated. Based on the semi-global stabilizability result from [114] we first show that the RHC is suboptimal. Then by an observability type estimate, we prove that the resulting receding horizon controlled system is semi-global exponentially stable. This requires techniques which differ from those which were employed in the previous chapters. At the end, we close the chapter with numerical experiments.

Chapter 6: Conclusion and Future Work In this chapter, we summarize the main contributions of the thesis and indicate possible directions for future work.

Chapter 2

A General RHC Scheme for Infinite-dimensional Controlled Systems

2.1 Introduction

Recall the following optimal control problem

$$\min_{u \in L^2(0,\infty;U)} J_\infty(u, y_0) := \int_0^\infty \ell(y(t), u(t))dt, \tag{2.1}$$

subject to

$$\begin{cases} \frac{d}{dt}y(t) = f(y(t)) + Bu(t) & \text{for } t > 0, \\ y(0) = y_0, \end{cases} \tag{2.2}$$

where $f(0) = 0$, $\ell(0,0) = 0$. The state $y(t)$ and the control $u(t)$ are respectively elements of spatially dependent function spaces H and U, and B is the control operator. Furthermore, the incremental cost function $\ell(\cdot,\cdot)$ is assumed to be uniformly positive definite in both the state and control variables.

In this chapter, we introduce a receding horizon framework to deal with infinite-dimensional optimal control problems of the form (2.1)-(2.2). For this framework we consider an appropriate functional analytic setting. The stabilizability of controlled system (2.2) is the key condition. Based on this condition and Bellman's principle, we develop an abstract setting which estimates the value of the cost J_∞ evaluated along the receding horizon control and trajectory in terms of the minimal value functional associated to (2.1)-(2.2). This property is called *suboptimality*.

To briefly recapture the receding horizon approach, we choose a sampling time $\delta > 0$ and an appropriate prediction horizon $T > \delta$. Then sampling instances $t_k := k\delta$ for $k = 0, 1, \ldots$ are defined. At every sampling instance t_k, an open-loop optimal control problem is solved over a finite prediction horizon $[t_k, t_k + T]$. The optimal control thus obtained is applied to steer the system from time t_k with the initial state $y_{rh}(t_k)$ until time $t_{k+1} := t_k + \delta$, at which point a new measurement of state is assumed to be available. The process is repeated starting from the new state: we obtain a new optimal control and a new predicted state trajectory by shifting the prediction horizon forward in time. Throughout, we denote the receding horizon state- and control variables by $y_{rh}(\cdot)$ and $u_{rh}(\cdot)$, respectively. Also, $(y_T^*(\cdot; y_0, t_0), u_T^*(\cdot; y_0, t_0))$ stands for the optimal state and control of the optimal control problem with finite time horizon T and initial function y_0 at initial time t_0. We next summarize the resulting Algorithm 2.1.

Algorithm 2.1 Receding Horizon Algorithm

Input: Let the prediction horizon T, the sampling time $\delta < T$, and the initial state $y_0 \in H$ be given.

1: Set $k := 0$, $t_0 := 0$, and $y_{rh}(t_0) := y_0$.

2: Find the optimal pair $(y_T^*(\cdot; y_{rh}(t_k), t_k), u_T^*(\cdot; y_{rh}(t_k), t_k))$ over the time horizon $[t_k, t_k + T]$ by solving the finite horizon open-loop problem

$$\min_{u \in L^2(t_k, t_k+T; U)} J_T(u, y_{rh}(t_k)) := \min_{u \in L^2(t_k, t_k+T; U)} \int_{t_k}^{t_k+T} \ell(y(t), u(t)) dt,$$

$$\text{subject to } \begin{cases} \frac{d}{dt} y(t) = f(y(t)) + Bu(t) & \text{for } t \in (t_k, t_k + T), \\ y(t_k) = y_{rh}(t_k) \end{cases}$$

3: Set

$$\begin{aligned} u_{rh}(\tau) &:= u_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta), \\ y_{rh}(\tau) &:= y_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta], \\ t_{k+1} &:= t_k + \delta, \\ k &:= k + 1. \end{aligned}$$

4: Go to step 2.

2.2 Suboptimality and Stability of RHC

Let $V \subset H = H^* \subset V^*$ be a Gelfand triple of real Hilbert spaces with V densely contained in H. Further let U denote the control space which is also assumed to be a real Hilbert space. For any $T > 0$ and $y_0 \in H$ we consider the controlled dynamical system

$$\begin{cases} \frac{d}{dt} y(t) = f(y(t)) + Bu(t) & \text{for } t \in (0, T), \\ y(0) = y_0, \end{cases} \tag{2.3}$$

where f is a continuous function from V to V^*, $f(0) = 0$, and $B \in \mathcal{L}(U, V^*)$. Here $\mathcal{L}(U, V^*)$ denotes the space of all continuous linear operators from U to V^*. Throughout the chapter, it is assumed that for any triple $(T, y_0, u) \in \mathbb{R}_+ \times H \times L^2(0,T;U)$ there exists a unique $y \in W(0,T)$, where

$$W(0,T) = L^2(0,T;V) \cap H^1(0,T;V^*), \tag{2.4}$$

satisfying

$$y(t) - y(0) = \int_0^t (f(y(s)) + Bu(s))ds \quad \text{in } V^*$$

for $t \in [0,T]$. For sufficient conditions on f we refer to, e.g., [131, Chapter II.3]. We recall that $W(0,T)$ is continuously embedded in $C([0,T];H)$, see e.g. [131, 139].

To define the optimal control problems we introduce the continuous incremental function $\ell : H \times U \to \mathbb{R}_+$ satisfying

$$\ell(y,u) \geq \alpha_\ell(\|y\|_H^2 + \|u\|_U^2) \tag{2.5}$$

for a number $\alpha_\ell > 0$ independent of $y \in H$ and $u \in U$, and $\ell(0,0) = 0$. For every $T > 0$ and $y_0 \in H$ consider the finite horizon optimal control problem

$$\begin{aligned} &\min_{u \in L^2(0,T;U)} J_T(u, y_0) := \min_{u \in L^2(0,T;U)} \int_0^T \ell(y(t), u(t))dt, \\ &\text{subject to} \\ &\begin{cases} \frac{d}{dt} y(t) = f(y(t)) + Bu(t) & \text{for } t \in (0,T), \\ y(0) = y_0. \end{cases} \end{aligned} \tag{P_T}$$

Throughout we fix a neighborhood $\mathcal{N}_0$ of the origin in H. We assume the following:

$$\begin{aligned} &(P_T) \text{ admits an optimal pair } (y_T^*(\cdot; y_0, 0), u_T^*(\cdot; y_0, 0)) \text{ for any} \\ &y_0 \in \mathcal{N}_0 \text{ and } T > 0. \end{aligned} \tag{A1}$$

Conditions on ℓ and f which imply (A1) are wellknown from, e.g., [134]. The functional J_∞ is defined as J_T in (P_T) with T replaced by ∞. With (A1) holding, the following definition is well-posed.

Definition 2.2.1. *For any $y_0 \in \mathcal{N}_0$ the infinite horizon value function $V_\infty(\cdot)$ is defined as the extended real valued function*

$$V_\infty(y_0) := \inf_{u \in L^2(0,\infty;U)} \{J_\infty(u, y_0) \text{ subject to } (2.3)\}.$$

Similarly, the finite horizon value function $V_T(\cdot)$ is defined by

$$V_T(y_0) := \min_{u \in L^2(0,T;U)} \{J_T(u, y_0) \text{ subject to } (2.3)\}.$$

The following notion of local stabilizability will be used.

Definition 2.2.2 (Local stabilizability). *The dynamical system* (2.3) *is called locally stabilizable if for every positive* T *and initial function* $y_0 \in \mathcal{N}_0$ *there exists a control* $\hat{u}(\cdot, y_0) \in L^2(0,T;U)$ *with*

$$V_T(y_0) \le J_T(\hat{u}, y_0) \le \gamma(T)\|y_0\|_H^2, \tag{2.6}$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous, nondecreasing and bounded function.*

If $\mathcal{N}_0$ can be chosen to be all of H then we call (2.3) globally stabilizable. We shall require the following two assumptions:

$$\begin{array}{l}\text{The dynamical system (2.3) is locally stabilizable for}\\ \text{the neighborhood } \mathcal{N}_0.\end{array} \tag{A2}$$

$$\left.\begin{array}{l}\text{For every } T>0 \text{ there exists a constant } c_T \ge 0 \text{ such that for every}\\ y_0 \in \mathcal{N}_0 \text{ and } u \text{ with } \|u\|_{L^2(0,T;U)} \le \sqrt{\gamma(T)/\alpha_\ell}\,\|y_0\|_H \text{ we have}\\ \|y(t)\|_H^2 \le \|y_0\|_H^2 + c_T\int_0^t \|y(s)\|_H^2 ds + c_T\int_0^t \|u(s)\|_U^2\, ds \quad \text{for all } t \in [0,T].\end{array}\right\} \tag{A3}$$

Below $\mathcal{B}_{d_1}(0)$ denotes a ball in H centered at 0 with radius d_1.

Lemma 2.2.1. *If* (A1)-(A3) *hold and* $T > \delta > 0$*, then there exists a neighborhood* $\mathcal{B}_{d_1}(0) \subset \mathcal{N}_0$ *with* $d_1 = d_1(T) > 0$ *such that for every* $y_0 \in \mathcal{B}_{d_1}(0)$ *the following inequalities hold:*

$$\begin{aligned} V_T(y_T^*(\delta; y_0, 0)) \le \int_\delta^{\tilde{t}} &\ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \\ &+ \gamma(T+\delta-\tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2 \quad \textit{for all } \tilde{t} \in [\delta, T], \end{aligned} \tag{2.7}$$

and

$$\int_{\tilde{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \le \gamma(T-\tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2 \quad \textit{for all } \tilde{t} \in [0,T]. \tag{2.8}$$

Proof. First observe that, due to (2.5), for every $y_0 \in \mathcal{N}_0$ and $\tilde{t} \in [0,T]$ we have by Bellman's principle

$$\begin{aligned} \alpha_\ell \int_0^{\tilde{t}} (\|y_T^*(t; y_0, 0)\|_H^2 + \|u_T^*(t; y_0, 0)\|_U^2)dt &\le \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \\ &= V_T(y_0) - V_{T-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0)), \end{aligned}$$

and as a consequence $\|u_T^*\|_{L^2(0,T;U)}^2 \le \frac{\gamma(T)}{\alpha_\ell}\|y_0\|_H^2$. By (2.6), (A3), and the above inequality we have

$$\begin{aligned} \|y_T^*(\tilde{t}; y_0, 0)\|_H^2 &\le \|y_0\|_H^2 + c_T\int_0^{\tilde{t}} \|y_T^*(t; y_0, 0)\|_H^2 dt + c_T\int_0^{\tilde{t}} \|u_T^*(t; y_0, 0)\|_U^2 dt \\ &\le \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell}(V_T(y_0) - V_{T-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0))) \\ &\le \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell}V_T(y_0) \le (1 + \frac{c_T}{\alpha_\ell}\gamma(T))\|y_0\|_H^2. \end{aligned}$$

Since $\mathcal{N}_0$ is a neighborhood of zero, it follows that there exists a ball $\mathcal{B}_{\delta_1}(0) \subseteq \mathcal{N}_0$. Choosing $d_1 := \sqrt{(1 + \frac{c_T}{\alpha_\ell}\gamma(T))^{-1}\delta_1^2}$ we obtain that for every $y_0 \in \mathcal{B}_{d_1}(0)$ we have

$$y_T^*(\tilde{t}; y_0, 0) \in \mathcal{N}_0 \qquad \text{for all } \tilde{t} \in [0, T].$$

We turn to the verification of (2.7). For simplicity of notation, we denote $y_T^*(\delta; y_0, 0)$ by $y^*(\delta)$, where $y_0 \in \mathcal{B}_{d_1}(0)$. Due to Bellman's optimality principle, we have for every $\tilde{t} \in [\delta, T]$

$$\begin{aligned} V_T(y^*(\delta)) &= \int_\delta^{T+\delta} \ell(y_T^*(t; y^*(\delta), \delta), u_T^*(t; y^*(\delta), \delta))dt \\ &= \int_\delta^{\tilde{t}} \ell(y_T^*(t; y^*(\delta), \delta), u_T^*(t; y^*(\delta), \delta))dt + V_{T+\delta-\tilde{t}}(y_T^*(\tilde{t}; y^*(\delta), \delta)). \end{aligned} \tag{2.9}$$

By optimality of $y_T^*(\cdot; y^*(\delta), \delta)$ as a solution on $[\delta, T+\delta]$ with initial state $y^*(\delta) \in \mathcal{N}_0$ at $t = \delta$ we obtain

$$\begin{aligned} V_T(y^*(\delta)) &= \int_\delta^{T+\delta} \ell(y_T^*(t; y^*(\delta), \delta), u_T^*(t; y^*(\delta), \delta))dt, \\ &= \int_\delta^{\tilde{t}} \ell(y_T^*(t; y^*(\delta), \delta), u_T^*(t; y^*(\delta), \delta))dt + V_{T+\delta-\tilde{t}}(y_T^*(\tilde{t}; y^*(\delta), \delta)), \\ &\le \int_\delta^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + V_{T+\delta-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0)) \\ &\le \int_\delta^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T+\delta-\tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2, \end{aligned}$$

where for the last inequality we used (2.6).

To prove the second inequality let $\tilde{t} \in [0, T]$ be arbitrary. By Bellman's principle and (2.6), we have

$$\begin{aligned} V_T(y_0) &= \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \int_{\tilde{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \\ &= \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + V_{T-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0)) \\ &\le \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T-\tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2. \end{aligned} \tag{2.10}$$

Therefore,

$$\int_{\tilde{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \le \gamma(T-\tilde{t})\|y_T^*(\tilde{t}; y_0, 0)\|_H^2 \quad \text{ for all } \tilde{t} \in [0, T],$$

as desired. □

Lemma 2.2.2. *Suppose that for some initial function $y_0 \in H$, properties* (2.7) *and* (2.8) *of Lemma 2.2.1 hold. Then for the choice of*

$$\theta_1 := 1 + \frac{\gamma(T)}{\alpha_\ell (T-\delta)}, \qquad \theta_2 := \frac{\gamma(T)}{\alpha_\ell \delta},$$

we have the following estimates

$$V_T(y_T^*(\delta; y_0, 0)) \le \theta_1 \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt, \tag{2.11}$$

and

$$\int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \le \theta_2 \int_0^\delta \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt. \tag{2.12}$$

Proof. To verify the inequality (2.11) recall that $y_T^*(\cdot; y_0, 0) \in C([0,T]; H)$. Hence there is a $\bar{t} \in [\delta, T]$ such that

$$\bar{t} = \arg\min_{t \in [\delta, T]} \|y_T^*(t; y_0, 0)\|_H^2.$$

By (2.7) we have

$$\begin{aligned} &V_T(y_T^*(\delta; y_0, 0)) \\ &\quad \le \int_\delta^{\bar{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T + \delta - \bar{t})\|y_T^*(\bar{t}; y_0, 0)\|_H^2 \\ &\quad \le \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \gamma(T)\|y_T^*(\bar{t}; y_0, 0)\|_H^2 \\ &\quad \le \int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt + \frac{\gamma(T)}{T-\delta}\int_\delta^T \|y_T^*(t; y_0, 0)\|_H^2 dt. \end{aligned} \tag{2.13}$$

Furthermore, by (2.5)

$$\int_\delta^T \|y_T^*(t; y_0, 0)\|_H^2 dt \le \frac{1}{\alpha_\ell}\int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt. \tag{2.14}$$

Now by using (2.13) and (2.14), we have

$$V_T(y_T^*(\delta; y_0, 0)) \le \left(1 + \frac{\gamma(T)}{\alpha_\ell(T-\delta)}\right)\int_\delta^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt.$$

Turning to (2.12) we define

$$\hat{t} = \arg\min_{t \in [0, \delta]} \|y_T^*(t; y_0, 0)\|_H^2.$$

Then by (2.8) we have

$$\begin{aligned}\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \leq& \int_{\hat{t}}^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt\\ \leq&\gamma(T-\hat{t})\|y_T^*(\hat{t};y_0,0)\|_H^2\\ \leq&\gamma(T)\|y_T^*(\hat{t};y_0,0)\|_H^2\\ \leq&\frac{\gamma(T)}{\delta}\int_0^\delta \|y_T^*(t;y_0,0)\|_H^2 dt.\end{aligned} \tag{2.15}$$

Moreover, we have

$$\frac{\gamma(T)}{\delta}\int_0^\delta \|y_T^*(t;y_0,0)\|_H^2 dt \leq \frac{\gamma(T)}{\alpha_\ell \delta}\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt. \tag{2.16}$$

By (2.15) and (2.16) we can estimate

$$\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \leq \frac{\gamma(T)}{\alpha_\ell \delta}\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt.$$

□

Proposition 2.2.1. *Suppose that* (A1)-(A3) *hold and that* $\delta > 0$. *Then there exist* $T^* > \delta$ *and* $\alpha \in (0,1)$ *such that the following inequality is satisfied*

$$V_T(y_T^*(\delta;y_0,0)) \leq V_T(y_0) - \alpha\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \tag{2.17}$$

for every $T \geq T^*$ *and* $y_0 \in \mathcal{B}_{d_1(T)}(0)$ *with* $d_1(T)$ *defined in Lemma 2.2.1.*

Proof. Since for θ_1 and θ_2 defined in Lemma 2.2.2 we have

$$1-\theta_2(\theta_1-1) = 1-\frac{\gamma^2(T)}{\alpha_\ell^2\delta(T-\delta)},$$

and

$$\frac{\gamma^2(T)}{\alpha_\ell^2\delta(T-\delta)} \to 0 \text{ as } T\to\infty,$$

there exist $T^* > \delta$ and $\alpha \in (0,1)$ such that $1-\theta_2(\theta_1-1) \geq \alpha$ for all $T \geq T^*$. Next let $T \geq T^*$ and $y_0 \in \mathcal{B}_{d_1(T)}(0)$. Then from the definition of $V_T(y_0)$, Lemma 2.2.1, and Lemma 2.2.2 we have

$$\begin{aligned}V_T(y_T^*(\delta;y_0,0)) - V_T(y_0) =& V_T(y_T^*(\delta;y_0,0)) - \int_0^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt\\ \leq&(\theta_1-1)\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt - \int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt\\ \leq&(\theta_2(\theta_1-1)-1)\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt\\ \leq&-\alpha\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt.\end{aligned}$$

This implies (2.17). □

Theorem 2.2.1 (Suboptimality)**.** *Suppose that* (A1)-(A3) *hold, and let a sampling time* $\delta > 0$ *be given. Then there exist numbers* $T^* > \delta$ *and* $\alpha \in (0,1)$ *such that for every fixed prediction horizon* $T \geq T^*$*, the receding horizon control* u_{rh} *obtained from Algorithm 2.1 satisfies*

$$\alpha V_\infty(y_0) \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq V_\infty(y_0) \tag{2.18}$$

for all $y_0 \in \mathcal{B}_{d_2}(0)$ *with some* $d_2 = d_2(T) > 0$*.*

Proof. The right and left inequalities are obvious; therefore we need to verify only the middle one. For fixed $\delta > 0$ choose T^* and α according to Proposition 2.2.1. Define $d_2 := \sqrt{((1 + \frac{c_T}{\alpha\alpha_\ell}\gamma(T))^{-1} d_1^2}$, where $T \geq T^*$ and d_1 is defined in Lemma 2.2.1. We proceed by induction with respect to the receding horizon sampling index k.

First, since $d_2 < d_1$, the assumptions of Proposition 2.2.1 are applicable due to Lemma 2.2.1, and we have

$$V_T(y_{rh}(t_1)) \leq V_T(y_0) - \alpha \int_0^{t_1} \ell(y_{rh}(t), u_{rh}(t))dt \tag{2.19}$$

and also

$$\begin{aligned}
\|y_{rh}(t_1)\|_H^2 &\overset{(A3)}{\leq} \|y_{rh}(0)\|_H^2 + c_T \int_0^{t_1} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2)dt \\
&\overset{(2.5)}{\leq} \|y_0\|_H^2 + \frac{c_T}{\alpha_\ell} \int_0^{t_1} \ell(y_{rh}(t), u_{rh}(t))dt \\
&\overset{(2.19)}{\leq} \|y_0\|_H^2 + \frac{c_T}{\alpha\alpha_\ell}(V_T(y_0) - V_T(y_{rh}(t_1))) \\
&\leq \|y_0\|_H^2 + \frac{c_T}{\alpha\alpha_\ell} V_T(y_0) \overset{(2.6)}{\leq} (1 + \frac{c_T}{\alpha\alpha_\ell}\gamma(T))\|y_0\|_H^2 \leq d_1^2.
\end{aligned}$$

Proceeding by induction, we assume that

$$y_{rh}(t_k) \in \mathcal{B}_{d_1}(0) \quad \text{for all } k = 0, \ldots, k', \tag{2.20}$$

and that

$$V_T(y_{rh}(t_{k'})) \leq V_T(y_0) - \alpha \int_0^{t_{k'}} \ell(y_{rh}(t), u_{rh}(t))dt \tag{2.21}$$

for $k' \in \mathbb{N}$.

Since $y_{rh}(t_{k'}) \in \mathcal{B}_{d_1}(0)$, by Lemma 2.2.1 and Proposition 2.2.1 we have

$$V_T(y_{rh}(t_{k'+1})) \leq V_T(y_{rh}(t_{k'})) - \alpha \int_{t_{k'}}^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt. \tag{2.22}$$

Combined with (2.21), this gives

$$V_T(y_{rh}(t_{k'+1})) \leq V_T(y_0) - \alpha \int_0^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt. \tag{2.23}$$

Moreover, by repeated use of (A3), which is applicable by (2.20) and due to (2.23), (2.5), and (2.6), we have

$$\begin{aligned}
\|y_{rh}(t_{k'+1})\|_H^2 &\overset{(A3)}{\leq} \|y_{rh}(t_{k'})\|_H^2 + c_T \int_{t_{k'}}^{t_{k'+1}} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2) dt \\
&\overset{(A3)}{\leq} \|y_{rh}(0)\|_H^2 + c_T \int_0^{t_{k'+1}} (\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_U^2) dt \\
&\overset{(2.5)}{\leq} \|y_{rh}(0)\|_H^2 + \frac{c_T}{\alpha_\ell} \int_0^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t)) dt \\
&\overset{(2.23)}{\leq} \|y_0\|_H^2 + \frac{c_T}{\alpha\alpha_\ell}(V_T(y_0) - V_T(y_{th}(t_{k'+1}))) \\
&\leq \|y_0\|_H^2 + \frac{c_T}{\alpha\alpha_\ell} V_T(y_0) \overset{(2.6)}{\leq} (1 + \frac{c_T}{\alpha\alpha_\ell}\gamma(T))\|y_0\|_H^2 \leq d_1^2.
\end{aligned}$$

Hence $y_{rh}(t_{k'+1}) \in \mathcal{B}_{d_1}(0)$ which concludes the induction step. Taking the limit $k' \to \infty$ we find

$$\alpha J_\infty(u_{rh}, y_0) = \alpha \int_0^\infty \ell(y_{rh}(t), u_{rh}(t)) dt \leq V_T(y_0), \tag{2.24}$$

which concludes the proof. □

Remark 2.2.1. If (2.3) is globally stabilizable, i.e., if (2.6) holds with $\mathcal{N}_0$ replaced by H and if also (A1) is satisfied for all $y_0 \in H$, then Theorem 2.2.1 holds for all $y_0 \in H$, without the need of (A3). In fact, (A3) was only used in the proof of Lemma 2.2.1 for the construction of $\mathcal{B}_{d_1}(0)$, which is no longer needed if (A2) holds globally.

In the following Theorem, we will show that the value function $V_{T-\delta}$ exponentially decays along the receding horizon trajectory y_{rh}.

Theorem 2.2.2 (Exponential decay)**.** *Suppose that* (A1)-(A3) *hold and let a sampling time $\delta > 0$ be given. Then there exist numbers $T^* > \delta$, $\alpha > 0$ such that for every prediction horizon $T \geq T^*$, and every $y_0 \in \mathcal{B}_{d_2}(0)$ with $d_2(T) > 0$, the receding horizon trajectory $y_{rh}(\cdot)$ satisfies*

$$V_T(y_{rh}(t_k)) \leq e^{-\zeta t_k} V_T(y_0), \tag{2.25}$$

where ζ is a positive number depending on α, δ, and T but independent of y_0. Moreover, for every positive t, we have

$$V_{T-\delta}(y_{rh}(t)) \leq c e^{-\zeta t} V_T(y_0) \tag{2.26}$$

with a positive constant c depending on α, δ, and T but independent of y_0.

Proof. Let $\delta > 0$ be arbitrary. Then according to Theorem 2.2.1 and (2.17), there exists a positive number T^* such that for every $T \geq T^*$ and $y_0 \in \mathcal{B}_{d_2}(0)$ with $d_2 > 0$ we have

$$V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \leq -\alpha \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t)) dt \quad \text{for every } k \in \mathbb{N}, \tag{2.27}$$

with a positive $\alpha < 1$. Moreover, by using (2.11) and (2.12), we have

$$\begin{aligned} V_T(y_{rh}(t_{k+1})) \leq &\theta_1 \int_{t_{k+1}}^{t_k+T} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))\, dt \\ \leq &\theta_1\theta_2 \int_{t_k}^{t_{k+1}} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))\, dt \\ = &\theta_1\theta_2 \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))\, dt, \end{aligned} \tag{2.28}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are defined in Lemma 2.2.2 and the last equality follows from Step 3 in Algorithm 2.1. Now, by using (2.27) and (2.28), we obtain

$$V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \leq \frac{-\alpha}{\theta_1\theta_2} V_T(y_{rh}(t_{k+1})) \quad \text{for every } k \in \mathbb{N}.$$

Therefore, by defining $\eta := (1 + \frac{\alpha}{\theta_1\theta_2})^{-1}$ for every $k \in \mathbb{N}$ we can write

$$V_T(y_{rh}(t_k)) \leq \eta V_T(y_{rh}(t_{k-1})) \leq \eta^2 V_T(y_{rh}(t_{k-2})) \leq \cdots \leq \eta^k V_T(y_0). \tag{2.29}$$

Now by defining $\zeta := \frac{|\ln \eta|}{\delta}$, we obtain the inequality (2.25).

Turning to the inequality (2.26) with $t > 0$ arbitrary, then there exists an index k such that $t \in [t_k, t_{k+1}]$. Now since $T - \delta \leq T + t_k - t$ and by using Bellman's optimality principle we have

$$\begin{aligned} V_{T-\delta}(y_{rh}(t)) &\leq V_{T+t_k-t}(y_{rh}(t)) \\ &= V_T(y_{rh}(t_k)) - \int_{t_k}^{t} \ell(y_T^*(s; y_{rh}(t_k), t_k), u_T^*(s; y_{rh}(t_k), t_k))\, ds \\ &\leq V_T(y_{rh}(t_k)). \end{aligned} \tag{2.30}$$

By using (2.29) and (2.30), we obtain

$$V_{T-\delta}(y_{rh}(t)) \leq V_T(y_{rh}(t_k)) \leq \frac{\eta^{k+1}}{\eta} V_T(y_0) = \frac{1}{\eta} e^{-\zeta t_{k+1}} V_T(y_0) \leq \frac{1}{\eta} e^{-\zeta t} V_T(y_0).$$

□

Remark 2.2.2. The above result is similar to the result obtained in [75, Theorem 2.4], if the value function $V_{T-\delta}$ is considered as a control Lyapunov function G. At every iteration k of Algorithm 2.1 for every open-loop optimal control problem we have

$$\begin{aligned} &\min_{u \in L^2(t_k, t_k+T; U)} J_T(u, y_{rh}(t_k)) \\ &= \int_{t_k}^{T+t_k} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))dt \\ &= \int_{t_k}^{\delta+t_k} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))dt + V_{T-\delta}(y_T^*(\delta + t_k; y_{rh}(t_k), t_k)) \\ &= \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))dt + V_{T-\delta}(y_{rh}(t_{k+1})). \end{aligned}$$

This means that the terminal cost $V_{T-\delta}$ is implicitly added to the objective function of every open-loop optimal control problem. Indeed $V_{t-\delta}$ can be interpreted as an approximation of the infinite horizon value function V_∞ which is incorporated in the objective function of every open-loop problem.

Remark 2.2.3 (Uniformly positiveness of the finite horizon value function)**.** Note that the inequality (2.26) does not imply the asymptotic stability of the RHC law defined by Algorithm 2.1 within the neighborhood $\mathcal{B}_{d_2}(0)$, unless the finite horizon value function $V_{t-\delta}$ is uniformly positive on the level-sets of $V_{t-\delta}$. That is, for every positive $r > 0$ we have

$$V_{T-\delta}(y) \geq C\|y\|_H^2 \quad \text{for all } y \in \Pi_r, \tag{UPV}$$

where C is a positive constant depending on the time horizon $T - \delta$ and Π_r is defined by

$$\Pi_r := \{y \in H \mid V_{T-\delta}(y) \leq r\}.$$

In the case of finite-dimensional controlled systems, the above condition was investigated in [136]. However for infinite-dimensional controlled systems, this condition holds only in special cases [58]. For instance, we will see in the next chapter that the conditions of type (UPV) hold globally for the optimal control problems governed by the linear wave equation and incremental functions of the form (3.5), see Lemmas 3.2.2, 3.3.3, and 3.4.2. Hence, for these cases the inequality (2.26) leads to the exponential stability of RHC defined by Algorithm 2.1. On the other hand, such an inequality of the form (UPV) does not hold for optimal control problems governed by parabolic PDE's, specifically not for the Burgers equation and the incremental functions of the form (4.5). As an example, consider the optimal control for the heat equation (linearized Burgers equation) defined on the open interval $(0, 1)$ with the incremental function defined by (4.5). It is known that V_T for this problem has the form

$$V_T(y) = \langle y, \Lambda_T y\rangle_{L^2(0,1)} \quad \text{for all } y \in L^2(0, 1), \tag{2.31}$$

where the linear operator Λ_T is the solution of the differential Riccati equation. One can show that the operator Λ_T is compact and, as a consequence, zero is the only accumulation point of the spectrum of Λ_T. Therefore, in Chapter 4 we need to show by different techniques that the receding horizon state y_{rh} tends to the origin as $t \to \infty$. See Theorems 4.2.1 and 4.3.1.

Theorem 2.2.3. *Suppose that for the finite horizon value function $V_T : H \to \mathbb{R}_+$ and the receding horizon trajectory $y_{rh} : \mathbb{R}_+ \to H$, the following assumptions hold:*

1. $\lim_{t\to\infty} V_T(y_{rh}(t)) = 0$.
2. *For all $y \in H$ such that $V_T(y) = 0$, we have $y = 0$.*
3. *V_T is weakly lower semi-continuous.*
4. *$\|y_{rh}(t)\|_H \leq C$ for all $t \geq 0$.*

Then we will show that $y_{th}(t) \rightharpoonup 0$ *as* $t \to 0$. *In other word, we will show the origin is weakly stable.*

Proof. Assume that $\{t_n\}_n$ is an arbitrary sequence of positive numbers such that $t_n \to \infty$ as $n \to \infty$. From Assumption 4, the sequence $y_{rh}(t_n)$ is uniformly bounded in H. Therefore it has at least a weak accumulation point. Moreover, let $\hat{y}$ be an arbitrary weak accumulation point of $\{y_{rh}(t_n)\}_n$, then there exists a subsequence $\{y_{rh}(t_{n_k})\}_k$ which converges weakly to $\hat{y}$. Now by using Assumptions 2 and 3 we have

$$0 \le V_T(\hat{y}) \le \liminf_{k\to\infty} V_T(y_{rh}(t_{n_k})) = 0, \tag{2.32}$$

and by using Assumption 1 we have

$$\hat{y} = 0. \tag{2.33}$$

Since 0 is the unique weak accumulation point of the sequence $\{y_{rh}(t_n)\}_n$, we obtain

$$y_{rh}(t_n) \rightharpoonup 0 \quad \text{in } H. \tag{2.34}$$

Since the sequence $\{t_n\}$ is arbitrary, we are finished with the proof. □

Chapter 3

On the global Stabilizability of the Wave Equation via RHC

3.1 Introduction

In this chapter we apply the receding horizon framework which has been introduced in Chapter 2 for the stabilization of the linear wave equation

$$\ddot{y} - \Delta y = 0,$$

where $y = y(t,x)$ is a real valued function of real variables t and x. Moreover, the double dots above the y stand for a second derivative with respect to time. Our RHC acts on a part of either the domain or Dirichlet boundary conditions or Neumann boundary conditions. The stabilization problem for the wave equation has been studied extensively by many authors, see for instance [5, 68, 71, 90, 94, 104, 112, 130] and the references cited therein. In these contributions the stabilization problem is obrained by means of a proper choice of a feedback control law. We peruse the control law computed by Algorithm 2.1 which rests on the solutions of a sequence of open-loop optimal control problems governed by the wave equation on finite intervals. To study the open-loop optimal control problems for the wave equation, numerically and analytically, we refer to [42, 69, 70, 85, 87, 88, 110]. Moreover we investigate the suboptimality and *exponential stability* of RHC for all the mentioned control types with respect to appropriate functional analytic settings. For all the cases, the key conditions are the observability conditions. By help of these conditions, we obtain not just the asymptotic stability but also the *exponential stability* of RHC.

To be more precise, we are concerned with minimizing an infinite horizon performance index

$$J_\infty(u;(y_0^1,y_0^2)) := \int_0^\infty \ell((y(t),\dot{y}(t)),u(t))dt \tag{3.1}$$

over all controls $u \in L^2(0,\infty;\mathcal{U})$ with an appropriate control space $\mathcal{U}$ and subject to the following cases:

1. **Distributed control:** In this case we consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = Bu & \text{in } (0,\infty) \times \Omega, \\ y = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega, \end{cases} \tag{3.2}$$

   where $\Omega \in \mathbb{R}^n$ is a bounded domain with the smooth boundary $\partial\Omega$, the control u is active on a nonempty open subset of Ω, and the control operator B is an extension-by-zero operator.

2. **Dirichlet control:** In this case the control acts on a part of Dirichlet boundary conditions

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,\infty) \times \Omega, \\ y = u & \text{on } (0,\infty) \times \Gamma_c, \\ y = 0 & \text{on } (0,\infty) \times \Gamma_0, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega, \end{cases} \tag{3.3}$$

   where, similar to the above case, $\Omega \in \mathbb{R}^n$ is a bounded domain with the smooth boundary $\partial\Omega$. Moreover, the two disjoint components Γ_c, Γ_0 are relatively open in $\partial\Omega$ and $int(\Gamma_c) \neq \emptyset$.

3. **Neumann control:** In this case, we are dealing with the following one-dimensional wave equation with a Neumann control action at one side of boundary

$$\begin{cases} \ddot{y} - y_{xx} = 0 & (t,x) \in (0,\infty) \times (0,L), \\ y(t,0) = 0 & t \in (0,\infty), \\ y_x(t,L) = u(t) & t \in (0,\infty), \\ y(0,x) = y_0^1, \quad \dot{y}(0,x) = y_0^2 & x \in (0,L), \end{cases} \tag{3.4}$$

   where $L > 0$.

By denoting $(y(t), \dot{y}(t))$ by $\mathcal{Y}(t)$, and choosing an appropriate control space $\mathcal{U}$, each controlled system in the above cases can be rewritten as the following first order controlled system in an abstract Hilbert space $\mathcal{H}$

$$\begin{cases} \dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{B}u & t \in (0,\infty), \\ \mathcal{Y}(0) = \mathcal{Y}_0 := (y_0^1, y_0^2), \end{cases} \tag{AP}$$

with an unbounded operator $\mathcal{A}$ and a control operator $\mathcal{B}$ which are defined according to the boundary conditions of the above cases, see, e.g., [92, 126, 132]. Now we can reformulate our infinite horizon problem as the following problem

$$\min\{J_\infty(u;\mathcal{Y}_0) \mid (\mathcal{Y},u) \text{ satisfies (AP)}, u \in L^2(0,\infty;\mathcal{U})\}. \tag{OP_∞}$$

The incremental function $\ell : \mathcal{H} \times \mathcal{U} \to \mathbb{R}_+$ is given by

$$\ell(\mathcal{Y},u) := \frac{1}{2}\|\mathcal{Y}\|_{\mathcal{H}}^2 + \frac{\beta}{2}\|u\|_{\mathcal{U}}, \tag{3.5}$$

where β is a positive constant.

To deal with the infinite horizon problem (OP_∞), one can employ the algebraic Riccati operator, see, e.g., [72, 93, 95]. But for the case of infinite-dimensional controlled systems, discretization leads to finite-dimensional Riccati equations of very large order and ultimately one is confronted with the curse of dimensionality. Model reduction techniques do not offer an efficient alternative either. In fact, the transfer function corresponding to the controlled system (3.2)-(3.4) has infinitely many unstable poles, and thus the model reduction based on balanced truncation will not produce finite $H_\infty-$error bounds, see e.g., [47]. For the sake of consistency in presentation, we reformulate Algorithm 2.1 for the problem OP_∞ and summarize the corresponding steps in Algorithm 3.1.

Algorithm 3.1 Receding Horizon Algorithm

Require: Let the prediction horizon T, the sampling time $\delta < T$, and the initial point $(y_0^1, y_0^2) \in \mathcal{H}$ be given. Then we proceed through the following steps:

1: $k := 0, \quad t_0 := 0$ and $\mathcal{Y}_{rh}(t_0) := (y_0^1, y_0^2)$.

2: Find the optimal pair $(\mathcal{Y}_T^*(\cdot; \mathcal{Y}_{rh}(t_k), t_k), u_T^*(\cdot; \mathcal{Y}_{rh}(t_k), t_k))$ over the time horizon $[t_k, t_k + T]$ by solving the finite horizon open-loop problem

$$\min_{u \in L^2(t_k, t_k+T; \mathcal{U})} J_T(u; \mathcal{Y}_{rh}(t_k)) := \min_{u \in L^2(t_k, t_k+T; \mathcal{U})} \int_{t_k}^{t_k+T} \ell(\mathcal{Y}(t), u(t))dt,$$

$$\text{subject to } \begin{cases} \dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{B}u & t \in (t_k, t_k + T) \\ \mathcal{Y}(t_k) = \mathcal{Y}_{rh}(t_k) \end{cases}$$

3: Set

$$\begin{aligned} u_{rh}(\tau) &:= u_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta), \\ \mathcal{Y}_{rh}(\tau) &:= \mathcal{Y}_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta], \\ t_{k+1} &:= t_k + \delta, \\ k &:= k + 1. \end{aligned}$$

4: Go to step 2.

The rest of this chapter is organized as follows: Sections 3.2, 3.3, and 3.4 deal, respectively, with the cases in which RHC enters as a distributed control, a Dirichlet boundary condition, and a Neumann boundary condition. In each of these sections, first well-posedness of the finite horizon optimal control problems are investigated, and then the suboptimality and exponential stability of RHC are analysed. Finally, in Section 3.5 we demonstrate numerical experiments in which Algorithm 3.1 is implemented for every type of the control actions. In addition, for each example the performance of RHC is evaluated and compared for different choices of the prediction horizon T and a fixed sampling time δ.

3.2 Distributed Control

In this section we consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = Bu & \text{in } (0,\infty)\times\Omega, \\ y = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega. \end{cases} \tag{3.6}$$

Here $\Omega \in \mathbb{R}^n$ is a bounded domain with the smooth boundary $\partial\Omega$ and the control operator B is the extension-by-zero operator defined by

$$(Bu)(x) := \begin{cases} u(x) & x \in \omega, \\ 0 & x \in \Omega\backslash\omega, \end{cases}$$

where the control domain ω is a nonempty open subset of Ω. We define $\mathcal{H}_1 := H_0^1(\Omega) \times L^2(\Omega)$, $\mathcal{U} := L^2(\omega)$, and the energy $\mathcal{E}(\cdot, y)$ along a trajectory y by

$$\mathcal{E}(t,y) := \|y(t)\|^2_{H_0^1(\Omega)} + \|\dot{y}(t)\|^2_{L^2(\Omega)}. \tag{3.7}$$

The incremental function $\ell : \mathcal{H}_1 \times L^2(\omega) \to \mathbb{R}_+$ is given by

$$\ell((y,z),u) := \frac{1}{2}\|(y,z)\|^2_{\mathcal{H}_1} + \frac{\beta}{2}\|u\|^2_{L^2(\omega)}. \tag{3.8}$$

For simplicity, in some places we denote the pair $(y(t), \dot{y}(t))$ by $\mathcal{Y}(t)$ at every time $t > 0$. Similarly, $\mathcal{Y}_0$ stands for the initial pair (y_0^1, y_0^2).

3.2.1 Existence and uniqueness of the solution

Consider the following linear wave equation

$$\begin{cases} \ddot{y} - \Delta y = f & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega. \end{cases} \tag{3.9}$$

Definition 3.2.1 (Weak solution)**.** *Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_1$, and $f \in L^2(0,T;L^2(\Omega))$. Then $(y,\dot{y}) \in C^0([0,T];\mathcal{H}_1)$ with $\ddot{y} \in L^2(0,T;H^{-1}(\Omega))$ is referred to as the weak solution of* (3.9)*, if for almost every $t \in (0,T)$ we have*

$$\langle \ddot{y}(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\nabla y(t), \nabla v)_{L^2(\Omega)} = (f(t), v)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega), \tag{3.10}$$

and also

$$(y(0), \dot{y}(0)) = (y_0^1, y_0^2) \text{ in } \mathcal{H}_1.$$

Definition 3.2.2 (Very weak solution). *Let $T > 0$, $(y_0^1, y_0^2) \in L^2(\Omega) \times H^{-1}(\Omega)$, and $f \in L^2(0,T; H^{-1}(\Omega))$ be given. A function $y \in L^2(0,T; L^2(\Omega))$ is referred to as the very weak solution of* (3.9)*, if the following inequality holds*

$$\int_0^T (g(t), y(t))_{L^2(\Omega)}\, dt = \\ -(y_0^1, \dot{\vartheta}(0))_{L^2(\Omega)} + \langle y_0^2, \vartheta(0)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^T \langle f(t), \vartheta(t)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, dt,$$

for all $g \in L^2(0,T; L^2(\Omega))$, with ϑ the weak solution of the following backward in time problem

$$\begin{cases} \ddot{\vartheta} - \Delta\vartheta = g & \text{in } (0,T)\times\Omega, \\ \vartheta = 0 & \text{on } (0,T)\times\partial\Omega, \\ \vartheta(T) = 0, \quad \dot{\vartheta}(T) = 0 & \text{on } \Omega. \end{cases}$$

The very weak solution is also called solution by transposition.

We recall the following results for (3.9), see, e.g., [97, 99].

Theorem 3.2.1 (Existence and regularity of solutions). *We have the following existence and regularity results for* (3.9)*:*

1. *Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_1$, and $f \in L^2(0,T; L^2(\Omega))$ be given. Then there exist a unique weak solution y with $(y, \dot{y}) \in C^0([0,T]; \mathcal{H}_1)$ to* (3.9) *satisfying $(y(0), \dot{y}(0)) = (y_0^1, y_0^2)$ in $\mathcal{H}_1$. Moreover, for this weak solution the following estimate holds*

$$\begin{aligned} &\|y\|_{C^0([0,T];H_0^1(\Omega))} + \|\dot{y}\|_{C^0([0,T];L^2(\Omega))} + \|\ddot{y}\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\le c_1 \left(\|y_0^1\|_{H_0^1(\Omega)} + \|y_0^2\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))} \right), \end{aligned} \tag{3.11}$$

   where the constant c_1 is independent of y_0^1, y_0^2, and f. Moreover, for this weak solution we have the following hidden regularity

$$\partial_\nu y \in L^2(0,T; L^2(\partial\Omega)),$$

   and the corresponding estimate

$$\|\partial_\nu y\|_{L^2(0,T;L^2(\partial\Omega))} \le c_N \left(\|y_0^1\|_{H_0^1(\Omega)} + \|y_0^2\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))} \right),$$

   where the constant c_N is independent of y_0^1, y_0^2, and f.

2. *For every $T > 0$, $f \in L^2(0,T; H^{-1}(\Omega))$, and every pair $(y_0^1, y_0^2) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique very weak solution to* (3.9)*. Moreover, this very weak solution belongs to the space*

$$C^1([0,T]; H^{-1}(\Omega)) \cap C^0([0,T]; L^2(\Omega)),$$

   and we have the following estimate

$$\begin{aligned} &\|y\|_{C^0([0,T];L^2(\Omega))} + \|\dot{y}\|_{C^0([0,T];H^{-1}(\Omega))} \\ &\qquad \le \bar{c}_1 \left(\|y_0^1\|_{L^2(\Omega)} + \|y_0^2\|_{H^{-1}(\Omega)} + \|f\|_{L^2(0,T;H^{-1}(\Omega))} \right), \end{aligned} \tag{3.12}$$

   where the constant $\bar{c}_1$ is independent of y_0^1, y_0^2, and f.

3.2.2 Existence of the optimal control

Since, in Algorithm 3.1, the solution of (OP_∞) is approximated by solving a sequence of the finite horizon open-loop optimal controls, one needs to be sure that any of these optimal control problems in Step 2 of Algorithm 3.1 is well-defined. Each of these optimal control problems can be rewritten as minimizing the following performance index function

$$J_T(u;(y_0^1,y_0^2)) := \int_0^T \ell((y(t),\dot{y}(t)),u(t))dt \tag{3.13}$$

over all $u \in L^2(0,T;L^2(\omega))$, subject to

$$\begin{cases} \ddot{y} - \Delta y = Bu & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega, \end{cases} \tag{3.14}$$

where $(y_0^1,y_0^2) \in \mathcal{H}_1$.

Proposition 3.2.1 (Existence and uniqueness of the optimal control)**.** *For every $T > 0$ and $(y_0^1,y_0^2) \in \mathcal{H}_1$, the optimal control problem*

$$\min\left\{J_T(u;(y_0^1,y_0^2)) \mid (y,u) \text{ satisfies } (3.14), u \in L^2(0,T;L^2(\omega))\right\} \tag{OP_T}$$

admits a unique solution.

Proof. According to Theorem 3.2.1, for every control $u \in L^2(0,T;L^2(\omega))$, there exists a unique weak solution y to (3.14) with $(y,\dot{y}) \in C^0([0,T];\mathcal{H}_1)$. As a result, the set of admissible controls is nonempty and by (3.8) we have

$$J_T(u;(y_0^1,y_0^2)) \geq \frac{\beta}{2}\|u\|^2_{L^2(0,T;L^2(\omega))}. \tag{3.15}$$

Let $((y^n,\dot{y}^n),u^n) \in C([0,T];\mathcal{H}_1) \times L^2(0,T;L^2(\omega))$ be a pair of minimizing sequences such that

$$\lim_{n\to\infty} J_T(u^n;(y_0^1,y_0^2)) = \sigma,$$

where y^n is the unique weak solution corresponding to u^n. By (3.11), (3.15), and due to the structure of ℓ, the set $\{((y^n,\dot{y}^n),u^n)\}_n$ is bounded in $C^0([0,T];\mathcal{H}_1)\times L^2(0,T;L^2(\omega))$. Therefore there exist subsequences y^n,$\dot{y}^n$, $\ddot{y}^n$, and u^n such that

$$\begin{aligned} y^n &\rightharpoonup^* y^* \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ \dot{y}^n &\rightharpoonup^* \dot{y}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \\ \ddot{y}^n &\rightharpoonup \ddot{y}^* \text{ in } L^2(0,T;H^{-1}(\Omega)), \\ u^n &\rightharpoonup u^* \text{ in } L^2(0,T;L^2(\omega)), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} y^* &\in L^\infty(0,T;H_0^1(\Omega)),\\ \dot{y}^* &\in L^\infty(0,T;L^2(\Omega)),\\ \ddot{y}^* &\in L^2(0,T;H^{-1}(\Omega)),\\ u^* &\in L^2(0,T;L^2(\omega)). \end{aligned}$$

Now it remains to show that y^* is the weak solution to (3.14) corresponding to the control u^*. By the definition of weak and weak-star convergence, for almost every $t \in [0,T]$ and for every $v \in H_0^1(\Omega)$, we have

$$\begin{aligned} \langle \ddot{y}^n(t) - \ddot{y}^*(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &\to 0,\\ (\nabla y^n(t) - \nabla y^*(t), \nabla v)_{L^2(\Omega)} &\to 0,\\ (Bu^n(t) - Bu^*(t), v)_{L^2(\Omega)} &\to 0. \end{aligned} \tag{3.17}$$

Due to the fact that $(y^*(0), \dot{y}^*(0)) \in \mathcal{H}_1$ and using (3.17) and (3.10) with $f = Bu$, we conclude that y^* is the weak solution to (3.14) corresponding to u^*. Since $u^n \to u^*$ weakly in $L^2(0,T;L^2(\omega))$ and $J_T(\cdot;(y_0^1,y_0^2))$ is weakly lower semi-continuous we have

$$0 \le J_T(u^*;(y_0^1,y_0^2)) \le \liminf_{n\to\infty} J_T(u^n;(y_0^1,y_0^2)) = \sigma,$$

and, as a consequence, the pair (y^*, u^*) is optimal. Uniqueness follows from the strict convexity of $J_T(\cdot;(y_0^1,y_0^2))$. □

3.2.3 Optimality conditions

Lemma 3.2.1. *Consider the following linear wave equation*

$$\begin{cases} \ddot{y} - \Delta y = f & \textit{in } (0,T)\times\Omega,\\ y = 0 & \textit{on } (0,T)\times\partial\Omega,\\ y(0) = 0, \quad \dot{y}(0) = 0 & \textit{on } \Omega, \end{cases} \tag{3.18}$$

with a forcing function $f \in L^2(0,T;L^2(\Omega))$. Moreover, let $g \in L^2(0,T;H^{-1}(\Omega))$ and $(p_T^1, p_T^2) \in L^2(\Omega)\times H^{-1}(\Omega)$. Then the weak solution to (3.18) *and the very weak solution p to*

$$\begin{cases} \ddot{p} - \Delta p = g & \textit{in } (0,T)\times\Omega,\\ p = 0 & \textit{on } (0,T)\times\partial\Omega,\\ p(T) = p_T^1, \quad \dot{p}(T) = p_T^2 & \textit{on } \Omega, \end{cases} \tag{3.19}$$

satisfy the following equality

$$\begin{aligned} &\int_0^T (f(t), p(t))_{L^2(\Omega)}\,dt\\ &= \int_0^T \langle g(t), y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt + (p_T^1, \dot{y}(T))_{L^2(\Omega)} - \langle p_T^2, y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \end{aligned} \tag{3.20}$$

Proof. First, due to Theorem 3.2.1 and the time reversibility of the linear wave equation, the solution p to (3.19) belongs to the space

$$C^1([0,T];H^{-1}(\Omega)) \cap C^0([0,T];L^2(\Omega)).$$

Moreover, equality (3.20) can be first established for a smooth solution of (3.19) by integration by parts and the Green formula. Moreover, for $(g, p_T^1, p_T^2) \in L^2(0,T;L^2(\Omega)) \times \mathcal{H}_1$ the solution to (3.19) belongs to the space $C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ (see, e.g., [97, 99]), and the spaces $L^2(0,T;L^2(\Omega))$ and $\mathcal{H}_1$ are dense in the spaces $L^2(0,T;H^{-1}(\Omega))$ and $L^2(\Omega) \times H^{-1}(\Omega)$, respectively. Next, (3.20) is derived by using density arguments and passing to the limit which is justified due to estimate (3.12). □

In the following, we derive the first-order optimality conditions for the finite horizon problems of the form (OP_T). Due to the presence of the tracking term for the velocity $\dot{y}(\cdot)$ in the performance index function (3.13), we will see that the solution of ajdoint equation exists in the very weak sense.

Theorem 3.2.2 (First-order optimality conditions)**.** *Let $(\bar{y}, \bar{u})$ be the optimal solution to (OP_T). Then for $(y_0^1, y_0^2) \in \mathcal{H}_1$ we have the following optimality conditions*

$$\begin{cases} \ddot{\bar{y}} - \Delta \bar{y} = B\bar{u} & \text{in } (0,T) \times \Omega, \\ \bar{y} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \bar{y}(0) = y_0^1, \quad \dot{\bar{y}}(0) = y_0^2 & \text{on } \Omega, \\ \ddot{\bar{p}} - \Delta \bar{p} = -\ddot{\bar{y}} - \Delta \bar{y} & \text{in } (0,T) \times \Omega, \\ \bar{p} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \bar{p}(T) = 0, \quad \dot{\bar{p}}(T) = -\dot{\bar{y}}(T) & \text{on } \Omega, \\ \beta \bar{u} = -B^* \bar{p} & \text{in } (0,T) \times \Omega, \end{cases}$$

where $p \in C^1([0,T];H^{-1}(\Omega)) \cap C^0([0,T];L^2(\Omega))$ is the solution of the adjoint equation.

Proof. For sake of simplicity in notation, we remove the overbar in the notation of $(\bar{y}, \bar{u})$. Let $(y_0^1, y_0^2) \in \mathcal{H}_1$ be given. Computing the directional derivative of $J_T(\cdot,(y_0^1, y_0^2))$ at u in the direction of an arbitrary $\delta u \in L^2(0,T;L^2(\omega))$ we obtain

$$\begin{aligned} & J_T'(u,(y_0^1,y_0^2))\delta u \\ &= \int_0^T (y(t), \delta y(t))_{H_0^1(\Omega)} dt + \int_0^T (\dot{y}(t), \dot{\delta y}(t))_{L^2(\Omega)} dt + \beta \int_0^T (u(t), \delta u(t))_{L^2(\omega)} dt, \\ &= \int_0^T \langle -\Delta y(t), \delta y(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt \\ &+ \int_0^T (\dot{y}(t), \dot{\delta y}(t))_{L^2(\Omega)} dt + \beta \int_0^T (u(t), \delta u(t))_{L^2(\omega)} dt, \end{aligned} \tag{3.21}$$

where $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Laplace operator with homogeneous Dirichlet boundary conditions and it is an isomorphism. Moreover $\delta y \in C^0([0,T];H_0^1(\Omega)) \cap$

$C^1([0,T];L^2(\Omega))$ is the weak solution of

$$\begin{cases} \ddot{\delta y} - \Delta \delta y = B\delta u & \text{in } (0,T)\times\Omega, \\ \delta y = 0 & \text{on } (0,T)\times\partial\Omega, \\ \delta y(0) = 0, \quad \dot{\delta y}(0) = 0 & \text{on } \Omega. \end{cases} \tag{3.22}$$

Next, we show that

$$\begin{aligned} &\int_0^T (\dot y(t), \dot{\delta y}(t))_{L^2(\Omega)} dt \\ &= \langle \dot y(T), \delta y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_0^T \langle \ddot y(t), \delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt. \end{aligned} \tag{3.23}$$

We proceed with the help of an approximation argument. Since the spaces $H^2(\Omega)\cap H_0^1(\Omega)$, $H_0^1(\Omega)$, and $L^2(0,T;H_0^1(\Omega))$ are dense in the spaces $H_0^1(\Omega)$, $L^2(\Omega)$, and $L^2(0,T;L^2(\Omega))$, respectively, there exist sequences $\{y_0^{1n}\}_n \subset H^2(\Omega)\cap H_0^1(\Omega)$, $\{y_0^{2n}\}_n \subset H_0^1(\Omega)$, and $\{f^n\}_n \subset L^2(0,T;H_0^1(\Omega))$ such that

$$\begin{aligned} y_0^{1n} &\to y_0^1 && \text{in } H_0^1(\Omega), \\ y_0^{2n} &\to y_0^2 && \text{in } L^2(\Omega), \\ f^n &\to Bu && \text{in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

Moreover, for any triple $(y_0^{1n}, y_0^{2n}, f^n) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times L^2(0,T;H_0^1(\Omega))$, the solution of y^n of (3.9) belongs to the space $C^0([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];H_0^1(\Omega))$ with $\ddot{y^n} \in L^2(0,T;L^2(\Omega))$ (see, e.g., [97, 99]), and due to (3.11) we have

$$\begin{aligned} &\|y^n - y\|_{C^0([0,T];H_0^1(\Omega))} + \|\dot{y^n} - \dot y\|_{C^0([0,T];L^2(\Omega))} + \|\ddot{y^n} - \ddot y\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\le c_1 \left(\|y_0^{1n} - y_0^1\|_{H_0^1(\Omega)} + \|y_0^{2n} - y_0^2\|_{L^2(\Omega)} + \|f^n - Bu\|_{L^2(0,T;L^2(\Omega))} \right). \end{aligned}$$

For the solution y^n of (3.9) with $(y_0^{1n}, y_0^{2n}, f^n) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times L^2(0,T;H_0^1(\Omega))$ and the solution δy of (3.22) we have

$$\begin{aligned} \int_0^T (\dot{y^n}(t), \dot{\delta y}(t))_{L^2(\Omega)} dt = \int_0^T \langle \dot{y^n}(t), \dot{\delta y}(t)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} dt = \\ (\dot{y^n}(T), \delta y(T))_{L^2(\Omega)} - \int_0^T \langle \ddot{y^n}(t), \delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt. \end{aligned}$$

By passing the limits we obtain

$$\begin{aligned} \int_0^T (\dot{y^n}(t), \dot{\delta y}(t))_{L^2(\Omega)} dt &\to \int_0^T (\dot y(t), \dot{\delta y}(t))_{L^2(\Omega)} dt, \\ (\dot{y^n}(T), \delta y(T))_{L^2(\Omega)} &\to (\dot y(T), \delta y(T))_{L^2(\Omega)} = \langle \dot y(T), \delta y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \\ \int_0^T \langle \ddot{y^n}(t), \delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt &\to \int_0^T \langle \ddot y(t), \delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt, \end{aligned}$$

and we are finished with the justification of (3.23). Now due to (3.21) and (3.23), the first order optimality condition is equivalent to the following equality

$$\begin{aligned}&\int_0^T \langle -\ddot{y}(t) - \Delta y(t), \delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt \\ &+ \langle \dot{y}(T), \delta y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \beta \int_0^T (u(t), \delta u(t))_{L^2(\omega)} dt = 0.\end{aligned} \tag{3.24}$$

Moreover, due to Lemma 3.2.1 and using equality (3.20) for equation (3.22) we have

$$\begin{aligned}&\int_0^T \langle g(t), \delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt \\ &+ (p_T^1, \dot{\delta y}(T))_{L^2(\Omega)} - \langle p_T^2, \delta y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_0^T (p(t), B\delta u)_{L^2(\Omega)}\, dt = 0,\end{aligned} \tag{3.25}$$

for a given $(g, p_T^1, p_T^2) \in L^2(0, T; H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega)$ and its corresponding very weak solution $p \in C^1([0, T]; H^{-1}(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ to (3.19). By comparing (3.24) with (3.25) and since $\delta u \in L^2(0, T; L^2(\omega))$ is arbitrary, we infer that

$$\begin{aligned}\beta u &= -B^* p && \text{in } L^2(0, T; L^2(\omega)),\\ p_T^1 &= 0 && \text{in } L^2(\Omega),\\ p_T^2 &= -\dot{y}(T) && \text{in } H^{-1}(\Omega),\\ g &= -\ddot{y} - \Delta y && \text{in } L^2(0, T; H^{-1}(\Omega)).\end{aligned}$$

□

3.2.4 Stabilizability

In this subsection we recall some aspects on the stabilizability of the wave equation with a distributed feedback law. Specifically, we consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = u(y) & \text{in } (0, \infty) \times \Omega,\\ y = 0 & \text{on } (0, \infty) \times \partial\Omega,\\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega, \end{cases} \tag{3.26}$$

with the feedback control u given by $u(y) := -a(x)\dot{y}$, where the function $a \in L^\infty(\Omega)$ satisfies

$$a_1 \ge a(x) \ge a_0 > 0 \text{ for almost every } x \in \omega, \text{ and } a(x) = 0 \text{ in } \Omega\backslash\omega. \tag{3.27}$$

The following observability conditions will be used later.

To specify the required *observability conditions*, for any $(\phi_0^1, \phi_0^2) \in \mathcal{H}_1$ we denote by ϕ the weak solution of the following system

$$\begin{cases} \ddot{\phi} - \Delta\phi = 0 & \text{in } (0, T) \times \Omega,\\ \phi = 0 & \text{on } (0, T) \times \partial\Omega,\\ \phi(0) = \phi_0^1, \quad \dot{\phi}(0) = \phi_0^2 & \text{on } \Omega. \end{cases} \tag{3.28}$$

Then we have the following observability conditions:

OB1. There exists $T_{ob1} > 0$ such that for every $T \geq T_{ob1}$, the weak solution ϕ to (3.28) with $(\phi, \dot{\phi}) \in C([0, T]; \mathcal{H}_1)$ satisfies the inequality

$$c_{ob1} \|(\phi_0^1, \phi_0^2)\|^2_{\mathcal{H}_1} \leq \int_0^T \int_\omega |\dot{\phi}|^2 dx dt \quad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_1,$$

where the positive constant c_{ob1} depends only on T and $\omega \subseteq \Omega$.

OB2. There exists $T_{ob2} > 0$ such that for every $T \geq T_{ob2}$, the weak solution ϕ to (3.28) with $(\phi, \dot{\phi}) \in C([0, T]; \mathcal{H}_1)$ satisfies the inequality

$$c_{ob2} \|(\phi_0^1, \phi_0^2)\|^2_{\mathcal{H}_1} \leq \int_0^T \int_{\Gamma_c} |\partial_\nu \phi|^2 dS dt \quad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_1,$$

where the positive constant c_{ob2} depends only on T and $\Gamma_c \subseteq \partial\Omega$.

The observability conditions OB1-OB2 are satisfied if and only if the Geometric Control Conditions (GCC) hold (see, for instance, Bardos, Lebeau, and Rauch [19] and Burq and Gérard [35]). Roughly speaking, GCC for $(\Omega, \omega, T_{ob1})$ (resp. $(\Omega, \Gamma_c, T_{ob2})$) states that all rays of the geometric optics should enter in the domain ω (resp. meet the boundary Γ_c) in a time smaller than T_{ob1} (resp. T_{ob2}). For a comprehensive study, we refer to Reference [19].

The following equivalence is frequently claimed in the literature. Since it is not straight forward to find a proof, we provide the arguments here.

Theorem 3.2.3 (Global stabilizability)**.** *Let* $(y_0^1, y_0^2) \in \mathcal{H}_1$ *and* $a \in L^\infty(\Omega)$ *satisfying* (3.27) *be given. Then for the controlled system* (3.26) *with the feedback law* $u(y) := -a\dot{y}$ *we have*

$$\mathcal{E}(t, y) \leq M e^{-\alpha t} \mathcal{E}(0, y) = M e^{-\alpha t} \|(y_0^1, y_0^2)\|^2_{\mathcal{H}_1} \tag{3.29}$$

for positive constants M*, and* α *independent of* (y_0^1, y_0^2)*, if and only if the observability condition OB1 holds.*

Proof. First we show that OB1 implies the exponential stabilizability. We set $u(y) := -a\dot{y}$ in (3.26). In this case the resulting closed-loop system is well-posed (see, e.g., [40]), and for its unique weak solution we have

$$y \in C^0([0, \infty); H_0^1(\Omega)) \cup C^1([0, \infty); L^2(\Omega))$$

Now, for an arbitrary $T > 0$ consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = -a\dot{y} & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega. \end{cases} \tag{3.30}$$

For the initial data $(y_0^1, y_0^2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, the strict solution of (3.30) belongs to the space $C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega))$. By taking L^2-inner product of (3.30) with $\dot{y}$, and integrating over $[0,T]$, we obtain the following estimate

$$\mathcal{E}(T,y) - \mathcal{E}(0,y) \leq -2a_0 \int_0^T \|\dot{y}(t)\|_{L^2(\omega)}^2 dt. \tag{3.31}$$

Now by using a density argument and the fact that $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ is dense in $H_0^1(\Omega) \times L^2(\Omega)$, it can be shown that the inequality (3.31) is also true for the weak solution of (3.30) with the initial data $(y_0^1, y_0^2) \in \mathcal{H}_1$.

Moreover the solution y of (3.30) can be expressed as $y := \psi + \phi$ where $\phi \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega))$ is the weak solution to (3.28) with $(\phi_0^1, \phi_0^2) = (y_0^1, y_0^2)$, and $\psi \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega))$ is the weak solution of

$$\begin{cases} \ddot{\psi} - \Delta\psi = -a\dot{y} & \text{in } (0,T) \times \Omega, \\ \psi = 0 & \text{on } (0,T) \times \partial\Omega, \\ \psi(0) = 0, \quad \dot{\psi}(0) = 0 & \text{on } \Omega. \end{cases} \tag{3.32}$$

Now by using the observability condition OB1, and estimate (3.11) for (3.32) we have

$$\begin{aligned} \mathcal{E}(0,y) = \|(y_0^1, y_0^2)\|_{\mathcal{H}_1}^2 &\leq \frac{1}{c_{ob1}} \int_0^{T_{ob1}} \|\dot{\phi}(t)\|_{L^2(\omega)}^2 dt \\ &\leq \frac{1}{c_{ob1}} \int_0^{T_{ob1}} \left(\|\dot{y}(t)\|_{L^2(\omega)}^2 + \|\dot{\psi}(t)\|_{L^2(\omega)}^2 \right) dt \\ &\leq c_1' \int_0^{T_{ob1}} \|\dot{y}(t)\|_{L^2(\omega)}^2 dt, \end{aligned} \tag{3.33}$$

for a constant $c_1' > 0$ which is independent of (y_0^1, y_0^2). By (3.31), (3.33) we obtain

$$\mathcal{E}(T_{ob1}, y) - \mathcal{E}(0,y) \leq -2a_0 \int_0^{T_{ob1}} \|\dot{y}(t)\|_{L^2(\omega)}^2 dt \leq -\frac{2a_0}{c_1'} \mathcal{E}(0,y) \leq -\frac{2a_0}{c_1'} \mathcal{E}(T_{ob1}, y).$$

Setting $\alpha := \frac{\ln(1+\frac{2a_0}{c_1'})}{T_{ob1}}$, we have for every $k = 1, 2, \ldots$

$$\mathcal{E}(kT_{ob1}, y) \leq e^{-\alpha T_{ob1}} \mathcal{E}((k-1)T_{ob1}, y),$$

and as a consequence for every $t \in [kT_{ob1}, (k+1)T_{ob1}]$ we infer that

$$\begin{aligned} \mathcal{E}(t,y) &\leq \mathcal{E}(kT_{ob1}, y) \leq e^{-\alpha k T_{ob1}} \mathcal{E}(0,y) \\ &= (1 + \frac{2a_0}{c_1'}) e^{-\alpha(k+1)T_{ob1}} \mathcal{E}(0,y) \leq (1 + \frac{2a_0}{c_1'}) e^{-\alpha t} \mathcal{E}(0,y). \end{aligned}$$

Thus we conclude (3.29).

Next we show that the stabilizability property (3.29) implies the observability condition OB1 for (3.28) with an arbitrary initial pair $(y_0^1, y_0^2) \in \mathcal{H}_1$. First, by setting

$u(y) := -a\dot{y}$ in (3.26) with a function $a \in L^\infty(\Omega)$ satisfying (3.27), taking the L^2-inner product of (3.26) with $\dot{y}$, and integrating over $[0,t]$ for $t > 0$, we obtain

$$\mathcal{E}(t,y) - \mathcal{E}(0,y) \geq -2a_1 \int_0^t \|\dot{y}(t)\|^2_{L^2(\omega)} dt. \tag{3.34}$$

where the constant a_1 defined in (3.27). Further by (3.29), for a large enough $T' > 0$ we have

$$2a_1 \int_0^{T'} \|\dot{y}(t)\|^2_{L^2(\omega)} dt \geq \frac{1}{2}\mathcal{E}(0,y). \tag{3.35}$$

Moreover, the solution ϕ to (3.28) with the initial pair (y_0^1, y_0^2) can be rewritten as $\phi := y - \psi$, where y is the weak solution to (3.30) and ψ is the weak solution to (3.32) for T' instead of T.

Now assume that the solution of (3.32) is smooth enough. Taking the L^2-inner product of (3.32) with $\dot{\psi}$ and integrating over $[0,T']$ we have

$$\begin{aligned} 0 \leq \frac{1}{2}\big(\|\dot{\psi}(T')\|^2_{L^2(\Omega)} + \|\nabla\psi(T')\|^2_{L^2(\Omega)}\big) &= \int_0^{T'} \int_\Omega -a\dot{y}\dot{\psi}\, dx dt \\ &= \int_0^{T'} \int_\omega -a(\dot{\psi} + \dot{\phi})\dot{\psi}\, dx dt. \end{aligned} \tag{3.36}$$

Note that for a forcing function $f \in L^2(0,T'; H_0^1(\Omega))$ instead of $-a\dot{y}$, the weak solution of (3.32) belongs to the space $C^0([0,T']; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T']; H_0^1(\Omega))$, see, e.g., [97, 99]. By using a density argument and the fact that the space $L^2(0,T'; H_0^1(\Omega))$ is dense in $L^2(0,T'; L^2(\Omega))$, it can be shown that the inequality (3.36) is also true for the weak solution of (3.32) with $-a\dot{y}$ as a forcing function. Moreover, (3.36) implies

$$\int_0^{T'} \|\dot{\psi}(t)\|^2_{L^2(\omega)} dt \leq \frac{a_1^2}{a_0^2} \int_0^{T'} \|\dot{\phi}(t)\|^2_{L^2(\omega)} dt. \tag{3.37}$$

Note also that

$$\int_0^{T'} \|\dot{\phi}(t)\|^2_{L^2(\omega)} dt + \int_0^{T'} \|\dot{\psi}(t)\|^2_{L^2(\omega)} dt \geq \int_0^{T'} \|\dot{y}(t)\|^2_{L^2(\omega)} dt. \tag{3.38}$$

Combining (3.35), (3.37), and (3.38), we complete the proof with

$$\frac{1}{4a_1}\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_1} = \frac{1}{4a_1}\mathcal{E}(0,y) \leq (1 + \frac{a_1^2}{a_0^2}) \int_0^{T'} \|\dot{\phi}(t)\|^2_{L^2(\omega)} dt.$$

□

3.2.5 Stability of RHC

Definition 3.2.3 (Value function). *For every pair $(y_0^1, y_0^2) =: \mathcal{Y}_0 \in \mathcal{H}_1$, the infinite horizon value function $V_\infty : \mathcal{H}_1 \to \mathbb{R}_+$ is defined as*

$$V_\infty(\mathcal{Y}_0) := \min_{u \in L^2(0,\infty; L^2(\omega))} \{J_\infty(u, \mathcal{Y}_0) \text{ subject to } (3.6)\}.$$

Similarly, the finite horizon value function $V_T : \mathcal{H}_1 \to \mathbb{R}_+$ is defined by

$$V_T(\mathcal{Y}_0) := \min_{u \in L^2(0,T;L^2(\omega))} \{J_T(u, \mathcal{Y}_0) \text{ subject to } (3.14)\}. \tag{3.39}$$

Lemma 3.2.2. *Suppose that the observability condition OB1 holds. For every $T > 0$, there exists a control $\hat{u} \in L^2(0,T;L^2(\omega))$ for (3.14) such that*

$$V_T(\mathcal{Y}_0) \leq J_T(\hat{u};\mathcal{Y}_0) \leq \gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \tag{3.40}$$

for every initial pair $(y_0^1, y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_1$, where $\gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous, and bounded function. Moreover, there exists a constant $\gamma_1(T) > 0$ depending on T such that

$$V_T(\mathcal{Y}_0) \geq \gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \tag{3.41}$$

for all $(y_0^1, y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_1$.

Proof. Let $(y_0^1, y_0^2) \in \mathcal{H}_1$ be given. By setting $u(t) := -\dot{y}(t)|_\omega$ in the controlled system (3.14), and using Theorem 3.2.3 for the choice

$$a(x) := \begin{cases} 1 & x \in \omega, \\ 0 & \textit{otherwise}, \end{cases}$$

we obtain

$$\mathcal{E}(t,y) \leq Me^{-\alpha t}\mathcal{E}(0,y) \quad \text{ for all } t \in [0,T].$$

Here the constants M and α were defined in Theorem 3.2.3. By integrating from 0 to T we have

$$\int_0^T \mathcal{E}(t,y)dt \leq \frac{M}{\alpha}(1 - e^{-\alpha T})\mathcal{E}(0,y).$$

By the definition of the value function V_T in (3.39) we have

$$\begin{aligned} V_T((y_0^1, y_0^2)) \leq \int_0^T \left(\frac{1}{2}\mathcal{E}(t,y) + \frac{\beta}{2}\|\dot{y}(t)\|^2_{L^2(\omega)}\right) dt \leq & \frac{(1+\beta)M}{2\alpha}(1 - e^{-\alpha T})\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_1} \\ = & \gamma_2(T)\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_1}, \end{aligned}$$

thus (3.40) holds.

To verify (3.41), we use the superposition argument for equation (3.14) with an arbitrary control $u \in L^2(0,T;L^2(\omega))$. We rewrite the solution of (3.14) as $y = \phi + \varphi$ where ϕ is the solution to (3.28) with the initial pair (y_0^1, y_0^2) instead of (ϕ_0^1, ϕ_0^2), and φ is the solution to the following equation

$$\begin{cases} \ddot{\varphi} - \Delta\varphi = Bu & \text{in } (0,T) \times \Omega, \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega, \\ \varphi(0) = 0, \quad \dot{\varphi}(0) = 0 & \text{on } \Omega. \end{cases} \tag{3.42}$$

By using the observability condition OB1 for (3.28) with the initial pair (y_0^1, y_0^2) and Ω instead of ω, and using the estimate (3.11) for (3.42) we obtain

$$\begin{aligned}
\|(y_0^1, y_0^2)\|_{\mathcal{H}_1}^2 &\leq \frac{1}{c_{ob1}(T)} \int_0^T \|\dot{\phi}(t)\|_{L^2(\Omega)}^2 dt \\
&\leq \frac{1}{c_{ob1}(T)} \int_0^T \left(\|\dot{y}(t)\|_{L^2(\Omega)}^2 + \|\dot{\varphi}(t)\|_{L^2(\Omega)}^2 \right) dt \\
&\leq \frac{1}{c_{ob1}(T)} \int_0^T \left(\|\dot{y}(t)\|_{L^2(\Omega)}^2 + Tc_1^2 \|u(t)\|_{L^2(\omega)}^2 \right) dt \\
&\leq c_1''(T) \int_0^T \left(\frac{1}{2} \|(y(t), \dot{y}(t))\|_{\mathcal{H}_1}^2 + \frac{\beta}{2} \|u(t)\|_{L^2(\omega)}^2 \right) dt \\
&= c_1''(T) \int_0^T \ell(\mathcal{Y}(t), u(t)) dt.
\end{aligned}$$

Since $u \in L^2(0, T; L^2(\omega))$ is arbitrary, we obtain (3.41) for a constant $c_1''(T)$ independent of u and (y_0^1, y_0^2). □

Remark 3.2.1. The property (3.41) is equivalent to the injectivity of the differential Recatti operator corresponding to (OP_T) which in turn is equivalent to the observability condition OB1, see, [57, Theorem 3.3].

Remark 3.2.2. Note that, as it has been shown in Lemma 3.2.2, the observability condition OB1 is equivalent to the stabilizability condition (3.40). The stabilizability condition (3.40) and well-posedness (Proposition 3.2.1) of open-loop problems in the form (OP_T) are equivalent to the conditions (A2) and (A1) in Chapter 2, respectively. Moreover, since the stabilizability condition (3.40) holds globally, the condition (A3) is no longer needed and we can use the receding horizon framework introduced in Chapter 2. In addition, by using the uniform positiveness of the value function V_T which has been established in (3.41) based on the observability condition OB1, we shall verify the exponential stability of RHC (see Remark 2.2.3).

From this point on, we denote $(y(t), \dot{y}(t))$ by $\mathcal{Y}(t)$ and we define $\alpha_\ell := \frac{\min(1,\beta)}{2}$. Furthermore, for the sake of convenience in reading we recall some of the lemmas from Chapter 2.

Lemma 3.2.3. *If the observability condition OB1 holds and $T > \delta > 0$, then for every $\mathcal{Y}_0 := (y_0^1, y_0^2) \in \mathcal{H}_1$ the following inequalities hold*

$$\begin{aligned}
V_T(\mathcal{Y}_T^*(\delta; \mathcal{Y}_0, 0)) \leq &\int_\delta^{\tilde{t}} \ell(\mathcal{Y}_T^*(t; \mathcal{Y}_0, 0), u_T^*(t; \mathcal{Y}_0, 0)) dt \\
&+ \gamma_2(T + \delta - \tilde{t}) \|\mathcal{Y}_T^*(\tilde{t}; \mathcal{Y}_0, 0)\|_{\mathcal{H}_1}^2 \quad \textit{for all } \tilde{t} \in [\delta, T],
\end{aligned} \tag{3.43}$$

and

$$\int_{\tilde{t}}^T \ell(\mathcal{Y}_T^*(t; \mathcal{Y}_0, 0), u_T^*(t; \mathcal{Y}_0, 0)) dt \leq \gamma(T - \tilde{t}) \|\mathcal{Y}_T^*(\tilde{t}; \mathcal{Y}_0, 0)\|_{\mathcal{H}_1}^2 \quad \textit{for all } \tilde{t} \in [0, T]. \tag{3.44}$$

Proof. The proof is similar to the proof of Lemma 2.2.1. □

Lemma 3.2.4. *Suppose that the observability condition OB1 holds and let $\mathcal{Y}_0 \in \mathcal{H}_1$ be given. Then for the choice*

$$\theta_1 := 1 + \frac{\gamma_2(T)}{\alpha_\ell(T-\delta)}, \qquad \theta_2 := \frac{\gamma_2(T)}{\alpha_\ell \delta},$$

we have the following estimates

$$V_T(\mathcal{Y}_T^*(\delta; \mathcal{Y}_0, 0)) \leq \theta_1 \int_\delta^T \ell(\mathcal{Y}_T^*(t; \mathcal{Y}_0, 0), u_T^*(t; \mathcal{Y}_0, 0))dt, \tag{3.45}$$

and

$$\int_\delta^T \ell(\mathcal{Y}_T^*(t; \mathcal{Y}_0, 0), u_T^*(t; \mathcal{Y}_0, 0))dt \leq \theta_2 \int_0^\delta \ell(\mathcal{Y}_T^*(t; \mathcal{Y}_0, 0), u_T^*(t; \mathcal{Y}_0, 0))dt. \tag{3.46}$$

Proof. The proof is similar to the proof of Lemma 2.2.2. □

Proposition 3.2.2. *Suppose that the observability condition OB1 holds and let $\delta > 0$ be given. Then there exist $T^* > \delta$ and $\alpha \in (0,1)$ such that the following inequality is satisfied*

$$V_T(\mathcal{Y}_T^*(\delta; \mathcal{Y}_0, 0)) \leq V_T(\mathcal{Y}_0) - \alpha \int_0^\delta \ell(\mathcal{Y}_T^*(t; \mathcal{Y}_0, 0), u_T^*(t; \mathcal{Y}_0, 0))dt \tag{3.47}$$

for every $T \geq T^*$ and $\mathcal{Y}_0 \in \mathcal{H}_1$.

Proof. The proof is similar to the proof of Proposition 2.2.1. □

Theorem 3.2.4 (Suboptimality and exponential decay)**.** *Suppose that the observability condition OB1 holds and let a sampling time $\delta > 0$ be given. Then there exist numbers $T^* > \delta$ and $\alpha \in (0,1)$, such that for every fixed prediction horizon $T \geq T^*$ and every $\mathcal{Y}_0 \in \mathcal{H}_1$, the receding horizon control u_{rh} obtained from Algorithm 3.1 for the stabilization of* (3.6) *satisfies the suboptimality inequality*

$$\alpha V_\infty(\mathcal{Y}_0) \leq \alpha J_\infty(u_{rh}, \mathcal{Y}_0) \leq V_T(\mathcal{Y}_0) \leq V_\infty(\mathcal{Y}_0), \tag{3.48}$$

and exponential stability

$$\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}_1} \leq c' e^{-\zeta t} \|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \quad \text{for } t \geq 0, \tag{3.49}$$

where the positive numbers ζ and c' depend on α, δ, and T, but are independent of $\mathcal{Y}_0$.

Proof. To show the suboptimality inequality, we refer to the proof of Theorem 2.2.1 in Chapter 2. Now we turn to inequality (3.49). It is of interest to verify this inequality in two different ways. First, due to Theorem 2.2.2 in Chapter 2, there exists a $T^* > 0$ such that for every $T \geq T^*$ we have

$$V_{T-\delta}(\mathcal{Y}_{rh}(t)) \leq ce^{-\zeta_1 t} V_T(\mathcal{Y}_0) \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}_1, \quad t > 0, \tag{3.50}$$

where the constants c and ζ_1 are given by

$$\zeta_1 := \frac{\ln(1 + \frac{\alpha}{1+\theta_1\theta_2})}{\delta}, \qquad c := (1 + \frac{\alpha}{1+\theta_1\theta_2}).$$

Here $\theta_1(T,\delta), \theta_2(T,\delta)$ are defined as in Lemma 3.2.4. Using Lemma 3.2.2 and (3.50) we obtain

$$\gamma_1(T-\delta)\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}_1} \leq V_{T-\delta}(\mathcal{Y}_{rh}(t)) \leq ce^{-\zeta_1 t} V_T(\mathcal{Y}_0) \leq c\gamma_2(T)e^{-\zeta_1 t}\|\mathcal{Y}_0\|^2_{\mathcal{H}_1}$$

for every $\mathcal{Y}_0 \in \mathcal{H}_1$. Setting $c_1' := \frac{c\gamma_2(T)}{\gamma_1(T-\delta)}$ we have

$$\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}_1} \leq c_1' e^{-\zeta_1 t}\|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}_1, \quad t > 0,$$

and thus (3.49) holds.

Second way: by Proposition 3.2.2, there exist a $T^* > 0$ and $\alpha \in (0,1)$ such that for every $T \geq T^*$, $\mathcal{Y}_0 \in \mathcal{H}_1$, and $k \in \mathbb{N}$ with $k \geq 1$, we have

$$\begin{aligned} V_T(\mathcal{Y}_{rh}(t_k)) - V_T(\mathcal{Y}(t_{k-1})) &\leq -\alpha \int_{t_{k-1}}^{t_k} \ell(\mathcal{Y}_{rh}(t), u_{rh}(t))dt \\ &\leq -\alpha V_\delta(\mathcal{Y}_{rh}(t_{k-1})), \end{aligned} \tag{3.51}$$

where $t_k = k\delta$ for $k = 0, 1, 2, \ldots$. Moreover, due to Lemma 3.2.2, for every $\mathcal{Y}_0 \in \mathcal{H}_1$ we obtain

$$V_\delta(\mathcal{Y}_0) \geq \gamma_1(\delta)\|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \geq \frac{\gamma_1(\delta)}{\gamma_2(T)} V_T(\mathcal{Y}_0). \tag{3.52}$$

Using (3.51) and (3.52) we can write

$$V_T(\mathcal{Y}_{rh}(t_k)) \leq \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right) V_T(\mathcal{Y}_{rh}(t_{k-1})) \text{ for every } k \geq 1.$$

Since $0 < \gamma_1(\delta) \leq \gamma_2(\delta) \leq \gamma_2(T)$ and $\alpha \in (0,1)$, we have $\eta := \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right) \in (0,1)$. Furthermore, by defining $\zeta_2 := \frac{|\ln\eta|}{\delta}$ and using the similar argument as above we can infer that

$$\gamma_1(T)\|\mathcal{Y}_{rh}(t_k)\|^2_{\mathcal{H}_1} \leq V_T(\mathcal{Y}_{rh}(t_k)) \leq e^{-\zeta_2 t_k} V_T(\mathcal{Y}_0) \leq \gamma_2(T)e^{-\zeta_2 t_k}\|\mathcal{Y}_0\|^2_{\mathcal{H}_1}$$

for every $k \geq 1$. Hence, by setting $c'' := \frac{\gamma_2(T)}{\gamma_1(T)}$ we can write

$$\|\mathcal{Y}_{rh}(t_k)\|^2_{\mathcal{H}_1} \leq c'' e^{-\zeta_2 k\delta}\|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \text{ for every } k \geq 1. \tag{3.53}$$

Now we consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = Bu_{rh} & \text{in } (t_k, t_{k+1}) \times \Omega, \\ y = 0 & \text{on } (t_k, t_{k+1}) \times \partial\Omega, \\ (y(t_k), \dot{y}(t_k)) = (y_{rh}(t_k), \dot{y}_{rh}(t_k)) \text{ for } k > 0 & \text{on } \Omega, \\ (y(t_k), \dot{y}(t_k) = (y_0^1, y_0^2) \text{ for } k = 0 & \text{on } \Omega, \end{cases} \tag{3.54}$$

with the solution $y_{rh}(t)$ for $t \in [t_k, t_{k+1}]$. First we assume that the solution y_{rh} of (3.54) is smooth enough. Taking the L^2-inner product of (3.54) with $\dot{y}$ and integrating over $[t_k, t_{k+1}]$, we have for $t \in [t_k, t_{k+1}]$

$$\begin{aligned} \|\mathcal{Y}_{rh}(t)\|_{\mathcal{H}_1}^2 &\leq \|\mathcal{Y}_{rh}(t_k)\|_{\mathcal{H}_1}^2 + \int_{t_k}^{t} \|\dot{y}_{rh}(t)\|_{L^2(\Omega)}^2 dt + \int_{t_k}^{t} \|u_{rh}(t)\|_{L^2(\omega)}^2 dt \\ &\leq \|\mathcal{Y}_{rh}(t_k)\|_{\mathcal{H}_1}^2 + \frac{1}{\alpha_\ell} \int_{t_k}^{t_{k+1}} \ell(\mathcal{Y}_{rh}(t), u_{rh}(t)) dt \\ &\leq \|\mathcal{Y}_{rh}(t_k)\|_{\mathcal{H}_1}^2 + \frac{1}{\alpha_\ell} V_T(\mathcal{Y}_{rh}(t_k)) \\ &\leq \left(1 + \frac{\gamma_2(T)}{\alpha_\ell}\right) \|\mathcal{Y}_{rh}(t_k)\|_{\mathcal{H}_1}^2. \end{aligned} \tag{3.55}$$

Since for a forcing function $f \in L^2(t_k, t_{k+1}; H_0^1(\Omega))$ instead of $Bu_{rh} \in L^2(t_k, t_{k+1}; L^2(\Omega))$ and initial data $(y(t_k), \dot{y}(t_k)) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$, the weak solution of (3.54) belongs to the space $C^0([t_k, t_{k+1}]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([t_k, t_{k+1}]; H_0^1(\Omega))$ (see, e.g., [97, 99]), by using a density argument and the fact that $L^2(t_k, t_{k+1}; H_0^1(\Omega))$ and $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ are, respectively, dense in $L^2(t_k, t_{k+1}; L^2(\Omega))$ and $\mathcal{H}_1$, it can be shown that inequality (3.55) is also true for the weak solution of (3.54).

Now for every $t > 0$, there exist a $k \in \mathbb{N}_0$ such that $t \in [t_k, t_{k+1}]$. Combining (3.53) and (3.55), we have

$$\begin{aligned} \|\mathcal{Y}_{rh}(t)\|_{\mathcal{H}_1}^2 \leq \left(1 + \frac{\gamma_2(T)}{\alpha_\ell}\right) \|\mathcal{Y}_{rh}(t_k)\|_{\mathcal{H}_1}^2 &\leq \frac{c''(\alpha_\ell + \gamma_2(T))}{\alpha_\ell} e^{-\zeta_2 t_k} \|\mathcal{Y}_0\|_{\mathcal{H}_1}^2 \\ &\leq \frac{c''(\alpha_\ell + \gamma_2(T))}{\alpha_\ell} \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right)^{-1} e^{-\zeta_2 t_{k+1}} \|\mathcal{Y}_0\|_{\mathcal{H}_1}^2, \\ &\leq \frac{c''(\alpha_\ell + \gamma_2(T))}{\alpha_\ell} \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right)^{-1} e^{-\zeta_2 t} \|\mathcal{Y}_0\|_{\mathcal{H}_1}^2, \end{aligned}$$

and by setting $c_2'' := \frac{c'(\alpha_\ell+\gamma_2(T))}{\alpha_\ell} \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right)^{-1}$ we have (3.49). □

3.3 Dirichlet Boundary Control

In this section, we consider the case when the control acts on a part of Dirichlet boundary conditions. To be more precise, we deal with the controlled system of the form

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,\infty)\times\Omega, \\ y = u & \text{on } (0,\infty)\times\Gamma_c, \\ y = 0 & \text{on } (0,\infty)\times\Gamma_0, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega. \end{cases} \tag{3.56}$$

Here $\Omega \in \mathbb{R}^n$ is a bounded domain with the smooth boundary $\partial\Omega := \overline{\Gamma_c} \cup \overline{\Gamma_0}$, where the two disjoint components Γ_c, Γ_0 are relatively open in $\partial\Omega$ and $int(\Gamma_c) \neq \emptyset$. Moreover, by setting $\mathcal{U} := L^2(\Gamma_c)$ and $\mathcal{H}_2 := L^2(\Omega) \times H^{-1}(\Omega)$, we are searching over all control functions $u \in L^2(0,\infty;\mathcal{U})$ for a given initial pair $(y_0^1, y_0^2) \in \mathcal{H}_2$. For simplicity, we denote the pair $(y(t), \dot{y}(t))$ with $\mathcal{Y}(t)$ for $t \geq 0$ and the initial function $\mathcal{Y}_0 := (y_0^1, y_0^2)$. Moreover, let $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the Laplace operator with homogeneous Dirichlet boundary conditions, and define the operator $\mathcal{G} : H^{-1}(\Omega) \to H_0^1(\Omega)$ by $\mathcal{G} := (-\Delta)^{-1}$. Further, we denote the unique linear extension of $\mathcal{G}$ with $\overline{\mathcal{G}} : (H^2(\Omega) \cap H_0^1(\Omega))^* \to L^2(\Omega)$, where $(H^2(\Omega) \cap H_0^1(\Omega))^*$ stands for the dual space of $H^2(\Omega) \cap H_0^1(\Omega)$. The incremental function $\ell : \mathcal{H}_2 \times L^2(\Gamma_c) \to \mathbb{R}_+$ is given by

$$\ell((y,z),u) := \frac{1}{2}\|(y,z)\|_{\mathcal{H}_2}^2 + \frac{\beta}{2}\|u\|_{L^2(\Gamma_c)}^2. \tag{3.57}$$

Moreover, we will use the space $H_{\Gamma_0}^1(\Omega) := \{q \in H^1(\Omega) : q|_{\Gamma_0} = 0\}$ and the control operator B_{bd} which is defined by

$$(B_{bd}u)(x) := \begin{cases} u(x) & x \in \Gamma_c, \\ 0 & x \in \Gamma_0. \end{cases}$$

3.3.1 Existence and uniqueness of the solution

Consider the following linear wave equation with the inhomogeneous Dirichlet boundary condition imposed on the whole of the boundary

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T)\times\Omega, \\ y = h & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega. \end{cases} \tag{3.58}$$

Definition 3.3.1 (Very weak solution). *Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_2$, and $h \in L^2(0,T;L^2(\partial\Omega))$ be given. A function $y \in L^\infty(0,T;L^2(\Omega))$ is referred to as the very weak solution of* (3.58)*, if the following inequality holds*

$$\begin{aligned} &\int_0^T (f(t), y(t))_{L^2(\Omega)} dt \\ &\quad = -(y_0^1, \dot{\vartheta}(0))_{L^2(\Omega)} + \langle y_0^2, \vartheta(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_0^T (\partial_\nu \vartheta(t), h(t))_{L^2(\partial\Omega)}\, dt, \end{aligned} \tag{3.59}$$

for all $f \in L^1(0,T;L^2(\Omega))$, with ϑ the weak solution of the following backward in time problem

$$\begin{cases} \ddot{\vartheta} - \Delta\vartheta = f & \text{in } (0,T)\times\Omega, \\ \vartheta = 0 & \text{on } (0,T)\times\partial\Omega, \\ \vartheta(T) = 0, \quad \dot{\vartheta}(T) = 0 & \text{on } \Omega. \end{cases} \tag{3.60}$$

The very weak solution is also called solution by transposition. We have the following result for the very weak solution of (3.58)*, see, e.g., [97, 99].*

Theorem 3.3.1 (Existence and regularity of the very weak solution)**.** *For every $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_2$, and $h \in L^2(0,T;L^2(\partial\Omega))$, there exists a unique very weak solution to* (3.58) *belonging to the space*

$$C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)),$$

and satisfies the following estimate

$$\|y\|_{C^0([0,T];L^2(\Omega))} + \|\dot{y}\|_{C^0([0,T];H^{-1}(\Omega))} + \|\ddot{y}\|_{L^2(0,T;(H_0^1(\Omega)\cap H^2(\Omega))^*)} \\ \leq c_2 \left(\|y_0^1\|_{L^2(\Omega)} + \|y_0^2\|_{H^{-1}(\Omega)} + \|h\|_{L^2(0,T;L^2(\partial\Omega))}\right) \tag{3.61}$$

for a constant c_2 depending on T and the domain Ω.

3.3.2 Existence of the optimal control

In Step 2 of Algorithm 3.1, each finite horizon optimal control problem can be rewritten as minimizing the following performance index function

$$J_T(u;(y_0^1,y_0^2)) := \int_0^T \ell((y(t),\dot{y}(t)),u(t))dt \tag{3.62}$$

over all $u \in L^2(0,T;L^2(\Gamma_c))$, subject to

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T)\times\Omega, \\ y = u & \text{on } (0,T)\times\Gamma_c, \\ y = 0 & \text{on } (0,T)\times\Gamma_0, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega. \end{cases} \tag{3.63}$$

where $(y_0^1, y_0^2) \in \mathcal{H}_2$. We have the following existence result.

Proposition 3.3.1 (Existence and uniqueness of the optimal control)**.** *For every $T > 0$ and $(y_0^1, y_0^2) \in \mathcal{H}_2$, the optimal control problem*

$$\min\left\{J_T(u;(y_0^1,y_0^2)) \mid (y,u) \text{ satisfies } (3.63), u \in L^2(0,T;L^2(\Gamma_c))\right\} \tag{OPD_T}$$

admits a unique solution.

Proof. We use the standard argument of calculus of variation. Since the objective function $J_T(u; (y_0^1, y_0^2))$ is bounded from below we have

$$\inf_{L^2(0,T;L^2(\Gamma_c))} J_T(u; (y_0^1, y_0^2)) = \sigma < \infty.$$

Therefore, there is a minimizing sequence $\{u^n\}_n \subset L^2(0,T; L^2(\Gamma_c))$ such that

$$\|u^n\|^2_{L^2(0,T;L^2(\Gamma_c))} \leq \frac{2}{\beta} J_T(u^n; (y_0^1, y_0^2)) < \infty, \tag{3.64}$$

and, as a consequence, the sequence has a weakly convergent subsequence $u^n \rightharpoonup u^*$ with limit $u^* \in L^2(0,T; L^2(\Gamma_c))$. Due to Theorem 3.3.1 and estimate (3.61) for (3.63), there exists a bounded sequence of very weak solutions $\{y^n\}_n \subset L^\infty(0,T; L^2(\Omega)) \cap W^{1,\infty}(0,T; H^{-1}(\Omega))$ to (3.63) corresponding to the control sequence $\{u^n\}_n$. Hence, there exist weakly-star convergent subsequences $\{y^n\}_n$ and $\{\dot{y}^n\}_n$ such that

$$\begin{aligned} &y^n \rightharpoonup^* y^* \text{ in } L^\infty(0,T; L^2(\Omega)),\\ &\dot{y}^n \rightharpoonup^* \dot{y}^* \text{ in } L^\infty(0,T; H^{-1}(\Omega)). \end{aligned}$$

Now it remains to show that y^* is the very weak solution to (3.63) corresponding to the control u^*. To see this, we only need to pass the limits in the very weak formulation (3.59) for the pair of sequences $(y^n, B_{bd}u^n)$ in the place of (y, h). This Follows from the fact that

$$\int_0^T (y^n(t) - y^*(t), f(t))_{L^2(\Omega)} dt \to 0 \text{ for every } f \in L^1(0,T; L^2(\Omega)),$$

$$\int_0^T (\partial_\nu \vartheta(t), B_{bd}(u^n(t) - u^*(t)))_{L^2(\partial\Omega)} dt \to 0 \text{ for any weak solution } \vartheta(f) \text{ to (3.60).}$$

Now since the solution operator $S : L^2(0,T; L^2(\Gamma_c)) \to L^2(0,T; \mathcal{H}_2)$ defined by $u \mapsto (y, \dot{y})$ is affine and continuous, the objective function $J_T(\cdot; (y_0^1, y_0^2))$ is weakly lower semi-continuous and we have

$$0 \leq J_T(u^*; (y_0^1, y_0^2)) \leq \liminf_{n\to\infty} J_T(u^n; (y_0^1, y_0^2)) = \sigma.$$

As a result, the pair (y^*, u^*) is optimal. Uniqueness follows from the strictly convexity of $J_T(\cdot; (y_0^1, y_0^2))$. □

3.3.3 Optimality conditions

Lemma 3.3.1. *Consider the following linear wave equation*

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T) \times \Omega,\\ y = u & \text{on } (0,T) \times \Gamma_c,\\ y = 0 & \text{on } (0,T) \times \Gamma_0,\\ y(0) = 0, \quad \dot{y}(0) = 0 & \text{on } \Omega, \end{cases} \tag{3.65}$$

with a function $u \in L^2(0,T;L^2(\Gamma_c))$ *and let* $g \in L^2(0,T;L^2(\Omega))$ *and* $(p_T^1, p_T^2) \in H_0^1(\Omega) \times L^2(\Omega)$*. Then the very weak solution to* (3.65) *and the weak solution* p *to*

$$\begin{cases} \ddot{p} - \Delta p = g & \text{in } (0,T) \times \Omega, \\ p = 0 & \text{on } (0,T) \times \partial\Omega, \\ p(T) = p_T^1, \quad \dot{p}(T) = p_T^2 & \text{on } \Omega, \end{cases} \tag{3.66}$$

satisfy the following equality

$$\begin{aligned} &\int_0^T (g(t), y(t))_{L^2(\Omega)}\,dt + \langle p_T^1, \dot{y}(T)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - (p_T^2, y(T))_{L^2(\Omega)} \\ &= -\int_0^T (u(t), \partial_\nu p(t))_{L^2(\Gamma_c)}\,dt. \end{aligned} \tag{3.67}$$

Proof. First, due to Theorem 3.3.1 the solution y to (3.65) belongs to the space

$$C^1([0,T];H^{-1}(\Omega)) \cap C^0([0,T];L^2(\Omega)).$$

Moreover, due to Theorem 3.2.1 and the time reversibility of the linear wave equation, the solution p to (3.66) belongs to the space

$$C^1([0,T];L^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega)),$$

and we have the following hidden regularity

$$\partial_\nu p \in L^2(0,T;L^2(\partial\Omega)).$$

Therefore, all the terms in equality (3.67) are well-defined. Further, the solution to (3.65) is equal to the solution of the following system

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T) \times \Omega, \\ y = B_{bd}u & \text{on } (0,T) \times \partial\Omega, \\ y(0) = 0, \quad \dot{y}(0) = 0 & \text{on } \Omega, \end{cases} \tag{3.68}$$

and equality (3.67) is equal to the following equality

$$\begin{aligned} &\int_0^T (g(t), y(t))_{L^2(\Omega)}\,dt + \langle p_T^1, \dot{y}(T)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - (p_T^2, y(T))_{L^2(\Omega)} \\ &= -\int_0^T (B_{bd}u(t), \partial_\nu p(t))_{L^2(\partial\Omega)}\,dt. \end{aligned} \tag{3.69}$$

Now equality (3.69) can be first established for a smooth solution of (3.68) by integration by parts and the Green formula. Next we approximate the solution by a sequence of regular functions $\{y^n\}_n$ which obtained by putting a sequence of regular functions $\{h^n\}_n$ in the place of $B_{bd}u$ in (3.68), see, e.g., [100, Proposition 3.3, page 102]. Finally, (3.69) is derived by using density arguments and passing to the limit which is justified due to estimate (3.61). □

In the following, we derive the first-order optimality conditions for the finite horizon problems of the form (OPD_T). Since the objective function in (OPD_T) involves the tracking term of the velocity $\dot{y}$ in the space $L^2(0,T;H^{-1}(\Omega))$, the solution to the adjoint equation gains more regularity than the one to (3.63) and this solution exists in the weak sense.

Theorem 3.3.2 (First-order optimality conditions)**.** *Let $(\bar{y},\bar{u})$ be the optimal solution to (OPD_T). Then for $(y_0^1,y_0^2)\in\mathcal{H}_2$ we have the following optimality conditions*

$$\begin{cases} \ddot{\bar{y}}-\Delta\bar{y}=0 & \text{in } (0,T)\times\Omega,\\ \bar{y}=\bar{u} & \text{on } (0,T)\times\Gamma_c,\\ \bar{y}=0 & \text{on } (0,T)\times\Gamma_0,\\ \bar{y}(0)=y_0^1, \quad \dot{\bar{y}}(0)=y_0^2 & \text{on } \Omega,\\ \ddot{\bar{p}}-\Delta\bar{p}=\bar{y}-\overline{\mathcal{G}}\ddot{\bar{y}} & \text{in } (0,T)\times\Omega,\\ \bar{p}=0 & \text{on } (0,T)\times\partial\Omega,\\ \bar{p}(0)=0, \quad \dot{\bar{p}}(T)=-\overline{\mathcal{G}}\dot{\bar{y}}(T) & \text{on } \Omega,\\ \beta\bar{u}=\partial_\nu\bar{p} & \text{on } (0,T)\times\Gamma_c, \end{cases}$$

where $p\in C^1([0,T];L^2(\Omega))\cap C^0([0,T];H_0^1(\Omega))$ is the solution of the adjoint equation

Proof. For sake of simplicity in notation, we remove the overbar in the notation of $(\bar{y},\bar{u})$. Let $(y_0^1,y_0^2)\in\mathcal{H}_2$ be given. Computing the directional derivative of $J_T(\cdot,(y_0^1,y_0^2))$ at $\bar{u}$ in the direction of an arbitrary $\delta u\in L^2(0,T;L^2(\Gamma_c))$ we obtain

$$\begin{aligned} &J_T'(u,(y_0^1,y_0^2))\delta u\\ &=\int_0^T (y(t),\delta y(t))_{L^2(\Omega)}dt+\int_0^T(\dot{y}(t),\dot{\delta y}(t))_{H^{-1}(\Omega)}dt+\beta\int_0^T(u(t),\delta u(t))_{L^2(\Gamma_c)}dt, \end{aligned} \tag{3.70}$$

where $\delta y\in C^1([0,T];H^{-1}(\Omega))\cap C^0([0,T];L^2(\Omega))$ is the very weak solution of

$$\begin{cases} \ddot{\delta y}-\Delta\delta y=0 & \text{in } (0,T)\times\Omega,\\ \delta y=\delta u & \text{on } (0,T)\times\Gamma_c,\\ \delta y=0 & \text{on } (0,T)\times\Gamma_0,\\ \delta y(0)=0, \quad \dot{\delta y}(0)=0 & \text{on } \Omega. \end{cases} \tag{3.71}$$

As defined $\overline{\mathcal{G}}:(H^2(\Omega)\cap H_0^1(\Omega))^*\to L^2(\Omega)$ denotes the unique linear extension of $\mathcal{G}:H^{-1}(\Omega)\to H_0^1(\Omega)$. Well-posedness of $\overline{\mathcal{G}}$ is justified since $H^{-1}(\Omega)$ is dense in $(H^2(\Omega)\cap H_0^1(\Omega))^*$. Moreover one can show that $\overline{\mathcal{G}}$ is the inverse of the operator $(-\widetilde{\Delta})^*: L^2(\Omega)\to(H^2(\Omega)\cap H_0^1(\Omega))^*$, where $-\widetilde{\Delta}:(H^2(\Omega)\cap H_0^1(\Omega))\to L^2(\Omega)$ is the Laplace operator with homogeneous Dirichlet boundary conditions. Next we show that

$$\int_0^T(\dot{y}(t),\dot{\delta y}(t))_{H^{-1}(\Omega)}dt=(\overline{\mathcal{G}}\dot{y}(T),\delta y(T))_{L^2(\Omega)}-\int_0^T(\overline{\mathcal{G}}\ddot{y}(t),\delta y(t))_{L^2(\Omega)}dt. \tag{3.72}$$

We proceed with the help of an approximation argument. The spaces $H_0^1(\Omega)$, $L^2(\Omega)$, and $H_0^2(0,T;H^{\frac{3}{2}}(\partial\Omega)) := \{q \in H^2(0,T;H^{\frac{3}{2}}(\partial\Omega)) : q(0) = \dot{q}(0) = 0\}$ are dense in the spaces $L^2(\Omega)$, $H^{-1}(\Omega)$, and $L^2(0,T;L^2(\partial\Omega))$, respectively, and the solutions of (3.63) (resp. (3.71)) is equal to the solution of (3.58) provided we choose $B_{bd}u \in L^2(0,T;L^2(\partial\Omega))$ (resp. $B_{bd}\delta u$) as the inhomogeneous Dirichlet part h and the pair (y_0^1, y_0^2) (resp. $(0,0)$) as the initial pair. Therefore, there exist sequences $\{y_0^{1n}\}_n \subset H_0^1(\Omega)$, $\{y_0^{2n}\}_n \subset L^2(\Omega)$, $\{h^n\}_n \subset H_0^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$, and $\{\delta h^n\}_n \subset H_0^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$ such that

$$\begin{aligned} y_0^{1n} &\to y_0^1 && \text{in } L^2(\Omega),\\ y_0^{2n} &\to y_0^2 && \text{in } H^{-1}(\Omega),\\ h^n &\to B_{bd}u && \text{in } L^2(0,T;L^2(\partial\Omega)),\\ \delta h^n &\to B_{bd}\delta u && \text{in } L^2(0,T;L^2(\partial\Omega)). \end{aligned}$$

Moreover, for any triple $(y_0^{1n}, y_0^{2n}, h^n) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$, the solution of y^n of (3.58) belongs to the space $C^1([0,T];L^2(\Omega)) \cap C^0([0,T];H^1(\Omega))$ with $\ddot{y^n} \in L^2(0,T;H^{-1}(\Omega))$ (see, e.g., [100]), and similarly, for any triple $(0,0,\delta h^n) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$, the solution of (3.58) belongs to the space $C^1([0,T];L^2(\Omega)) \cap C^0([0,T];H^1(\Omega))$. By using estimate (3.61) we have

$$\begin{aligned} &\|y^n - y\|_{C^0([0,T];L^2(\Omega))} + \|\dot{y^n} - \dot{y}\|_{C^0([0,T];H^{-1}(\Omega))} + \|\ddot{y^n} - \ddot{y}\|_{L^2(0,T;(H_0^1(\Omega))\cap H^2(\Omega))^*)}\\ &\qquad \le c_2 \left(\|y_0^{1n} - y_0^1\|_{L^2(\Omega)} + \|y_0^{2n} - y_0^2\|_{H^{-1}(\Omega)} + \|h^n - B_{bd}u\|_{L^2(0,T;L^2(\partial\Omega))}\right), \end{aligned}$$

and

$$\|\delta y^n - \delta y\|_{C^0([0,T];L^2(\Omega))} + \|\dot{\delta y^n} - \dot{\delta y}\|_{C^0([0,T];H^{-1}(\Omega))} \le c_2 \|\delta h^n - B_{bd}\delta u\|_{L^2(0,T;L^2(\partial\Omega))}.$$

For a solution y^n of (3.58) with $(y_0^{1n}, y_0^{2n}, h^n) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$ and a solution δy^n of (3.58) with $(0,0,\delta h^n) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$, we have

$$\begin{aligned} \int_0^T (\dot{y^n}(t), \dot{\delta y^n}(t))_{H^{-1}(\Omega)} dt &= \int_0^T \langle \mathcal{G}\dot{y^n}(t), \dot{\delta y^n}(t)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} dt =\\ &(\mathcal{G}\dot{y^n}(T), \delta y^n(T))_{L^2(\Omega)} - \int_0^T (\mathcal{G}\ddot{y^n}(t), \delta y^n(t))_{L^2(\Omega)} dt. \end{aligned}$$

By passing the limit, we obtain

$$\begin{aligned} &\int_0^T (\dot{y^n}(t), \dot{\delta y^n}(t))_{H^{-1}(\Omega)} dt \to \int_0^T (\dot{y}(t), \dot{\delta y}(t))_{H^{-1}(\Omega)} dt,\\ &\quad (\mathcal{G}\dot{y^n}(T), \delta y^n(T))_{L^2(\Omega)} \to (\mathcal{G}\dot{y}(T), \delta y(T))_{L^2(\Omega)} = (\overline{\mathcal{G}}\dot{y}(T), \delta y(T))_{L^2(\Omega)},\\ &\int_0^T (\mathcal{G}\ddot{y^n}(t), \delta y^n(t))_{L^2(\Omega)} dt \to \int_0^T (\overline{\mathcal{G}}\ddot{y}(t), \delta y(t))_{L^2(\Omega)} dt, \end{aligned}$$

and we are finished with the justification of (3.72). Now due to (3.70) and (3.72), the first order optimality condition is equivalent to the following equality

$$\int_0^T (y(t) - \overline{\mathcal{G}}\ddot{y}(t), \delta y(t))_{L^2(\Omega)} dt + (\overline{\mathcal{G}}\dot{y}(T), \delta y(T))_{L^2(\Omega)} + \beta \int_0^T (u(t), \delta u(t))_{L^2(\Gamma_c)} dt = 0. \tag{3.73}$$

Moreover, due to Lemma 3.3.1 and by using equality (3.67) for equation (3.71), we have

$$\begin{aligned} \int_0^T (g(t), \delta y(t))_{L^2(\Omega)}\, dt + \langle p_T^1, \dot{\delta y}(T)) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ - (p_T^2, \delta y(T))_{L^2(\Omega)} + \int_0^T (\partial_\nu p(t), \delta u(t))_{L^2(\Gamma_c)}\, dt = 0, \end{aligned} \tag{3.74}$$

for an arbitrary triple $(g, p_T^1, p_T^2) \in L^2(0,T; L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ and its corresponding weak solution $p \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ to (3.66). By comparing (3.73) with (3.74) and since $\delta u \in L^2(0,T; L^2(\Gamma_c))$ is arbitrary, we infer that

$$\begin{aligned} \beta u &= \partial_\nu p && \text{in } L^2(0,T; L^2(\Gamma_c)),\\ p_T^1 &= 0 && \text{in } H_0^1(\Omega),\\ p_T^2 &= -\overline{\mathcal{G}}\dot{y}(T) && \text{in } L^2(\Omega),\\ g &= y - \overline{\mathcal{G}}\ddot{y} && \text{in } L^2(0,T; L^2(\Omega)). \end{aligned}$$

□

3.3.4 Stabilizability

Similar to the previous section, we show that for the controlled system (3.56) there exists a feedback law $u(y)$ that stabilizes the system with respect to the energy

$$\mathcal{E}(t, y) := \|y(t)\|_{L^2(\Omega)}^2 + \|\dot{y}(t)\|_{H^{-1}(\Omega)}^2, \tag{3.75}$$

which is defined along a trajectory y.

Lemma 3.3.2 (Equivalence of the observability conditions)**.** *The observability condition OB1 is equivalent to the following observability inequality:*

OB3. For every $T \geq T_{ob1}$, the very weak solution ϕ to (3.28) *with $(\phi, \dot{\phi}) \in C^0([0,T]; \mathcal{H}_2)$ satisfies the inequality*

$$c_{ob1} \|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_2}^2 \leq \int_0^T \int_\omega |\phi|^2 dx dt \quad \textit{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_2,$$

where the constants c_{ob1}, T_{ob1} have been defined in the observability condition OB1.

Similarly, the observability condition OB2 is equivalent to the following observability condition:

OB4. For every $T \geq T_{ob2}$, the very weak solution ϕ to (3.28) *with* $(\phi, \dot{\phi}) \in C^0([0,T]; \mathcal{H}_2)$ *satisfies the inequality*

$$c_{ob2}\|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_2}^2 \leq \int_0^T \int_{\Gamma_c} |\partial_\nu \mathcal{G}\dot{\phi}|^2 dSdt \quad \textit{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_2,$$

where the constants c_{ob2}, T_{ob2} have been defined in the observability condition OB2.

Proof. Similar results can be found in the literature e.g., [4, 98], but for sake of completeness we give a proof.

Proof of the first equivalence ($OB1 \iff OB3$): Let $v(t) := \int_0^t \phi(s)ds - \mathcal{G}\phi_0^2$ for $t \geq 0$, with ϕ the very weak solution of (3.28) for the initial pair $(\phi_0^1, \phi_0^2) \in \mathcal{H}_2$. Then, for every $T \geq T_{ob1}$, v is the weak solution of the following problem

$$\begin{cases} \ddot{v} - \Delta v = 0 & \text{in } (0,T) \times \Omega, \\ v = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(0) = -\mathcal{G}\phi_0^2, \quad \dot{v}(0) = \phi_0^1 & \text{on } \Omega. \end{cases} \tag{3.76}$$

Using the observability condition OB1 for (3.76), we obtain

$$c_{ob1}\|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_2}^2 = c_{ob1}\|(-\mathcal{G}\phi_0^2, \phi_0^1)\|_{\mathcal{H}_1}^2 \leq \int_0^T \int_\omega |\dot{v}|^2 dxdt = \int_0^T \int_\omega |\phi|^2 dxdt$$

for a very weak solution ϕ of (3.28) with an arbitrary $(\phi_0^1, \phi_0^2) \in \mathcal{H}_2$. Using a similar argument, one can show the converse equivalence.

We turn to the second assertion i.e. the equivalence ($OB2 \iff OB4$). Similarly we set $v := \mathcal{G}\dot{\phi}$ with ϕ the very weak solution of (3.28) for the initial pair $(\phi_0^1, \phi_0^2) \in \mathcal{H}_2$. Then for every $T \geq T_{ob2}$, v is the weak solution to the following problem

$$\begin{cases} \ddot{v} - \Delta v = 0 & \text{in } (0,T) \times \Omega, \\ v = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(0) = \mathcal{G}\phi_0^2, \quad \dot{v}(0) = -\phi_0^1 & \text{on } \Omega. \end{cases} \tag{3.77}$$

By using the observability condition OB2 for (3.77), we obtain

$$\begin{aligned} c_{ob2}\|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_2}^2 &= c_{ob2}\|(\mathcal{G}\phi_0^2, -\phi_0^1)\|_{\mathcal{H}_1}^2 \\ &\leq \int_0^T \int_{\Gamma_c} |\partial_\nu v|^2 dSdt = \int_0^T \int_{\Gamma_c} |\partial_\nu (\mathcal{G}\dot{\phi})|^2 dSdt \end{aligned}$$

for a very weak solution ϕ of (3.28) with an arbitrary $(\phi_0^1, \phi_0^2) \in \mathcal{H}_2$. By a similar argument, the converse equivalence can be shown. □

Proposition 3.3.2. *Suppose that $T > 0$ and $u \in L^2(0,T; L^2(\Gamma_c))$. Then the following linear wave equation*

$$\begin{cases} \ddot{\psi} - \Delta\psi = 0 & \textit{in } (0,T) \times \Omega, \\ \psi = u & \textit{on } (0,T) \times \Gamma_c, \\ \psi = 0 & \textit{on } (0,T) \times \Gamma_0, \\ \psi(0) = 0, \quad \dot{\psi}(0) = 0 & \textit{on } \Omega, \end{cases} \tag{3.78}$$

admits a unique very weak solution $\psi \in C^0([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$. *Moreover,* $\partial_\nu(\mathcal{G}\psi) \in H^1(0,T;L^2(\Gamma_c))$ *and we have the following estimate*

$$\|\partial_\nu(\mathcal{G}\psi)\|_{H^1(0,T;L^2(\Gamma_c))} \leq c_2' \|u\|_{L^2(0,T;L^2(\Gamma_c))}, \tag{3.79}$$

where the constant c_2' *depends only on* T.

Proof. The proof can be found in, e.g., [4]. □

The proof of the first direction in the following equivalence can be found in, e.g., [4]. Nevertheless, we provide here a proof for completeness.

Theorem 3.3.3 (Global stabilizability). *Suppose that* $(y_0^1, y_0^2) \in \mathcal{H}_2$ *is given. Then the solution of the controlled system* (3.56) *with the feedback law* $u(y) := \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$ *converges exponentially to zero with respect to* $\mathcal{H}_2$, *i.e.*

$$\mathcal{E}(t,y) \leq Me^{-\alpha t}\mathcal{E}(0,y) = Me^{-\alpha t}\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_2} \tag{3.80}$$

for positive constants M, α *independent of* (y_0^1, y_0^2), *if and only if the observability condition OB2 holds.*

Proof. First assume that condition OB2 holds. We show the exponential decay inequality (3.80).

Let $(y_0^1, y_0^2) \in \mathcal{H}_2$ be given. Setting $u(y) := \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$ in (3.56) we obtain a closed-loop system. This system is well-posed (see, e.g., [4, 91]), and for its unique solution we have

$$y \in C([0,\infty);L^2(\Omega)) \cap C^1([0,\infty);H^{-1}(\Omega)),$$

and $\partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c} \in L^2(0,\infty;L^2(\Gamma_c))$.

Now, for an arbitrary $T > 0$ consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T)\times\Omega, \\ y = \partial_\nu(\mathcal{G}\dot{y}) & \text{on } (0,T)\times\Gamma_c, \\ y = 0 & \text{on } (0,T)\times\Gamma_0, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega. \end{cases} \tag{3.81}$$

Suppose that the solution y of (3.81) is smooth enough. Taking L^2-inner product of (3.81) with $\mathcal{G}\dot{y}$, formally, and integrating over $[0,T]$, we obtain the following estimate

$$\|(y(T),\dot{y}(T))\|^2_{\mathcal{H}_2} - \|(y(0),\dot{y}(0))\|^2_{\mathcal{H}_2} = -2\int_0^T \|\partial_\nu(\mathcal{G}\dot{y}(t))\|^2_{L^2(\Gamma_c)}dt. \tag{3.82}$$

We can approximate the pair $(y_0^1, y_0^2) \in \mathcal{H}_2$ by a sequence of pairs $(y_0^{1n}, y_0^{2n}) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ for which the corresponding solutions y^n of (3.81), by using the standard semi group theory, belong to the space $C^0([0,T];H^1_{\Gamma_0}(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and converge to y in the space $C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$ as n tends to infinity. Then by passing to the limits, it can be shown that equality (3.82) is also true for the initial pair $(y_0^1, y_0^2) \in \mathcal{H}_2$ and its corresponding solution y.

Further, the solution y of (3.81) can be rewritten as $y = \phi + \psi$, where ϕ is the solution of (3.28) with the initial pair (y_0^1, y_0^2) in the place of (ϕ_0^1, ϕ_0^2), and ψ is the solution to (3.78) with $u = \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$. Since the condition OB2 is satisfied and due to Lemma 3.3.2, the observability condition OB2 is equivalent to condition OB4, we are allowed to use the observability condition OB4.

By using the observability condition OB4 for (3.28) with the initial pair (y_0^1, y_0^2), and estimate (3.79) for ψ with $u = \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$, we obtain

$$\begin{aligned} \|(y_0^1, y_0^2)\|_{\mathcal{H}_2}^2 &\leq \frac{1}{c_{ob2}} \int_0^{T_{ob2}} \int_{\Gamma_c} |\partial_\nu(\mathcal{G}\dot{\phi})|^2 dSdt \\ &\leq \frac{1}{c_{ob2}} \left(\int_0^{T_{ob2}} \int_{\Gamma_c} |\partial_\nu(\mathcal{G}\dot{y})|^2 dSdt + \int_0^{T_{ob2}} \int_{\Gamma_c} |\partial_\nu(\mathcal{G}\dot{\psi})|^2 dSdt \right) \\ &\leq \frac{1 + {c_2'}^2}{c_{ob2}} \int_0^{T_{ob2}} \int_{\Gamma_c} |\partial_\nu(\mathcal{G}\dot{y})|^2 dSdt, \end{aligned} \tag{3.83}$$

for $T_{ob2} > 0$ defined in the observability condition OB2. Combining (3.82) and (3.83), we have

$$\begin{aligned} \|(y(T_{ob2}), \dot{y}(T_{ob2}))\|_{\mathcal{H}_2}^2 - \|(y(0), \dot{y}(0))\|_{\mathcal{H}_2}^2 &= -2 \int_0^{T_{ob2}} \|\partial_\nu(\mathcal{G}\dot{y}(t))\|_{L^2(\Gamma_c)}^2 dt \\ &\leq \frac{-2c_{ob2}}{1 + {c_2'}^2} \|(y(0), \dot{y}(0))\|_{\mathcal{H}_2}^2 \\ &\leq \frac{-2c_{ob2}}{1 + {c_2'}^2} \|(y(T_{ob2}), \dot{y}(T_{ob2}))\|_{\mathcal{H}_2}^2. \end{aligned}$$

As a result, we have

$$\mathcal{E}(t, y) \leq M e^{-\alpha t} \mathcal{E}(0, y) \qquad \text{for every } t > 0,$$

where $\alpha := \frac{\ln(1 + \frac{2c_{ob2}}{1 + {c_2'}^2})}{T_{ob2}}$ and $M := (1 + \frac{2c_{ob2}}{1 + {c_2'}^2})$.

Next we show that the stabilizability property (3.80) implies the observability condition OB2. Due to Lemma 3.3.2, it is sufficient to show that the stabilizability property (3.80) implies the observability condition OB4 for a given pair $(y_0^1, y_0^2) \in \mathcal{H}_2$.

Now, let inequality (3.80) holds for the pair $(y_0^1, y_0^2) \in \mathcal{H}_2$. Then due to (3.80) and (3.82), there exists a $T' > 0$ such that

$$\int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{y}(t))\|_{L^2(\Gamma_c)}^2 dt \geq \frac{1}{4} \mathcal{E}(0, y). \tag{3.84}$$

Moreover, the very weak solution ϕ to (3.28) with the initial pair $(y_0^1, y_0^2) \in \mathcal{H}_2$ can be rewritten as $\phi := y - \psi$, where y is the solution to (3.81) and ψ is the solution to (3.78) with $u = \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$for T' instead of T.

Next, we will show that for the very weak solution ψ to (3.78) with $u = \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c} \in L^2(0, T'; L^2(\Gamma_c))$, we have the following inequality

$$
\begin{aligned}
0 \leq \frac{1}{2}\big(\|\dot{\psi}(T')\|^2_{H^{-1}(\Omega)} + \|\psi(T')\|^2_{L^2(\Omega)}\big) &= \int_0^{T'}\int_{\Gamma_c} -\partial_\nu(\mathcal{G}\dot{y})\partial_\nu(\mathcal{G}\dot{\psi})dSdt \\
&= \int_0^{T'}\int_{\Gamma_c} -\partial_\nu(\mathcal{G}\dot{\psi} + \mathcal{G}\dot{\phi})\partial_\nu(\mathcal{G}\dot{\psi})dSdt.
\end{aligned}
\tag{3.85}
$$

First, the solution to (3.78) with $u = \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$ is equal to the solution of the following systems

$$
\begin{cases}
\ddot{\psi} - \Delta\psi = 0 & \text{in } (0, T') \times \Omega, \\
\psi = B_{bd}\left(\partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}\right) & \text{on } (0, T') \times \partial\Omega, \\
\psi(0) = 0, \quad \dot{\psi}(0) = 0 & \text{on } \Omega.
\end{cases}
\tag{3.86}
$$

Moreover, the space $H^2_0(0, T'; H^{\frac{3}{2}}(\partial\Omega))$ is dense in the space $L^2(0, T'; L^2(\partial\Omega))$, and for every $h \in H^2_0(0, T'; H^{\frac{3}{2}}(\partial\Omega))$ instead of $B_{bd}\left(\partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}\right) \in L^2(0, T'; L^2(\partial\Omega))$, as the inhomogenous Drichlet part in (3.86), the very weak solution of (3.86) belongs to the space $C^0([0, T']; H^1(\Omega)) \cap C^1([0, T']; L^2(\Omega))$, see, e.g., [100]. Therefore, there exists a sequence $\{h^n\}_n \subset H^2_0(0, T'; H^{\frac{3}{2}}(\partial\Omega))$ such that

$$
h^n \to B_{bd}\left(\partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}\right) \text{ in } L^2(0, T'; L^2(\partial\Omega)),
\tag{3.87}
$$

and corresponding to this sequence, there exists a sequence of solutions

$$
\{\psi^n\}_n \subset C^0([0, T']; H^1(\Omega)) \cap C^1([0, T']; L^2(\Omega))
$$

which satisfy

$$
\begin{cases}
\ddot{\psi^n} - \Delta\psi^n = 0 & \text{in } (0, T') \times \Omega, \\
\psi^n = h^n & \text{on } (0, T') \times \partial\Omega, \\
\psi^n(0) = 0, \quad \dot{\psi^n}(0) = 0 & \text{on } \Omega.
\end{cases}
\tag{3.88}
$$

Moreover, due to the estimate (3.61), we obtain

$$
\begin{aligned}
\|\psi^n - \psi\|_{C^0([0,T'];L^2(\Omega))} + \|\dot{\psi^n} - \dot{\psi}\|_{C^0([0,T'];H^{-1}(\Omega))} \leq \\
c_2\|h^n - B_{bd}\left(\partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}\right)\|_{L^2(0,T';L^2(\partial\Omega))}.
\end{aligned}
\tag{3.89}
$$

Now, by taking L^2-inner product of (3.88) with $\mathcal{G}\dot{\psi^n}$, and integrating over $[0, T']$, we obtain

$$
0 \leq \frac{1}{2}\big(\|\dot{\psi^n}(T')\|^2_{H^{-1}(\Omega)} + \|\psi^n(T')\|^2_{L^2(\Omega)}\big) = \int_0^{T'}\int_{\partial\Omega} -h^n\partial_\nu(\mathcal{G}\dot{\psi^n})dSdt.
\tag{3.90}
$$

Now by using (3.89), passing the limits in (3.90), and using the following equality

$$
\int_0^{T'}\int_{\partial\Omega}(-B_{bd}\left(\partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}\right))\partial_\nu(\mathcal{G}\dot{\psi})dSdt = \int_0^{T'}\int_{\Gamma_c} -\partial_\nu(\mathcal{G}\dot{y})\partial_\nu(\mathcal{G}\dot{\psi})dSdt,
\tag{3.91}
$$

we conclude the inequality (3.85) for the very weak solution ψ to (3.78) with $u = \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c} \in L^2(0,T';L^2(\Gamma_c))$.

Next, (3.85) implies that

$$\int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\psi}(t))\|^2_{L^2(\Gamma_c)}dt \le \int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\phi}(t))\|^2_{L^2(\Gamma_c)}dt. \tag{3.92}$$

Using (3.84), (3.92), and the following inequality

$$\int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\phi}(t))\|^2_{L^2(\Gamma_c)}dt \ge \int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{y}(t))\|^2_{L^2(\Gamma_c)}dt - \int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\psi}(t))\|^2_{L^2(\Gamma_c)}dt,$$

we complete the proof with

$$\int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\phi}(t))\|^2_{L^2(\Gamma_c)}dt \ge \frac{1}{8}\mathcal{E}(0,y).$$

□

3.3.5 Stability of RHC

From this point on, we denote $(y(t),\dot{y}(t))$ by $\mathcal{Y}(t)$ and we define $\alpha_\ell := \frac{\min(1,\beta)}{2}$.

Definition 3.3.2 (Value function)**.** *For every pair $(y_0^1,y_0^2) =: \mathcal{Y}_0 \in \mathcal{H}_2$, the infinite horizon value function $V_\infty : \mathcal{H}_2 \to \mathbb{R}_+$ is defined as*

$$V_\infty(\mathcal{Y}_0) := \min_{u\in L^2(0,\infty;L^2(\Gamma_c))} \{J_\infty(u,\mathcal{Y}_0) \text{ subject to } (3.56)\}.$$

Similarly, the finite horizon value function $V_T : \mathcal{H}_2 \to \mathbb{R}_+$ is defined by

$$V_T(\mathcal{Y}_0) := \min_{u\in L^2(0,T;L^2(\Gamma_c))} \{J_T(u,\mathcal{Y}_0) \text{ subject to } (3.63)\}. \tag{3.93}$$

Lemma 3.3.3. *Suppose that the observability conditions OB1-OB2 hold. Then for every $T > 0$, there exists a control $\hat{u} \in L^2(0,T;L^2(\Gamma_c))$ for (3.63) such that*

$$V_T(\mathcal{Y}_0) \le J_T(\hat{u};\mathcal{Y}_0) \le \gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \tag{3.94}$$

for every initial pair $(y_0^1,y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_2$, where $\gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous, and bounded function. Moreover, there exists a constant $\gamma_1(T) > 0$ depending on T such that

$$V_T(\mathcal{Y}_0) \ge \gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \tag{3.95}$$

for all $(y_0^1,y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_2$.

Proof. Let $T > 0$ and an initial pair $(y_0^1, y_0^2) \in \mathcal{H}_2$ be given. Since the observability condition OB2 holds, by setting $u(t) := \partial_\nu(\mathcal{G}\dot{y}(t))|_{\Gamma_c}$ in the controlled system (3.63), and using Theorem 3.3.3, we obtain

$$\|(y(t), \dot{y}(t))\|_{\mathcal{H}_2}^2 \le Me^{-\alpha t}\|(y(0), \dot{y}(0))\|_{\mathcal{H}_2}^2 \quad \text{for all } t \in [0, T],$$

where the constants M and α were defined in Theorem 3.3.3. By integrating from 0 to T we have

$$\int_0^T \|(y(t), \dot{y}(t))\|_{\mathcal{H}_2}^2 dt \le \frac{M}{\alpha}(1 - e^{-\alpha T})\|(y(0), \dot{y}(0))\|_{\mathcal{H}_2}^2.$$

Moreover, by (3.80) and (3.82) we have

$$\begin{aligned}\int_0^T \|u(t)\|_{L^2(\Gamma_c)}^2 dt &= \int_0^T \|\partial_\nu(\mathcal{G}\dot{y}(t))\|_{L^2(\Gamma_c)}^2 dt \\ &\le \frac{1}{2}\left(\mathcal{E}(0, y) + \mathcal{E}(T, y)\right) \le \frac{(1+M)}{2}\|(y_0^1, y_0^2)\|_{\mathcal{H}_2}^2.\end{aligned} \tag{3.96}$$

By (3.57), (3.96), and the definition of the value function V_T we have

$$\begin{aligned}V_T(y_0^1, y_0^2) &\le \int_0^T \left(\frac{1}{2}\|(y(t), \dot{y}(t))\|_{\mathcal{H}_2}^2 + \frac{\beta}{2}\|\partial_\nu(\mathcal{G}\dot{y}(t))\|_{L^2(\Gamma_c)}^2\right) dt \\ &\le \left(\frac{M}{2\alpha}(1 - e^{-\alpha T}) + \frac{\beta(1+M)}{4}\right)\|(y_0^1, y_0^2)\|_{\mathcal{H}_2}^2 = \gamma_2(T)\|(y_0^1, y_0^2)\|_{\mathcal{H}_2}^2,\end{aligned}$$

which gives (3.94).

Now to verify (3.95), we use the superposition argument for (3.63) with an arbitrary control $u \in L^2(0, T; L^2(\Gamma_c))$. We rewrite the solution of (3.63) as $y = \phi + \psi$ where ϕ is the solution to (3.28) with the initial data (y_0^1, y_0^2) and ψ is the solution to (3.78). Since the condition OB1 is satisfied and due to Lemma 3.3.2, the observability condition OB1 is equivalent to the condition OB3, we are allowed to use the observability condition OB3. Now by using the observability condition OB3 for (3.28) with the initial data (y_0^1, y_0^2) and Ω instead of ω, and using estimate (3.61) for (3.78), we obtain

$$\begin{aligned}\|(y_0^1, y_0^2)\|_{\mathcal{H}_2}^2 &\le \frac{1}{c_{ob1}}\int_0^T \|\phi(t)\|_{L^2(\Omega)}^2 dt \\ &\le \frac{1}{c_{ob1}}\int_0^T \left(\|y(t)\|_{L^2(\Omega)}^2 + \|\varphi(t)\|_{L^2(\Omega)}^2\right) dt \\ &\le \frac{1}{c_{ob1}}\int_0^T \left(\|y(t)\|_{L^2(\Omega)}^2 + Tc_2^2\|u(t)\|_{L^2(\Gamma_c)}^2\right) dt \\ &\le c_2''(T)\int_0^T \left(\frac{1}{2}\|(y(t), \dot{y}(t))\|_{\mathcal{H}_2}^2 + \frac{\beta}{2}\|u(t)\|_{L^2(\Gamma_c)}^2\right) dt \\ &= c_2''(T)\int_0^T \ell(\mathcal{Y}(t), u(t)) dt.\end{aligned}$$

Since $u \in L^2(0, T; L^2(\Gamma_c))$ is arbitrary, we obtain (3.95) for a constant $c_2''(T)$ independent of u and (y_0^1, y_0^2). □

Remark 3.3.1. The property (3.95) is equivalent to the injectivity of the differential Recatti operator corresponding to (OPD_T) which in turn is equivalent to the observability condition OB1, see, [57, Theorem 3.3].

Remark 3.3.2. Note that, as it has been shown in Lemma 3.3.3, the observability condition OB2 is equivalent to the stabilizability condition (3.94). The stabilizability condition (3.94) and well-posedness (Proposition 3.3.1) of open-loop problems in the form (OPD_T) are equivalent to the conditions (A2) and (A1) in Chapter 2, respectively. Moreover, since the stabilizability condition (3.94) holds globally, the condition (A3) is no longer needed and we can use the receding horizon framework introduced in Chapter 2. In addition, by using the uniform positiveness of the value function V_T which has been established in (3.95) based on the observability condition OB1, we shall verify the exponential stability of RHC (see Remark 2.2.3).

Theorem 3.3.4 (Suboptimality and exponential decay)**.** *Suppose that the observability conditions OB1-OB2 hold and let a sampling time $\delta > 0$ be given. Then there exist numbers $T^* > \delta$ and $\alpha \in (0, 1)$, such that for every fixed prediction horizon $T \geq T^*$ and every $\mathcal{Y}_0 \in \mathcal{H}_2$, the receding horizon control u_{rh} obtained from Algorithm 3.1 for the stabilization of* (3.56) *satisfies the suboptimality inequality*

$$\alpha V_\infty(\mathcal{Y}_0) \leq \alpha J_\infty(u_{rh}, \mathcal{Y}_0) \leq V_T(\mathcal{Y}_0) \leq V_\infty(\mathcal{Y}_0),$$

and exponential stability

$$\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}_2} \leq c' e^{-\zeta t} \|\mathcal{Y}_0\|^2_{\mathcal{H}_2}, \tag{3.97}$$

where the positive numbers ζ and c' depend on α, δ, and T, but are independent from $\mathcal{Y}_0$.

Proof. Let $\delta > 0$ be given. To show the suboptimality inequality we refer to the proof of Theorem 2.2.1 in Chapter 2. Now we turn to inequality (3.97). It is of interest to compute the constant ζ and c' in two different ways. First, due to Theorem 2.2.2 in Chapter 2, there exists a $T^* > 0$ such that for every $T \geq T^*$ we have

$$V_{T-\delta}(\mathcal{Y}_{rh}(t)) \leq c e^{-\zeta_1 t} V_T(\mathcal{Y}_0) \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}_2,$$

where the constants c and ζ_1 are given by

$$\zeta_1 := \frac{\ln(1 + \frac{\alpha}{1+\theta_1\theta_2})}{\delta}, \qquad c := (1 + \frac{\alpha}{1 + \theta_1\theta_2}),$$

and $\theta_1(T, \delta), \theta_2(T, \delta)$ are defined by

$$\theta_1 := 1 + \frac{\gamma_2(T)}{\alpha_\ell(T - \delta)}, \qquad \theta_2 := \frac{\gamma_2(T)}{\alpha_\ell \delta}.$$

Due to Lemma 3.3.3, and using (3.94) and (3.95) we obtain

$$\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}_2} \leq \frac{c\gamma_2(T)}{\gamma_1(T - \delta)} e^{-\zeta_1 t} \|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}_2.$$

Second way: similar to the proof of Theorem 3.2.4, one can show that there exist a $T^* > 0$ such that for every $T \geq T^*$ and $k \geq 1$ we have

$$\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}_2} \leq c'_2 e^{-\zeta_2 k\delta} \|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \text{ for every } \mathcal{Y}_0 \in \mathcal{H}_2, \tag{3.98}$$

where $\zeta_2 := \frac{|\ln \eta|}{\delta}$ with $\eta := \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right) \in (0,1)$, and $c'_2 := \frac{\gamma_2(T)}{\gamma_1(T)}$. Moreover, for every $t > 0$ there exists a $k \in \mathbb{N}$ such that $t \in [k\delta, (k+1)\delta]$. Using (3.61), (3.94), and (3.98), we have for $t > 0$,

$$\begin{aligned}
\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}_2} &\leq 3c_2^2 \left(\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}_2} + \int_{k\delta}^{(k+1)\delta} \|u_{rh}(t)\|^2_{L^2(\Gamma_c)} dt \right) \\
&\leq 3c_2^2 \left(\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}_2} + \frac{2}{\beta} V_T(\mathcal{Y}_{rh}(k\delta)) \right) \\
&\leq 3c_2^2 \left(\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}_2} + \frac{2\gamma_2(T)}{\beta} \|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}_2} \right) \\
&\leq 3c_2^2 c'_2 (1 + \frac{2\gamma_2(T)}{\beta}) e^{-\zeta_2 k\delta} \|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \\
&\leq 3c_2^2 c'_2 (1 + \frac{2\gamma_2(T)}{\beta}) \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right)^{-1} e^{-\zeta_2 (k+1)\delta} \|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \\
&\leq 3c_2^2 c'_2 (1 + \frac{2\gamma_2(T)}{\beta}) \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right)^{-1} e^{-\zeta_2 t} \|\mathcal{Y}_0\|^2_{\mathcal{H}_2},
\end{aligned}$$

and the proof is complete. □

3.4 Neumann Boundary Control

In this section, we are dealing with the following one-dimensional wave equation with a Neumann control action at one side of boundary

$$\begin{cases}
\ddot{y} - y_{xx} = 0 & (t,x) \in (0,\infty) \times (0,L), \\
y(t,0) = 0 & t \in (0,\infty), \\
y_x(t,L) = u(t) & t \in (0,\infty), \\
y(0,x) = y_0^1, \quad \dot{y}(0,x) = y_0^2 & x \in (0,L),
\end{cases} \tag{3.99}$$

where $L > 0$, $u \in L^2(0,\infty)$, and $(y_0^1, y_0^2) \in V \times L^2(0,L)$ with $V := \{q \in H^1(0,L) : q(0) = 0\}$. The functional space V is equipped with the following scalar product

$$(\phi, \psi) := \int_0^L \phi_x \psi_x dx.$$

Moreover, V^* stands for the dual space of V. Similar to the previous sections we define the functional space $\mathcal{H}_3 := V \times L^2(0,L)$ with its corresponding energy

$$\mathcal{E}(t,y) := \|y(t)\|^2_V + \|\dot{y}(t)\|^2_{L^2(0,L)},$$

along a trajectory y. The incremental function $\ell : V \times L^2(0,L) \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\ell((y,z),u) := \frac{1}{2}\|(y,z)\|^2_{\mathcal{H}_3} + \frac{\beta}{2}u^2. \tag{3.100}$$

Moreover, later we will use the space $V^2 := \{q \in H^2(0,L) \cap V : q_x(L) = 0\}$.

Remark 3.4.1. Note that, for the case $dim(\Omega) \geq 2$, the generalization of controlled system (3.99) has the form

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,\infty) \times \Omega, \\ \partial_\nu y = u & \text{on } (0,\infty) \times \Gamma_c, \\ y = 0 & \text{on } (0,\infty) \times \Gamma_0, \\ y(0) = y_0^1, \quad \dot{y}(0) = y_0^2 & \text{on } \Omega, \end{cases} \tag{3.101}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with the smooth boundary $\partial\Omega := \overline{\Gamma_c} \cup \overline{\Gamma_0}$, two disjoint components Γ_c, Γ_0 are relatively open in $\partial\Omega$, and $int(\Gamma_c) \neq \emptyset$. Moreover $u \in L^2(0,\infty; L^2(\Gamma_c))$, $(y_0^1, y_0^2) \in \mathcal{H}_3$ with $\mathcal{H}_3 := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, and the functional space $H^1_{\Gamma_0}(\Omega)$ is defined by

$$H^1_{\Gamma_0}(\Omega) := \{f \in H^1(\Omega) : f|_{\Gamma_0} = 0\}.$$

For every $T > 0$, the solution operator $L : L^2(0,T; L^2(\Gamma_c)) \to C^0([0,T]; \mathcal{H}_3)$ defined by $u \mapsto (y, \dot{y})$ is not continuous, see [92] for an example. For our framework we would require that the solution operator is continuous from $L^2(0,T; L^2(\Gamma_c))$ to $C^0([0,T]; \mathcal{H}_3)$. However, this property does not hold as it was shown in [92]. In fact, the solution $(y(\cdot), \dot{y}(\cdot))$, depending on the geometry of Ω, on any level belongs to a strictly larger space than $\mathcal{H}_3$. Nevertheless, stabilization with respect to the energy $\mathcal{H}_3$ is of great interest from physical point of view, and to the best of our knowledge, most of results concerning the stabilization and controllability for the controlled system (3.101) are dealing with the energy in $\mathcal{H}_3$.

3.4.1 Existence and uniqueness of the solution

Consider the following one dimensional wave equation with an inhomogeneous Neumann boundary condition

$$\begin{cases} \ddot{y} - y_{xx} = 0 & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0 & t \in (0,T), \\ y_x(t,L) = u(t) & t \in (0,T), \\ y(0,x) = y_0^1, \quad \dot{y}(0,x) = y_0^2 & x \in (0,L). \end{cases} \tag{3.102}$$

Definition 3.4.1 (Weak solution). *Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_3$, and $u \in L^2(0,T)$ be given. Then y is referred to as the weak solution to* (3.102) *if $(y, \dot{y}) \in C^0([0,T]; \mathcal{H}_3)$,*

$(y(0), \dot{y}(0)) = (y_0^1, y_0^2)$, *and for every* $\vartheta \in C^1([0,T] \times [0,L])$ *with* $\vartheta(0,\tau) = 0, \forall \tau \in [0,T]$, *it satisfies*

$$\begin{aligned}\int_0^L \dot{y}(t,x)\vartheta(t,x)dx - \int_0^L\ y_0^2(x)\vartheta(0,x)dx \\ + \int_0^t \int_0^L \left(-\dot{\vartheta}\dot{y} + \vartheta_x y_x\right) dxd\tau - \int_0^t u(\tau)\vartheta(\tau,L)d\tau = 0\end{aligned} \tag{3.103}$$

for almost every $t \in [0,T]$.

Theorem 3.4.1 (Existence and uniqueness of the weak solution)**.** *Let* $T > 0$*,* $L > 0$*,* $(y_0^1, y_0^2) \in \mathcal{H}_3$*, and* $u \in L^2(0,T)$ *be given. Then there exists an unique weak solution* y *with*

$$y \in C^0([0,T];V) \cap C^1([0,T];L^2(0,L)),$$

and for this weak solution we have the estimate

$$\|y\|_{C^0([0,T];V)} + \|\dot{y}\|_{C^0([0,T];L^2(0,L))} + \|\ddot{y}\|_{L^2(0,T;V^*)} \le c_3 \left(\|y_0^1\|_V + \|y_0^2\|_{L^2(0,L)} + \|u\|_{L^2(0,T)}\right), \tag{3.104}$$

where the constant c_3 *depends only on* T *and* L*. Furthermore,* $y(\cdot,L) \in H^1(0,T)$ *and we have*

$$\|\dot{y}(\cdot,L)\|_{L^2(0,T)} \le c_4 \left(\|y_0^1\|_V + \|y_0^2\|_{L^2(0,L)} + \|u\|_{L^2(0,T)}\right), \tag{3.105}$$

for a constant c_4 *depending only on* L *and* T*.*

Proof. The proof is given in, e.g., [45, page 68]. □

We will later need the following auxiliary problem.

$$\begin{cases} \ddot{y} - y_{xx} = f & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0 & t \in (0,T), \\ y_x(t,L) = 0 & t \in (0,T), \\ y(0,x) = y_0^1, \quad \dot{y}(0,x) = y_0^2 & x \in (0,L). \end{cases} \tag{3.106}$$

Definition 3.4.2 (Very weak solution)**.** *Let* $L > 0$*,* $T > 0$*,* $(y_0^1, y_0^2) \in L^2(0,L) \times V^*$*, and* $f \in L^2(0,T;V^*)$ *be given. A function* $y \in L^2(0,T;L^2(0,L))$ *is referred to as the very weak solution of* (3.106)*, if the following inequality holds*

$$\int_0^T (g(t), y(t))_{L^2(0,L)}\, dt = -(y_0^1, \dot{\vartheta}(0))_{L^2(0,L)} + \langle y_0^2, \vartheta(0)\rangle_{V^*,V} + \int_0^T \langle f(t), \vartheta(t)\rangle_{V^*,V}\, dt,$$

for all $g \in L^2(0,T;L^2(0,L))$ *and* $\vartheta \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];V)$ *the weak solution of the following backward in time problem*

$$\begin{cases} \ddot{\vartheta} - \vartheta_{xx} = g & (t,x) \in (0,T) \times (0,L), \\ \vartheta(t,0) = 0 & t \in (0,T), \\ \vartheta_x(t,L) = 0 & t \in (0,T), \\ \vartheta(T,x) = 0, \quad \dot{\vartheta}(T,x) = 0 & x \in (0,L). \end{cases}$$

The very weak solution is also called solution by transposition.

We have the following existence and regularity results for the very weak solution of (3.106), see, e.g., [97, 99].

Theorem 3.4.2 (Existence and uniqueness of the very weak solution). *For every $L > 0$, $T > 0$, $f \in L^2(0,T;V^*)$, and every pair $(y_0^1, y_0^2) \in L^2(\Omega) \times V^*$, there exists a unique very weak solution to* (3.106). *Moreover, this very weak solution belongs to the space*

$$C^1([0,T];V^*) \cap C^0([0,T];L^2(0,L)),$$

and we have the following estimate

$$\|y\|_{C^0([0,T];L^2(0,L))} + \|\dot{y}\|_{C^0([0,T];V^*)} \leq \bar{c}_4 \left(\|y_0^1\|_{L^2(0,L)} + \|y_0^2\|_{V^*} + \|f\|_{L^2(0,T;V^*)} \right), \quad (3.107)$$

where the constant $\bar{c}_4$ is independent of y_0^1, y_0^2, and f.

3.4.2 Existence of the optimal control

Consider the following optimal control problem

$$\min \left\{ J_T(u;(y_0^1, y_0^2)) \mid (y,u) \text{ satisfies } (3.102), u \in L^2(0,T) \right\}. \qquad (OPN_T)$$

Proposition 3.4.1 (Existence and uniqueness of the optimal control). *For every $T > 0$ and $(y_0^1, y_0^2) \in \mathcal{H}_3$, the optimal control problem (OPN_T) admits a unique solution.*

Proof. We use the standard argument of calculus of variation. Since the objective function $J_T(u;(y_0^1, y_0^2))$ is bounded from below we have

$$\inf_{u \in L^2(0,T)} J_T(u;(y_0^1, y_0^2)) = \sigma < \infty.$$

Therefore, there is a minimizing sequence $\{u^n\}_n \subset L^2(0,T)$ such that

$$\|u^n\|^2_{L^2(0,T)} \leq \frac{2}{\beta} J_T(u^n;(y_0^1, y_0^2)) < \infty,$$

and, as a consequence, the sequence has a weakly convergent subsequence $u^n \rightharpoonup u^*$ with the limit $u^* \in L^2(0,T)$. Due to Theorem 3.4.1 and estimate (3.104), there exists a bounded sequence of weak solutions $\{y^n\}_n \subset L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(0,L))$ to (3.102) corresponding to the control sequence $\{u^n\}_n$. Hence, there are weakly-star convergent subsequences of $\{y^n\}_n$, $\{\dot{y}^n\}_n$, and $\{\ddot{y}^n\}_n$ such that

$$\begin{aligned} y^n &\rightharpoonup^* y^* \text{ in } L^\infty(0,T;V), \\ \dot{y}^n &\rightharpoonup^* \dot{y}^* \text{ in } L^\infty(0,T;L^2(0,L)), \\ \ddot{y}^n &\rightharpoonup \ddot{y}^* \text{ in } L^2(0,T;V^*). \end{aligned}$$

Now it remains to show that y^* is the weak solution to (3.102) corresponding to the control u^*. To see this, we only need to pass the limits in the weak formulation (3.103) for the pair of sequences (y^n, u^n). For every $\vartheta \in C^1([0,T]\times[0,L])$ such that $\vartheta(0,\tau)=0$ for all $\tau \in [0,T]$, we have for every $t \in [0,T]$

$$\int_0^t \int_0^L (\dot{y}^n(\tau,x) - \dot{y}^*(\tau,x))\vartheta(\tau,x)dxd\tau \to 0,$$
$$\int_0^t \int_0^L (\dot{y}^n(\tau,x) - \dot{y}^*(\tau,x))\dot{\vartheta}(\tau,x)dxd\tau \to 0,$$
$$\int_0^t \int_0^L ((y^n)_x(\tau,x) - \dot{y}^*_x(\tau,x))\vartheta_x(\tau,x)dxd\tau \to 0,$$
$$\int_0^t (u^n(\tau) - u^*(\tau))\vartheta(\tau,L)d\tau \to 0.$$

Moreover, due to estimate (3.104), for every $t \in [0,T]$ the sequence $\{\dot{y}^n(t)\}_n$ is bounded in $L^2(0,L)$. Hence, it has a weakly convergent subsequence $\dot{y}^n(t) \rightharpoonup \bar{y}_t$ with limit $\bar{y}_t \in L^2(0,L)$. We define the time-point evaluation operator $\mathcal{I}_t : H^1(0,T;V^*) \to V^*$ at a time $t \in [0,T]$ by $p \mapsto p(t)$. This operator is continuous. Moreover, for every $t \in [0,T]$ and $q \in V$ we have

$$\begin{aligned}(\bar{y}_t, q)_{L^2(0,L)} = \lim_{n\to\infty} \langle \mathcal{I}_t \dot{y}^n, q\rangle_{V^*,V} &= \lim_{n\to\infty} \langle \dot{y}^n, \mathcal{I}_t^* q\rangle_{H^1(0,T;V^*),(H^1(0,T;V^*))^*} \\ &= \langle \dot{y}^*, \mathcal{I}_t^* q\rangle_{H^1(0,T;V^*),(H^1(0,T;V^*))^*} \\ &= \langle \mathcal{I}_t \dot{y}^*, q\rangle_{V^*,V},\end{aligned} \tag{3.108}$$

where $\mathcal{I}_t^* : V \to (H^1(0,T;V^*))^*$ is the adjoint operator to $\mathcal{I}_t$. Therefore, for every $\vartheta \in C^1([0,T]\times[0,L])$ such that $\vartheta(0,\tau)=0$ for all $\tau \in [0,T]$, we have for almost every $t \in [0,T]$

$$\int_0^L \dot{y}^n(t,x)\vartheta(t,x)dx \to \int_0^L \dot{y}^*(t,x)\vartheta(t,x)dx, \tag{3.109}$$

and, as a consequence, y^* is the weak solution to (3.102) corresponding to the control u^*. Now since the solution operator $S : L^2(0,T) \to L^\infty(0,T;\mathcal{H}_3)$ defined by $u \mapsto (y,\dot{y})$ is affine and continuous, the objective function $J_T(\cdot; y_0^1, y_0^2)$ is weakly lower semi-continuous and we have

$$0 \le J_T(u^*; (y_0^1, y_0^2)) \le \liminf_{n\to\infty} J_T(u^n; (y_0^1, y_0^2)) = \sigma.$$

As a result, the pair (y^*, u^*) is optimal. Uniqueness follows from the strictly convexity of $J_T(\cdot; (y_0^1, y_0^2))$. □

3.4.3 Optimality conditions

Lemma 3.4.1. *Consider the following linear wave equation*

$$\begin{cases} \ddot{y} - y_{xx} = 0 & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = 0 & t \in (0,T), \\ y_x(t,L) = u(t) & t \in (0,T), \\ y(0,x) = 0, \quad \dot{y}(0,x) = 0 & x \in (0,L), \end{cases} \tag{3.110}$$

with an inhomogeneous Neumann part $u \in L^2(0,T)$*. Moreover, let* $g \in L^2(0,T;V^*)$ *and* $(p_T^1, p_T^2) \in L^2(0,L) \times V^*$*. Then the weak solution* y *to* (3.110) *and the very weak solution* p *to*

$$\begin{cases} \ddot{p} - p_{xx} = g & (t,x) \in (0,T) \times (0,L), \\ p(t,0) = 0 & t \in (0,T), \\ p_x(t,L) = 0 & t \in (0,T), \\ p(T,x) = p_T^1, \quad \dot{p}(T,x) = p_T^2 & x \in (0,L), \end{cases} \tag{3.111}$$

satisfy the following equality

$$\int_0^T u(t)p(t,L)\,dt = \int_0^T \langle g(t), y(t)\rangle_{V^*,V}\,dt + (p_T^1, \dot{y}(T))_{L^2(0,L)} - \langle p_T^2, y(T)\rangle_{V^*,V}. \tag{3.112}$$

Proof. First due to Theorem 3.4.2 and the time reversibility of the linear wave equation, the very weak solution p of (3.111) belongs to the space

$$C^1([0,T];V^*) \cap C^0([0,T];L^2(0,L)).$$

Moreover, due to Theorem 3.4.1, the weak solution y of (3.110) belongs to the space $C^1([0,T];L^2(0,L)) \cap C^0([0,T];V)$. Therefore, the right hand side of (3.112) is well-posed. We show that $p(\cdot,L) \in L^2(0,T)$ is well-defined and can be associated to the very weak solution p to (3.111). Consider the following linear functional

$$\ell_{g,p_T^1,p_T^2}(u) := \int_0^T \langle g(t), y(t)\rangle_{V^*,V}\,dt + (p_T^1, \dot{y}(T))_{L^2(0,L)} - \langle p_T^2, y(T)\rangle_{V^*,V}, \tag{3.113}$$

where $y(u)$ is the solution of (3.110) depending on $u \in L^2(0,T)$. Moreover, due to (3.113) and estimate (3.104) we have

$$\begin{aligned} |\ell_{g,p_T^1,p_T^2}(u)| \leq& \|g\|_{L^2(0,T;V^*)}\|y\|_{L^2(0,T;V)} + \|\dot{y}(T)\|_{L^2(0,L)}\|p_T^1\|_{L^2(0,L)} + \|y(T)\|_V\|p_T^2\|_{V^*}, \\ \leq& \hat{c}_4 \left(\|g\|_{L^2(0,T;V^*)} + \|p_T^1\|_{L^2(0,L)} + \|p_T^2\|_{V^*}\right)\|u\|_{L^2(0,T)}, \end{aligned} \tag{3.114}$$

for a constant $\hat{c}_4$ depending on T and L. Therefore, $\ell_{g,p_T^1,p_T^2} : L^2(0,T) \to \mathbb{R}$ is a continuous linear functional. By using the Riesz representation theorem and (3.114), there exist an unique object $p(\cdot,L) \in L^2(0,T)$ such that

$$\int_0^T u(t)p(t,L)\,dt = \ell_{g,p_T^1,p_T^2}(u), \tag{3.115}$$

and we have

$$\|p(\cdot,L)\|_{L^2(0,T)} \leq \hat{c}_4 \left(\|g\|_{L^2(0,T;V^*)} + \|p_T^1\|_{L^2(0,L)} + \|p_T^2\|_{V^*}\right). \tag{3.116}$$

Next, we show that $p(\cdot, L)$ is the trace of the solution p to (3.111). Since the spaces V, $L^2(0, L)$, and $L^2(0, T; L^2(0, L))$ are dense in the spaces $L^2(0, L)$, V^*, and $L^2(0, T; V^*)$, respectively, there exist sequences $\{p_T^{1n}\}_n \subset V$, $\{p_T^{2n}\}_n \subset L^2(0, L)$, and $\{g^n\}_n \subset L^2(0, T; L^2(0, L))$ such that

$$\begin{aligned} p_T^{1n} &\to p_T^1 \quad \text{in } L^2(0, L), \\ p_T^{2n} &\to p_T^2 \quad \text{in } V^*, \\ g^n &\to g \quad \text{in } L^2(0, T; V^*). \end{aligned}$$

Moreover, for any triple $(p_T^{1n}, p_T^{2n}, g^n) \in V \times L^2(0, L) \times L^2(0, T; L^2(0, T))$, the solution of p^n of (3.111) belongs to the space $C^1([0, T]; L^2(0, L)) \cap C^0([0, T]; V)$ (see, e.g., [99]). Now, by using (3.107) we have

$$\begin{aligned} &\|p^n - p\|_{C^0([0,T];L^2(0,L))} + \|\dot{p^n} - \dot{p}\|_{C^0([0,T];V^*)} \\ &\qquad \le \bar{c}_4 \left(\|p_T^{1n} - p_T^1\|_{L^2(0,L)} + \|p_T^{2n} - p_T^2\|_{V^*} + \|g^n - g\|_{L^2(0,T;V^*)} \right). \end{aligned} \tag{3.117}$$

For the solution p^n of (3.111) with $(p_T^{1n}, p_T^{2n}, g^n) \in V \times L^2(0, L) \times L^2(0, T; L^2(0, L))$ and the solution y of (3.110) we have

$$\int_0^T u(t) p^n(t, L)\, dt = \int_0^T \langle g^n(t), y(t) \rangle_{V^*, V}\, dt + (p_T^{1n}, \dot{y}(T))_{L^2(0,L)} - \langle p_T^{2n}, y(T) \rangle_{V^*, V}. \tag{3.118}$$

Moreover, due to (3.116), we have

$$\|p^n(\cdot, L)\|_{L^2(0,T)} \le \hat{c}_4 \left(\|g^n\|_{L^2(0,T;V^*)} + \|p_T^{1n}\|_{L^2(0,L)} + \|p_T^{2n}\|_{V^*} \right).$$

Therefore the sequence $p^n(\cdot, L)$ is bounded in $L^2(0, T)$ and, as a consequence, there is a weakly convergent subsequence $\{p^n(\cdot, L)\}_n$ such that $p^n(\cdot, L) \rightharpoonup p^*(\cdot, L)$ with a function $p^*(\cdot, L) \in L^2(0, T)$. Now by passing the limits we have

$$\begin{aligned} \int_0^T \langle g^n(t), y(t) \rangle_{V^*, V}\, dt &\to \int_0^T \langle g(t), y(t) \rangle_{V^*, V}\, dt, \\ (p_T^{1n}, \dot{y}(T))_{L^2(0,L)} &\to (p_T^1, \dot{y}(T))_{L^2(0,L)}, \\ \langle p_T^{2n}, y(T) \rangle_{V^*, V} &\to \langle p_T^2, y(T) \rangle_{V^*, V}, \\ \int_0^T u(t) p^n(t, L)\, dt &\to \int_0^T u(t) p^*(t, L)\, dt, \end{aligned}$$

and, as a consequence, by using (3.118) we obtain

$$\int_0^T u(t) p^*(t, L)\, dt = \int_0^T \langle g(t), y(t) \rangle_{V^*, V}\, dt + (p_T^1, \dot{y}(T))_{L^2(0,L)} - \langle p_T^2, y(T) \rangle_{V^*, V}. \tag{3.119}$$

Moreover, due to (3.117), we infer that

$$p^n \to p \text{ in } C^1([0, T]; V^*) \cap C^0([0, T]; L^2(0, L)).$$

Finally, (3.113), (3.115), and (3.119) imply

$$\int_0^T u(t)p^*(t,L)\,dt = \int_0^T u(t)p(t,L)\,dt = \ell_{g,p_T^1,p_T^2}(u). \tag{3.120}$$

Since equality (3.120) holds for all $u \in L^2(0,T)$, we conclude $p^*(\cdot,L) = p(\cdot,L)$ in $L^2(0,T)$. □

In the following, we derive the first-order optimality conditions for the finite horizon problems of the form (OPN_T). Due to the presence of the tracking term for the velocity $\dot{y} \in L^2(0,T;L^2(0,L))$ in the objective function of (OPN_T), we will see that the solution to the ajdoint equation has less regularity than the one to (3.102) and it exists in the very weak sense.

Theorem 3.4.3 (First-order optimality conditions)**.** *Let $(\bar{y},\bar{u})$ be the optimal solution to (OPN_T). Then for $(y_0^1,y_0^2) \in \mathcal{H}_3$ we have the following optimality conditions*

$$\begin{cases} \ddot{\bar{y}} - \bar{y}_{xx} = 0 & (t,x) \in (0,T)\times(0,L),\\ \bar{y}(t,0) = 0 & t \in (0,T),\\ \bar{y}_x(t,L) = \bar{u}(t) & t \in (0,T),\\ \bar{y}(0,x) = y_0^1, \quad \dot{\bar{y}}(0,x) = y_0^2 & x \in (0,L),\\ \ddot{\bar{p}} - \bar{p}_{xx} = \ddot{\bar{y}} + \bar{y}_{xx} & (t,x) \in (0,T)\times(0,L),\\ \bar{p}(t,0) = 0 & t \in (0,T),\\ \bar{p}_x(t,L) = 0 & t \in (0,T),\\ \bar{p}(T,x) = 0, \quad \dot{\bar{p}}(T,x) = \dot{\bar{y}}(T) & x \in (0,L),\\ \beta\bar{u}(t,L) = \bar{p}(t,L) & t \in (0,T), \end{cases}$$

where p the solution of the adjoint equation belongs to the space $C^0([0,T];L^2(0,L)) \cap C^1([0,T];V^*)$.

Proof. For sake of simplicity in notation, we remove the overbar in the notation of $(\bar{y},\bar{u})$. Let $(y_0^1,y_0^2) \in \mathcal{H}_3$ be given. Computing the directional derivative of $J_T(\cdot,(y_0^1,y_0^2))$ at u in the direction of an arbitrary $\delta u \in L^2(0,T)$ we obtain

$$\begin{aligned} &J_T'(u,(y_0^1,y_0^2))\delta u\\ &= \int_0^T (y(t),\delta y(t))_V dt + \int_0^T (\dot{y}(t),\dot{\delta y}(t))_{L^2(0,L)} dt + \beta\int_0^T u(t)\delta u(t)dt,\\ &= \int_0^T \langle -y_{xx}(t),\delta y(t)\rangle_{V^*,V} dt + \int_0^T (\dot{y}(t),\dot{\delta y}(t))_{L^2(0,L)} dt + \beta\int_0^T u(t)\delta u(t)dt, \end{aligned} \tag{3.121}$$

where the operator $-\partial_{xx} : V \to V^*$ defined by $q \mapsto -q_{xx}$ with the boundary conditions $q_x(L) = 0$ and $q(0) = 0$ is an isomorphism, and $\delta y \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];V)$

is the weak solution of

$$\begin{cases} \ddot{\delta y} - \delta y_{xx} = 0 & (t,x) \in (0,T)\times(0,L), \\ \delta y(t,0) = 0 & t \in (0,T), \\ \delta y_x(t,L) = \delta u(t) & t \in (0,T), \\ \delta y(0,x) = 0, \quad \dot{\delta y}(0,x) = 0 & x \in (0,L). \end{cases} \tag{3.122}$$

Next, we show that

$$\int_0^T (\dot{y}(t), \dot{\delta y}(t))_{L^2(0,L)} dt = (\dot{y}(T), \delta y(T))_{L^2(0,L)} - \int_0^T \langle \ddot{y}(t), \delta y(t)\rangle_{V^*,V} dt. \tag{3.123}$$

We proceed with the help of an approximation argument. Since the spaces V^2, V, and $\mathcal{Z}$ with $\mathcal{Z} := \{q \in C^3([0,T]) : u(0) = \dot{u}(0) = 0\}$ are dense in the spaces V, $L^2(0,L)$, and $L^2(0,T)$, respectively, there exist sequences $\{y_0^{1n}\}_n \subset V^2$, $\{y_0^{2n}\}_n \subset V$, and $\{u^n\}_n \subset \mathcal{Z}$ such that

$$\begin{aligned} y_0^{1n} &\to y_0^1 && \text{in } V, \\ y_0^{2n} &\to y_0^2 && \text{in } L^2(0,L), \\ u^n &\to u && \text{in } L^2(0,T). \end{aligned}$$

Moreover, for any triple $(y_0^{1n}, y_0^{2n}, u^n) \in V^2 \times V \times \mathcal{Z}$, the solution of y^n of (3.102) belongs to the space $C^1([0,T];V) \cap C^0([0,T];V^2)$ with $\ddot{y^n} \in L^2(0,T;L^2(0,L))$ (see, e.g., [45, page 69]), and due to (3.104) we have

$$\begin{aligned} &\|y^n - y\|_{C^0([0,T];V)} + \|\dot{y^n} - \dot{y}\|_{C^0([0,T];L^2(0,L))} + \|\ddot{y^n} - \ddot{y}\|_{L^2(0,T;V^*)} \\ &\qquad \le c_3 \left(\|y_0^{1n} - y_0^1\|_V + \|y_0^{2n} - y_0^2\|_{L^2(0,L)} + \|u^n - u\|_{L^2(0,T)} \right). \end{aligned}$$

For the solution y^n of (3.102) with $(y_0^{1n}, y_0^{2n}, u^n) \in V^2 \times V \times \mathcal{Z}$ and the solution δy of (3.122) we have

$$\begin{aligned} \int_0^T (\dot{y^n}(t), \dot{\delta y}(t))_{L^2(0,L)} dt &= \int_0^T \langle \dot{y^n}(t), \dot{\delta y}(t)\rangle_{V,V^*} dt = \\ &(\dot{y^n}(T), \delta y(T))_{L^2(0,L)} - \int_0^T \langle \ddot{y^n}(t), \delta y(t)\rangle_{V^*,V} dt. \end{aligned}$$

By passing the limits we obtain

$$\begin{aligned} \int_0^T (\dot{y^n}(t), \dot{\delta y}(t))_{L^2(0,L)} dt &\to \int_0^T (\dot{y}(t), \dot{\delta y}(t))_{L^2(0,L)} dt, \\ (\dot{y^n}(T), \delta y(T))_{L^2(0,L)} &\to (\dot{y}(T), \delta y(T))_{L^2(0,L)} = \langle \dot{y}(T), \delta y(T)\rangle_{V^*,V}, \\ \int_0^T \langle \ddot{y^n}(t), \delta y(t)\rangle_{V^*,V} dt &\to \int_0^T \langle \ddot{y}(t), \delta y(t)\rangle_{V^*,V} dt, \end{aligned}$$

and, as a consequence, we are finished with the justification of (3.123). Now due to (3.121) and (3.123), the first order optimality condition is equivalent to the following equality

$$\int_0^T \langle -\ddot{y}(t) - y_{xx}(t), \delta y(t)\rangle_{V^*,V} dt + \langle \dot{y}(T), \delta y(T)\rangle_{V^*,V} + \beta \int_0^T \delta u(t) u(t)\, dt = 0. \quad (3.124)$$

Moreover, due to Lemma 3.4.1, and using equality (3.112) for equation (3.122), we have

$$-\int_0^T \langle g(t), \delta y(t)\rangle_{V^*,V}\, dt - (p_T^1, \dot{\delta y}(T))_{L^2(0,L)} + \langle p_T^2, \delta y(T)\rangle_{V^*,V} + \int_0^T \delta u(t) p(t,L)\, dt = 0, \quad (3.125)$$

for a given $(g, p_T^1, p_T^2) \in L^2(0,T;V^*) \times L^2(0,L) \times V^*$ and its corresponding very weak solution $p \in C^1([0,T];V^*) \cap C^0([0,T];L^2(0,L))$ to (3.111). By comparing (3.124) with (3.125), and since $\delta u \in L^2(0,T)$ is arbitrary, we infer that

$$\begin{aligned}
\beta u &= p(\cdot, L) && \text{in } L^2(0,T),\\
p_T^1 &= 0 && \text{in } L^2(0,L),\\
p_T^2 &= \dot{y}(T) && \text{in } V^*,\\
g &= \ddot{y} + y_{xx} && \text{in } L^2(0,T;V^*).
\end{aligned}$$

□

3.4.4 Stabilizability

To specify the required *observability conditions*, for any given $(\phi_0^1, \phi_0^2) \in \mathcal{H}_3$, we denote by ϕ the weak solution of the following system

$$\begin{cases}
\ddot{\phi} - \phi_{xx} = 0 & (t,x) \in (0,T) \times (0,L),\\
\phi(t,0) = 0 & t \in (0,T),\\
\phi_x(t,L) = 0 & t \in (0,T),\\
\phi(0,x) = \phi_0^1, \quad \dot{\phi}(0,x) = \phi_0^2 & x \in (0,L).
\end{cases} \quad (3.126)$$

Then we have the following observability inequalities:

OB5. There exists $T_{ob3} > 0$ such that for every $T \geq T_{ob3}$, the weak solution ϕ to (3.126) with $(\phi, \dot{\phi}) \in C^0([0,T];\mathcal{H}_3)$ satisfies the inequality

$$c_{ob3} \|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_3}^2 \leq \int_0^T |\dot{\phi}(t,L)|^2 dt \qquad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_3,$$

where the positive constant c_{ob3} depends only on T and L.

OB6. There exists $T_{ob4} > 0$ such that for every $T \geq T_{ob4}$, the weak solution ϕ to (3.126) with $(\phi, \dot{\phi}) \in C^0([0,T]; \mathcal{H}_3)$ satisfies the inequality

$$c_{ob4} \|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_3}^2 \leq \int_0^T \int_\omega |\dot{\phi}|^2 dx dt \quad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_3,$$

where the positive constant c_{ob4} depends only on T and $\omega \subset (0, L)$.

The proof of the following equivalence can be found in, e.g., [130]. Nevertheless, we provide here a proof for completeness.

Theorem 3.4.4 (Global stabilizability). *Suppose that $(y_0^1, y_0^2) \in \mathcal{H}_3$ is given. Then the solution of the controlled system (3.99) with the feedback law $u(t) := -\dot{y}(t, L)$ converges exponentially to zero with respect to $\mathcal{H}_3$, i.e.*

$$\mathcal{E}(t, y) \leq M e^{-\alpha t} \mathcal{E}(0, y) = M e^{-\alpha t} \|(y_0^1, y_0^2)\|_{\mathcal{H}_3}^2 \tag{3.127}$$

for positive constants M and α independent of (y_0^1, y_0^2), if and only if the observability condition OB5 holds.

Proof. First assume that OB5 holds. We show the exponential decay inequality (3.127).

Let $(y_0^1, y_0^2) \in \mathcal{H}_3$ be given. Setting $u(t) := -\dot{y}(t, L)$ in (3.99), we obtain a closed-loop system which is well-posed in the space $\mathcal{H}_3$ and the unique weak solution y of this system belongs to the space

$$C^0([0,\infty); V) \cap C^1([0,\infty); L^2(0, L)),$$

and we have $\dot{y}(\cdot, L) \in L^2(0, \infty)$, see, e.g., [133]. Now, for an arbitrary $T > 0$, consider the following controlled system

$$\begin{cases} \ddot{y} - y_{xx} = 0 & (t, x) \in (0, T) \times (0, L), \\ y(t, 0) = 0 & t \in (0, T), \\ y_x(t, L) = -\dot{y}(t, L) & t \in (0, T), \\ y(0, x) = y_0^1, \quad \dot{y}(0, x) = y_0^2 & x \in (0, L). \end{cases} \tag{3.128}$$

Suppose that the solution y of (3.128) is smooth enough. By taking L^2-inner product of (3.128) with $\dot{y}$, formally, and integrating over $[0, T]$, we obtain

$$\|(y(T), \dot{y}(T))\|_{\mathcal{H}_3}^2 - \|(y(0), \dot{y}(0))\|_{\mathcal{H}_3}^2 = -2 \int_0^T |\dot{y}(t, L)|^2 dt. \tag{3.129}$$

We can approximate the pair $(y_0^1, y_0^2) \in \mathcal{H}_3$ by a sequence of pairs $(y_0^{1n}, y_0^{2n}) \in (H^2(0, L) \cap V) \times V$ for which, by using the standard semi group theory, the corresponding solutions $\{y^n\}_n$ of (3.128) belong to the space $C^0([0,T]; H^2(0, L) \cap V) \cap C^1([0,T]; V)$ and converge to y in the space $C^0([0,T]; V) \cap C^1([0,T]; L^2(0, L))$, as n tends to infinity. Then by passing to the limits, it can be shown that equality (3.129) is also true for the initial pair $(y_0^1, y_0^2) \in \mathcal{H}_3$ and its corresponding solution y.

Further, the solution y of (3.128) can be rewritten as $y = \phi + \psi$, where ϕ is the solution of (3.126) with the initial pair (y_0^1, y_0^2) in place of (ϕ_0^1, ϕ_0^2) and ψ is the solution of the following problem

$$\begin{cases} \ddot{\psi} - \psi_{xx} = 0 & (t,x) \in (0,T) \times (0,L), \\ \psi(t,0) = 0 & t \in (0,T), \\ \psi_x(t,L) = -\dot{y}(t,L) & t \in (0,T), \\ \psi(0,x) = 0 \quad \dot{\psi}(0,x) = 0 & x \in (0,L). \end{cases} \tag{3.130}$$

Using the observability condition OB5 for (3.126) with the initial pair (y_0^1, y_0^2), and estimate (3.105) for ψ with $u = -\dot{y}(\cdot, L)$, we obtain

$$\begin{aligned} \|(y_0^1, y_0^2)\|_{\mathcal{H}_3}^2 &\le \frac{1}{c_{ob3}} \int_0^{T_{ob3}} |\dot{\phi}(t,L)|^2 dt \\ &\le \frac{1}{c_{ob3}} \left(\int_0^{T_{ob3}} |\dot{y}(t,L)|^2 dt + \int_0^{T_{ob3}} |\dot{\psi}(t,L)|^2 dt \right) \\ &\le \frac{1+c_4^2}{c_{ob3}} \int_0^{T_{ob3}} |\dot{y}(t,L)|^2 dt, \end{aligned} \tag{3.131}$$

for a T_{ob3} depending on L. Combining (3.129) and (3.131), we have

$$\begin{aligned} \|(y(T_{ob3}), \dot{y}(T_{ob3}))\|_{\mathcal{H}_3}^2 - \|(y(0), \dot{y}(0))\|_{\mathcal{H}_3}^2 &= -2 \int_0^{T_{ob3}} |\dot{y}(t,L)|^2 dt \\ &\le \frac{-2c_{ob3}}{1+c_4^2} \|(y(0), \dot{y}(0))\|_{\mathcal{H}_3}^2 \\ &\le \frac{-2c_{ob3}}{1+c_4^2} \|(y(T_{ob3}), \dot{y}(T_{ob3}))\|_{\mathcal{H}_3}^2. \end{aligned}$$

As a result, we have

$$\mathcal{E}(t,y) \le M e^{-\alpha t} \mathcal{E}(0,y) \quad \text{for every } t > 0,$$

where $\alpha := \frac{\ln(1+\frac{2c_{ob3}}{1+c_4^2})}{T_{ob3}}$ and $M := (1 + \frac{2c_{ob3}}{1+c_4^2})$.

Next we show that the stabilizability property (3.127) implies the observability condition OB5 for (3.126) with an arbitrary initial pair $(y_0^1, y_0^2) \in \mathcal{H}_3$.

Let inequality (3.127) holds for the pair $(y_0^1, y_0^2) \in \mathcal{H}_3$. Then by using (3.127) and (3.129), there exists a $T' > 0$ such that

$$\int_0^{T'} |\dot{y}(t,L)|^2 dt \ge \frac{1}{4} \mathcal{E}(0,y). \tag{3.132}$$

Moreover, the solution ϕ to (3.126) with the initial pair (y_0^1, y_0^2) can be rewritten as $\phi := y - \psi$, where y is the weak solution to (3.128) and ψ is the weak solution to (3.130) for T' instead of T.

Now, by taking L^2-inner product of (3.130) with $\dot{\psi}$, formally, and integrating over $[0, T']$, we obtain

$$0 \leq \frac{1}{2}\big(\|\dot{\psi}(T')\|^2_{L^2(0,L)} + \|\psi(T')\|^2_V\big) = \int_0^{T'} -\dot{y}(t,L)\dot{\psi}(t,L)dt$$
$$= \int_0^{T'} -(\dot{\psi}(t,L) + \dot{\phi}(t,L))\dot{\psi}(t,L)dt, \tag{3.133}$$

Note that for $u \in \mathcal{Z}$ with $\mathcal{Z} := \{q \in C^3([0,T']) : u(0) = \dot{u}(0) = 0\}$ as the inhomogeneous Neumann part in (3.130), instead of $-\dot{y}(t,L) \in L^2(0,T')$, the solution of (3.130) belongs to the space $C^0([0,T']; V^2) \cap C^1([0,T']; V)$ (see, e.g., [45, page 69]). Therefore, by using a density argument and the fact that $\mathcal{Z}$ is dense $L^2(0,T')$, it can be shown that the inequality (3.133) is also true for the weak solution of (3.130) with $-\dot{y}(\cdot,L) \in L^2(0,T')$ as the inhomogeneous Neumann boundary condition. Moreover, (3.133) implies that

$$\int_0^{T'} |\dot{\psi}(t,L)|^2 dt \leq \int_0^{T'} |\dot{\phi}(t,L)|^2 dt. \tag{3.134}$$

Using (3.132), (3.134), and the following inequality

$$\int_0^{T'} |\dot{\phi}(t,L)|^2 dt \geq \int_0^{T'} |\dot{y}(t,L)|^2 dt - \int_0^{T'} |\dot{\psi}(t,L)|^2 dt,$$

we complete the proof with

$$\int_0^{T'} |\dot{\phi}(t,L)|^2 dt \geq \frac{1}{8}\mathcal{E}(0,y).$$

□

3.4.5 Stability of RHC

From this point on, we denote $(y(t), \dot{y}(t))$ by $\mathcal{Y}(t)$.

Definition 3.4.3 (Value function)**.** *for every pair $(y_0^1, y_0^2) =: \mathcal{Y}_0 \in \mathcal{H}_3$, the infinite horizon value function $V_\infty : \mathcal{H}_3 \to \mathbb{R}_+$ is defined as*

$$V_\infty(\mathcal{Y}_0) := \min_{u \in L^2(0,\infty)} \{J_\infty(u, \mathcal{Y}_0) \text{ subject to } (3.99)\}.$$

Similarly, the finite horizon value function $V_T : \mathcal{H}_3 \to \mathbb{R}_+$ is defined by

$$V_T(\mathcal{Y}_0) := \min_{u \in L^2(0,T)} \{J_T(u, \mathcal{Y}_0) \text{ subject to } (3.102)\}. \tag{3.135}$$

Lemma 3.4.2. *Suppose that the observability conditions OB5-OB6 hold. For every $T > 0$, there exists a control $\hat{u} \in L^2(0,T)$ for (3.102) such that*

$$V_T(\mathcal{Y}_0) \leq J_T(\hat{u}; \mathcal{Y}_0) \leq \gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_3} \tag{3.136}$$

for every initial pair $(y_0^1, y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_3$, where $\gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous, and bounded function. Moreover, there exists a constant $\gamma_1(T) > 0$ depending on T such that

$$V_T(\mathcal{Y}_0) \geq \gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_3} \tag{3.137}$$

for all $(y_0^1, y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_3$.

Proof. Let an initial pair $(y_0^1, y_0^2) \in \mathcal{H}_3$ be given. By setting $u(t) := -\dot{y}(t, L)$ in the controlled system (3.102), and using Theorem 3.4.4 we obtain

$$\|(y(t), \dot{y}(t))\|^2_{\mathcal{H}_3} \leq M e^{-\alpha t}\|(y(0), \dot{y}(0))\|^2_{\mathcal{H}_3} \quad \text{for all } t \in [0, T],$$

where the constants M and α were defined in Theorem 3.4.4. By integrating from 0 to T we have

$$\int_0^T \|(y(t), \dot{y}(t))\|^2_{\mathcal{H}_3} dt \leq \frac{M}{\alpha}(1 - e^{-\alpha T})\|(y(0), \dot{y}(0))\|^2_{\mathcal{H}_3}.$$

Moreover, by (3.127) and (3.129) we have

$$\begin{aligned} \int_0^T |u(t)|^2 dt &= \int_0^T |\dot{y}(t, L)|^2 dt \\ &\leq \frac{1}{2}\left(\mathcal{E}(0, y) + \mathcal{E}(T, y)\right) \leq \frac{(1 + M)}{2}\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_3}. \end{aligned} \tag{3.138}$$

By (3.100), (3.138), and the definition of value function V_T, we have

$$\begin{aligned} V_T(y_0^1, y_0^2) &\leq \int_0^T \left(\frac{1}{2}\|(y(t), \dot{y}(t))\|^2_{\mathcal{H}_3} + \frac{\beta}{2}|\dot{y}(t, L)|^2\right) dt \\ &\leq \left(\frac{M}{2\alpha}(1 - e^{-\alpha T}) + \frac{\beta(1 + M)}{4}\right)\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_3} \\ &= \gamma_2(T)\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_3}. \end{aligned}$$

Now to verify (3.137), we use the superposition argument for (3.102) with an arbitrary control $u \in L^2(0, T)$. We rewrite the solution of (3.102) as $y = \phi + \psi$ where ϕ is the solution to (3.126) with the initial pair (y_0^1, y_0^2), and ψ is the solution to the following problem

$$\begin{cases} \ddot{\psi} - \psi_{xx} = 0 & (t, x) \in (0, T) \times (0, L), \\ \psi(t, 0) = 0 & t \in (0, T), \\ \psi_x(t, L) = u(t) & t \in (0, T), \\ \psi(0, x) = 0 \quad \dot{\psi}(0, x) = 0 & x \in (0, L). \end{cases} \tag{3.139}$$

By using the observability condition OB6 for (3.126) with the initial pair (y_0^1, y_0^2) and $(0, L)$ instead of ω, and using estimate (3.104) for (3.139) we obtain

$$\begin{aligned}
\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_3} &\leq \frac{1}{c_{ob4}} \int_0^T \|\dot{\phi}(t)\|^2_{L^2(0,L)} dt \\
&\leq \frac{1}{c_{ob4}} \int_0^T \left(\|\dot{y}(t)\|^2_{L^2(0,L)} + \|\dot{\varphi}(t)\|^2_{L^2(0,L)} \right) dt \\
&\leq \frac{1}{c_{ob4}} \int_0^T \left(\|\dot{y}(t)\|^2_{L^2(0,L)} + T c_3^2 |u(t)|^2 \right) dt \\
&\leq c'(T) \int_0^T \left(\frac{1}{2} \|(y(t), \dot{y}(t))\|^2_{\mathcal{H}_3} + \frac{\beta}{2} |u(t)|^2 \right) dt \\
&= c'(T) \int_0^T \ell(\mathcal{Y}(t), u(t)) dt.
\end{aligned}$$

Since $u \in L^2(0, T)$ is arbitrary, we obtain (3.137) for a constant $c'(T)$ independent of u and (y_0^1, y_0^2). □

Remark 3.4.2. The property (3.137) is equivalent the injectivity of the differential Recatti operator corresponding to (OPN_T) which in turn is equivalent to the observability condition OB6, see, [57, Theorem 3.3].

Remark 3.4.3. Note that, as it has been shown in Lemma 3.4.2, the observability condition OB5 is equivalent to the stabilizability condition (3.136). The stabilizability condition (3.136) and well-posedness (Proposition 3.4.1) of open-loop problems in the form (OPN_T) are equivalent to the conditions (A2) and (A1) in Chapter 2, respectively. Moreover, since the stabilizability condition (3.136) holds globally, the condition (A3) is no longer needed and we can use the receding horizon framework introduced in Chapter 2. In addition, by using the uniform positiveness of the value function V_T which has been established in (3.137) based on the observability condition OB6, we shall verify the exponential stability of RHC.

Theorem 3.4.5 (Suboptimality and exponential decay). *Suppose that the observability conditions OB5-OB6 hold and let a sampling time $\delta > 0$ be given. Then there exist numbers $T^* > \delta$ and $\alpha \in (0, 1)$, such that for every fixed prediction horizon $T \geq T^*$ and every $\mathcal{Y}_0 \in \mathcal{H}_3$, the receding horizon control u_{rh} obtained from Algorithm 3.1 for the stabilization of* (3.99) *satisfies the suboptimality inequality*

$$\alpha V_\infty(\mathcal{Y}_0) \leq \alpha J_\infty(u_{rh}, \mathcal{Y}_0) \leq V_T(\mathcal{Y}_0) \leq V_\infty(\mathcal{Y}_0),$$

and exponential stability

$$\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}_3} \leq c' e^{-\zeta t} \|\mathcal{Y}_0\|^2_{\mathcal{H}_3},$$

where the positive numbers ζ, c' depend on α, δ, and T, but are independent of $\mathcal{Y}_0$.

Proof. The proof is similar to that of Theorem 3.3.4. □

3.5 Numerical Experiments

This section is devoted to numerical simulations. In order to justify our theoretical results for the receding horizon Algorithms 3.1, we give numerical results for all the cases: Distributed control, Dirichlet boundary control, and Neumann boundary control. We give also a short description about the descretization of control and state, optimization algorithm, and the implementation of Algorithm 3.1.

3.5.1 Discretization

To study the finite elements method for the wave equation we can mention the works [13, 14, 17, 18, 80, 81]. We follow the finite element framework which was investigated for the wave equation by [18] and applied for optimal control problems governed by the linear wave equations [85]. In this framework, the open-loop problems discretized temporally and spatially by appropriate finite elements frameworks, for which the approaches optimize-discretize and discretize-optimize are identical; see, e.g., [22, 109]. In all cases, for the discretization of the state we write the equation as a system of first order equations in time. The spatial discretization was done by a conforming linear finite element scheme using the continuous piecewise linear basis functions over a uniform mesh. This uniform mesh was generated by triangulation. For the temporal discretization of the state equation, a Petrov-Galerkin scheme based on continuous piecewise linear basis functions for the trial space and piecewise constant test function employed. By doing so, the resulting discretized system is equivalent to the space-discretized system to which the Crank-Nicolson time stepping method has been applied. Since the temporal test functions have been chosen to be piecewise constant, it is natural to discretize the adjoint equation and also control by these functions. This implies that the approximated gradient is consistent with both continuous functional and the discrete functional. In the case of the Dirichlet boundary control, the inhomogeneous Dirichlet condition $y|_{\Gamma_c} = u$ was treated by interpreting u as the trace of sufficiently smooth function $\hat{y}$ and solving the equation for $v = y - \hat{y}$ instead of y with homogeneous Dirichlet boundary conditions, see, e.g., [53, page 376] for more detail.

3.5.2 Optimization

Every discretized open-loop problem was first formulated in the reduced problem. This unconstrained optimization problem consists of minimizing a reduced objective function which depends only on the control variable u. Then these reduced problems were solved by applying the Barzilai-Borwein (BB) method [20, 118] equipped with a nonmonotone line search [48]. Moreover, the optimization algorithm was terminated as the $L^2(0, T; \mathcal{U})$-norm of the gradient for the reduced objective function was less than the tolerance 10^{-6}.

3.5.3 Implementation of RHC

Turning to our numerical experiment, we considered three examples corresponding to the cases: distributed control (3.2), Dirichlet boundary control (3.3), and Neumann boundary control (3.4). We applied Algorithm 3.2 which is based on Algorithm 3.1. For

Algorithm 3.2 RHC($\mathcal{Y}_0, T_\infty$)

Input: Let a final computational time horizon T_∞, and an initial state $\mathcal{Y}_0 := (y_0^1, y_0^2) \in \mathcal{H}$ be given.

1: Choose a prediction horizon $T < T_\infty$ and a sampling time $\delta \in (0, T]$.

2: Consider a grid $0 = t_0 < t_1 < \cdots < t_r = T_\infty$ on the interval $[0, T_\infty]$ where $t_i = i\delta$ for $i = 0, \ldots, r$.

3: **for** $i = 0, \ldots, r-1$ **do**

Solve the open-loop subproblem on $[t_i, t_i + T]$

$$\min \frac{1}{2} \int_{t_i}^{t_i+T} \|\mathcal{Y}(t)\|_{\mathcal{H}}^2 dt + \frac{\beta}{2} \int_{t_i}^{t_i+T} \|u(t)\|_{\mathcal{U}}^2 dt$$

subject to

$$\begin{cases} \dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{B}u \quad t \in (t_k, t_k + T), \\ \mathcal{Y}(t_i) = \mathcal{Y}_T^*(t_i) \text{ if } i \geq 1 \text{ or } \mathcal{Y}(t_i) = (y_0^1, y_0^2) \text{ if } i = 0, \end{cases}$$

where $\mathcal{Y}_T^*(\cdot)$ is the solution to the previous subproblem on $[t_{i-1}, t_{i-1} + T]$.

4: The receding horizon pair $(\mathcal{Y}_{rh}^*(\cdot), u_{rh}^*(\cdot))$ is the concatenation of the optimal pairs $(\mathcal{Y}_T^*(\cdot), u_T^*(\cdot))$ on the finite horizon intervals $[t_i, t_{i+1}]$ with $i = 0, \ldots, r-1$.

a given initial pair $(y_0^1, y_0^2) =: \mathcal{Y}_0 \in \mathcal{H}$ and a constant T_∞ defined as the final computation time, we ran Algorithm 3.2 for all the above mentioned cases. For every example, the receding horizon control u_{rh} was computed for the fixed sampling time $\delta = 0.25$ and different values of the prediction horizon T. In each example, the performance of the computed receding horizon controls for different prediction horizons are compared with each other. Moreover, in order to get more intuition about the stabilization problem, the results related to uncontrolled problem are also reported. As performance criteria for our comparison, we considered the following quantities:

1. $J_{T_\infty}(u_{rh}, \mathcal{Y}_0) := \frac{1}{2} \int_0^{T_\infty} \|\mathcal{Y}_{rh}(t)\|_{\mathcal{H}}^2 dt + \frac{\beta}{2} \int_0^{T_\infty} \|u_{rh}(t)\|_{\mathcal{U}}^2 dt$,
2. $\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H})}$,
3. $\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}}$,
4. iter : the *total* number of iterations (BB-gradient steps) that the optimizer needs for all open-loop problems on the intervals $(t_i, t_i + T)$ for $i = 0, \ldots, r-1$.

3.5.4 Numerical examples

For the cases distributed control (3.2) and Dirichlet control (3.3), we considered the unit square $(0,1)^2 \subset \mathbb{R}^2$ as the spatial domain Ω. This spatial domain was discretized by using $N = 4225$ cells as it has been explained above. Moreover for the case of Neumann control (3.4), the string equation is defined on the interval $(0,1)$. In this case, the spatial discretization was also done by the standard Galerkin method based on piecewise linear and continuous basis functions with the mesh-size $h = 0.0125$ and the time discretization was proceeded as it has been described in the subsection 3.5.1. For all examples, the step-size $\Delta t = 0.0025$ was chosen for time discretization. The numerical simulations were carried out on the MATLAB platform.

Example 3.5.1 (Distributed control)**.** In this example we applied Algorithm 3.2 to the infinite horizon problem (3.1)-(3.2) with ℓ defined by (3.8). We set $\mathcal{U} := L^2(\omega)$, $\beta = 0.1$, $T_\infty = 15$, and

$$y_0^1(x) := 5e^{-20((x_1-0.5)^2+(x_2-0.5)^2)}, \qquad y_0^2(x) = 0,$$

where $x := (x_1, x_2) \in \Omega$. Before applying Algorithm 3.2, we investigate the uncontrolled system. For this case we obtained the following quantities:

$$\|\mathcal{Y}\|_{L^2(0,T_\infty;\mathcal{H}_1)} = 1.17 \times 10^3, \qquad \|\mathcal{Y}(T_\infty)\|_{\mathcal{H}_1} = 78.57.$$

In fact, for this system $\mathcal{H}_1$-energy is conserved in time, i.e.,

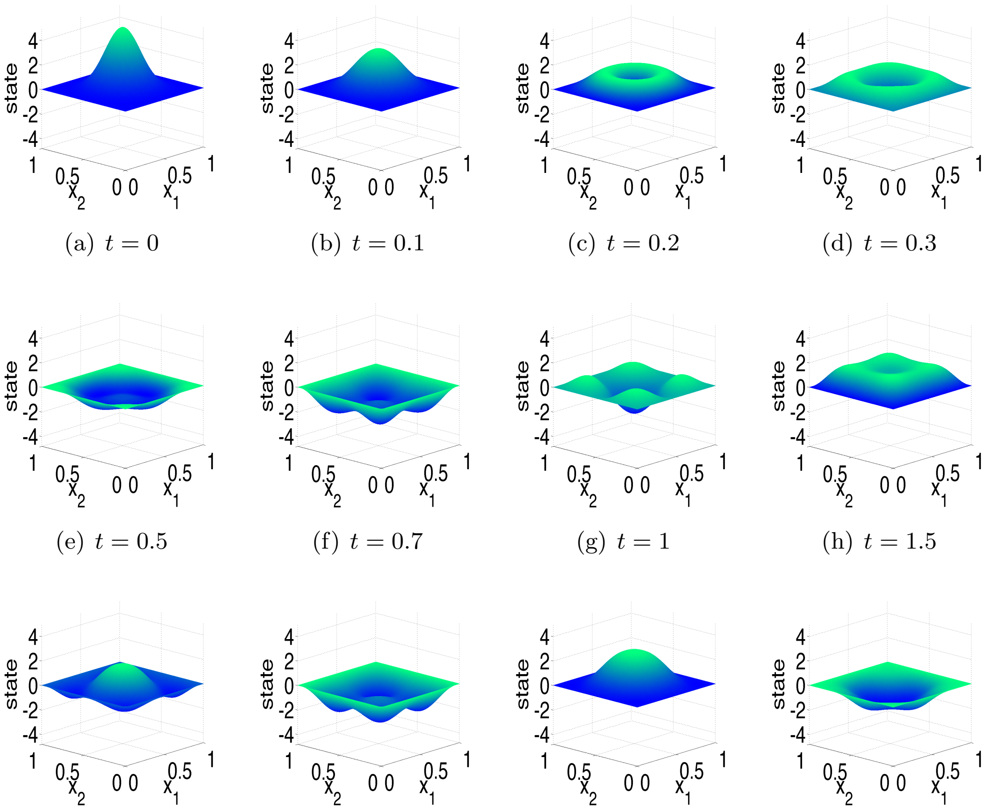
$$\|\mathcal{Y}(t)\|_{\mathcal{H}_1} = \|(y_0^1, y_0^2)\|_{\mathcal{H}_1} = 78.57 \qquad \text{for all } t \in [0, T_\infty],$$

where $\mathcal{H}_1 = H_0^1(\Omega) \times L^2(\Omega)$. As it is depicted by Figure 3.1, a single wave propagates and moves from the center of the domain to the boundaries. While moving to the boundaries, it decomposes into several small waves. After hitting the boundaries, the resulting small waves propagate and join together to form a single wave at the center of domain. This process repeats constantly, as time progresses. We employed RHC computed by Algorithm 3.2 for different choices of the prediction horizon T and the fixed sampling time $\delta = 0.25$. The control domain is described in Figure 3.2 and the corresponding results are gathered in Table 3.1. Moreover, Figure 3.4 demonstrates the evolution of the $\mathcal{H}_1$-energy of the receding horizon states for the different choices of T and fixed $\delta = 0.25$. The evolution of the $L^2(\omega)$-norm of the corresponding RHCs are plotted in Figure 3.3. Figure 3.5 shows the receding horizon state at different time points for the choice of $T = 1.5$.

Prediction horizon	J_{T_∞}	$\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H}_1)}$	$\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}_1}$	iter
$T = 1.5$	8.20×10^2	40.19	2.62×10^{-8}	1515
$T = 1$	1.13×10^3	47.40	3.03×10^{-6}	847
$T = 0.5$	3.13×10^3	79.10	2.00×10^{-3}	550
$T = 0.25$	1.94×10^4	197.43	3.79×10^{-1}	373

Table 3.1: Numerical results for Example 3.5.1

Example 3.5.2 (Dirichlet control)**.** Here we considered the stabilization of the wave equation (3.3) by Dirichlet boundary control. We set $\mathcal{U} := L^2(\Gamma_c)$, $T_\infty = 10$, $\beta = 1$, and



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.2$ (d) $t = 0.3$

(e) $t = 0.5$ (f) $t = 0.7$ (g) $t = 1$ (h) $t = 1.5$

(i) $t = 2.5$ (j) $t = 5$ (k) $t = 10$ (l) $t = 15$

Figure 3.1: Several snapshots of the uncontrolled state corresponding to Example 3.5.1

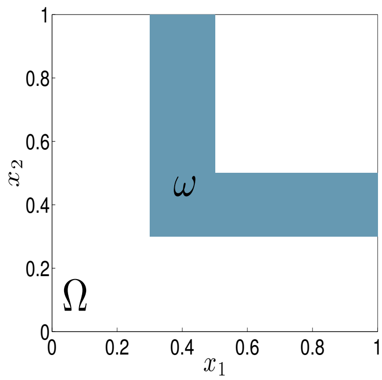
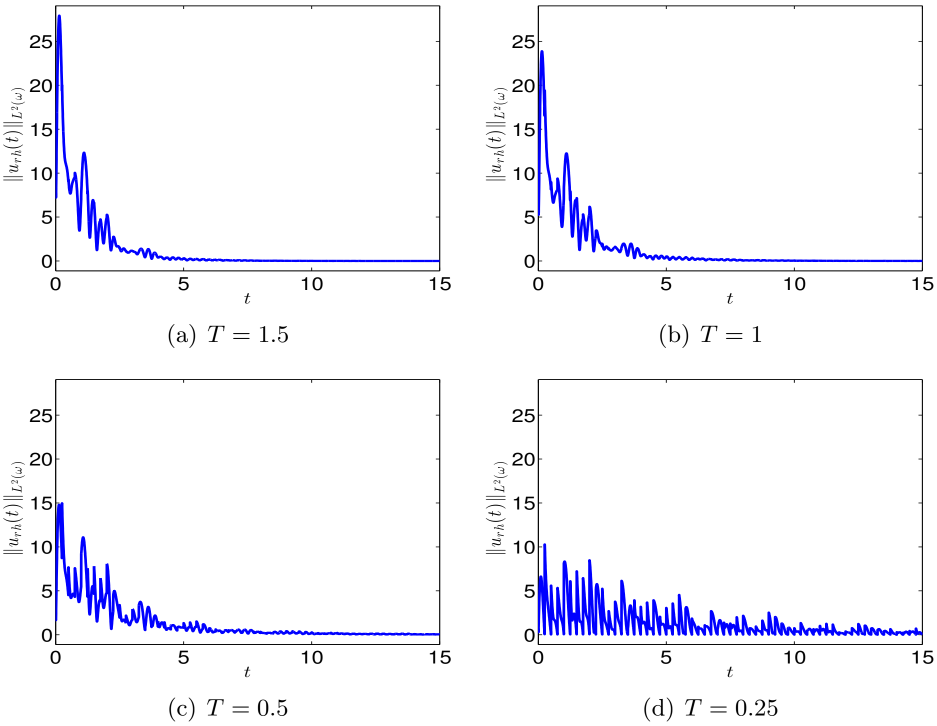


Figure 3.2: Control domain for Example 3.5.1



(a) $T = 1.5$ (b) $T = 1$

(c) $T = 0.5$ (d) $T = 0.25$

Figure 3.3: Evolution of $L^2(\omega)$-norm for RHC corresponding to Example 3.5.1 with different prediction horizons T

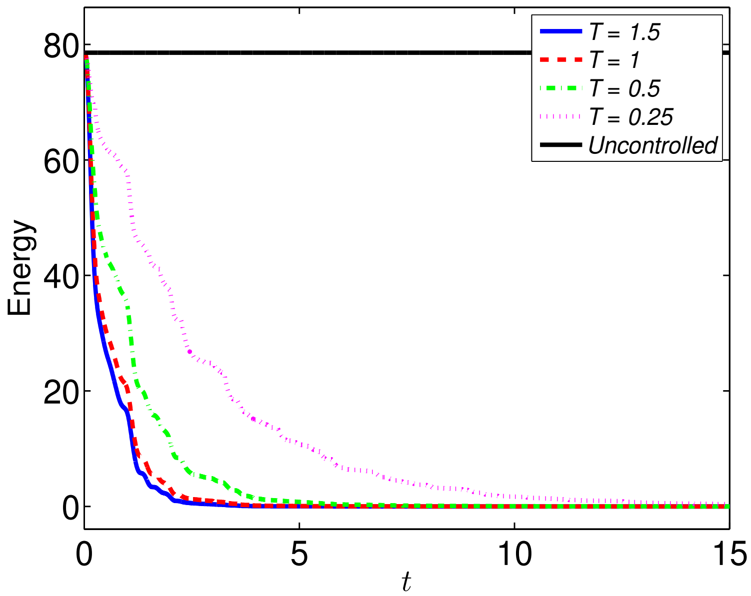
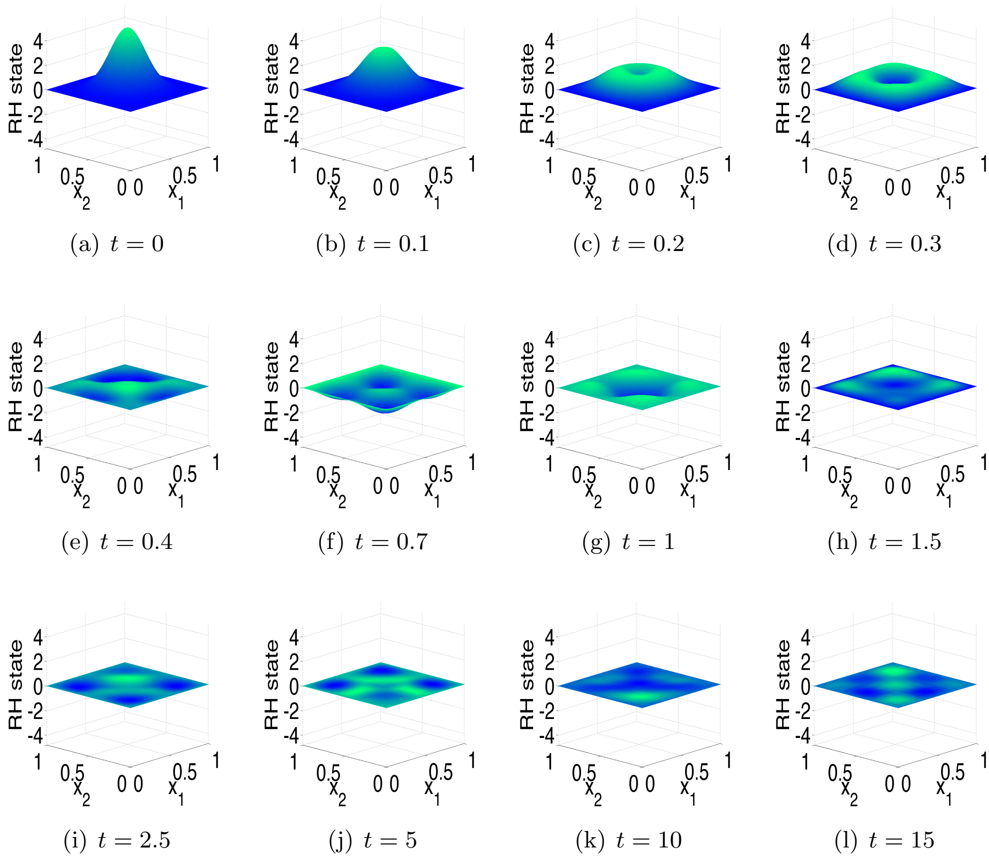


Figure 3.4: Evolution of $\|\mathcal{Y}_{rh}(t)\|_{\mathcal{H}_1}$ corresponding to Example 3.5.1 for different choices of T



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.2$ (d) $t = 0.3$

(e) $t = 0.4$ (f) $t = 0.7$ (g) $t = 1$ (h) $t = 1.5$

(i) $t = 2.5$ (j) $t = 5$ (k) $t = 10$ (l) $t = 15$

Figure 3.5: Several snapshots of receding horizon state for the choice of $T = 1.5$ corresponding to Example 3.5.1

chose the same initial pair (y_0^1, y_0^2) as in the previous example. For this example, the stabilization task was done with respect to the energy $\mathcal{H}_2 = L^2(\Omega) \times H^{-1}(\Omega)$ which is different from one in the previous example. For the uncontrolled state, the $\mathcal{H}_2$-energy is conserved over the time. More precisely, we have

$$\|\mathcal{Y}(t)\|_{\mathcal{H}_2} = 1.96 \qquad \text{for all } t \in [0, T_\infty],$$

and also $\|\mathcal{Y}\|_{L^2(0,T_\infty;\mathcal{H}_2)} = 19.60$. The receding horizon Dirichlet control is active on a subset $\Gamma_c \subset \partial\Omega$ as it is illustrated in Figure 3.6. Similar to the previous example, we

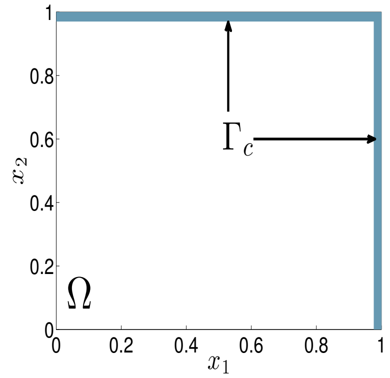


Figure 3.6: Control domain for Example 3.5.2

implemented Algorithm 3.2 for different values of the prediction horizon T and the fixed sampling time $\delta = 0.25$. The corresponding results are summarized in Table 3.2, Figure 3.7, and Figure 3.8. Figure 3.9 shows the receding horizon state at different time points

Prediction horizon	J_{T_∞}	$\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H}_2)}$	$\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}_2}$	iter
$T = 1.5$	2.20	1.93	2.11×10^{-6}	715
$T = 1$	2.75	2.23	3.42×10^{-5}	599
$T = 0.5$	6.77	3.64	6.00×10^{-3}	445
$T = 0.25$	33.75	8.20	2.36×10^{-1}	359

Table 3.2: Numerical results for Example 3.5.2

for the choice of $T = 1.5$ and $\delta = 0.25$.

Example 3.5.3 (Neumann control)**.** In this example, we deal with the controlled system (3.4) governed by the one-dimensional wave equation on the interval $(0, 1)$, with a homogeneous Dirichlet boundary condition at zero, and a Neumann boundary control action at one. We chose $\mathcal{U} := \mathbb{R}$, $\beta = 15$, $T_\infty = 15$, and

$$y_0^1(x) := 5e^{-20(x-0.5)^2}, \qquad y_0^2(x) = 0,$$

as the initial data. In the uncontrolled case, we have a vibrating string which is fixed at one end of the boundary, but whose other end keeps moving up and down in a periodic

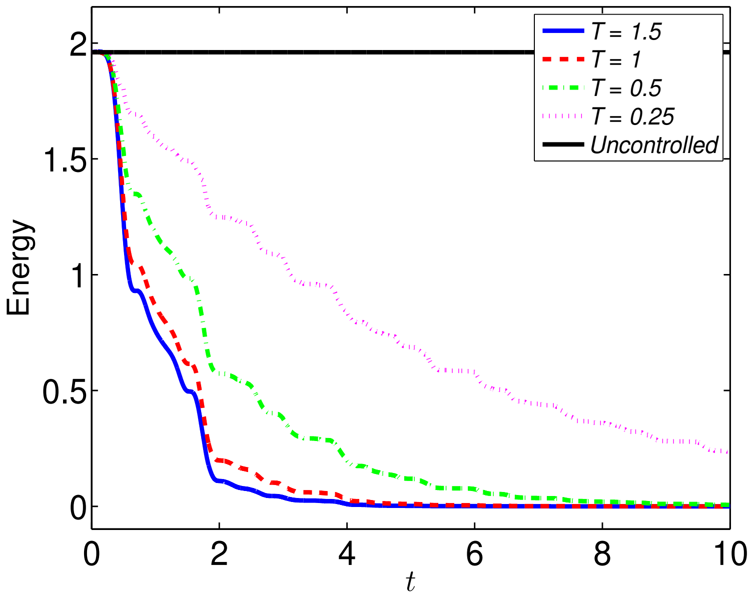


Figure 3.7: Evolution of $\|\mathcal{Y}_{rh}(t)\|_{\mathcal{H}_2}$ corresponding to Example 3.5.2 for different choices of T

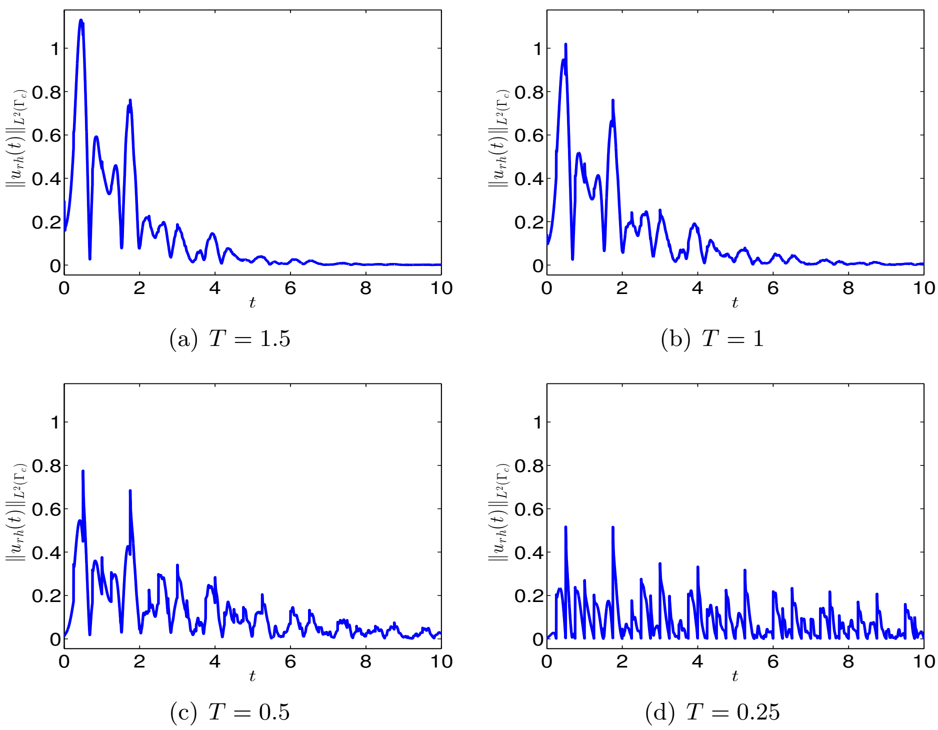
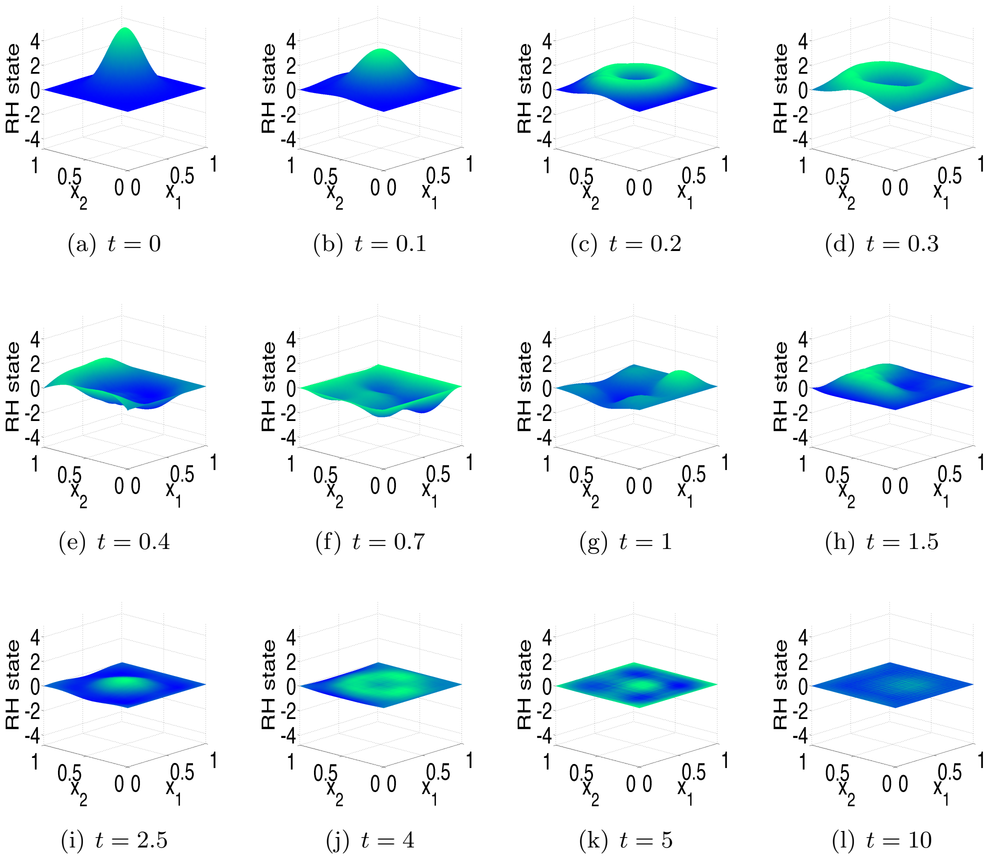


Figure 3.8: Evolution of $L^2(\Gamma_c)$-norm for RHC corresponding to Example 3.5.2 for different choices of T



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.2$ (d) $t = 0.3$

(e) $t = 0.4$ (f) $t = 0.7$ (g) $t = 1$ (h) $t = 1.5$

(i) $t = 2.5$ (j) $t = 4$ (k) $t = 5$ (l) $t = 10$

Figure 3.9: Several snapshots of receding horizon state for the choice of $T = 1.5$ corresponding to Example 3.5.2

fashion. Similar to the previous example for the uncontrolled system, the $\mathcal{H}_3$-energy with $\mathcal{H}_3 = V \times L^2(0,1)$ is conserved for all times. Further we have

$$\|\mathcal{Y}\|_{L^2(0,T_\infty;\mathcal{H}_3)} = 2.10 \times 10^3, \quad \|\mathcal{Y}(T_\infty)\|_{\mathcal{H}_3} = 140.13.$$

The uncontrolled solution can be seen from Figure 3.10. The numerical results of RHC

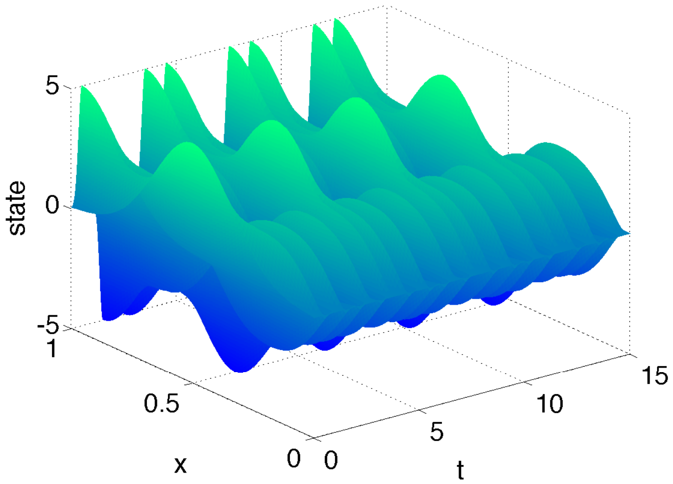


Figure 3.10: Uncontrolled solution for Example 3.5.3

computed by Algorithm 3.2 for the different choices of the prediction horizon T and the fixed sampling time $\delta = 0.25$, are revealed by Table 3.3, and Figures 3.11 and 3.12.

Prediction horizon	J_{T_∞}	$\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H}_3)}$	$\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}_3}$	iter
$T = 1.5$	1.30×10^4	161.47	3.85×10^{-6}	5348
$T = 1$	1.67×10^4	182.97	7.08×10^{-5}	3303
$T = 0.5$	3.92×10^4	280.22	4.91×10^{-2}	1507
$T = 0.25$	2.41×10^5	694.40	9.26	823

Table 3.3: Numerical results for Example 3.5.3

Figure 3.13 shows the receding horizon state and control for the choice of $T = 1.5$.

By observing Tables 3.1-3.3 and Figures 3.4, 3.7, and 3.11, we can assert that the results corresponding to the performance criteria are reasonable. Except for the case that $\delta = T$, for all prediction horizons $T > \delta$ the underlying system was successfully stabilized as the theory in the previous sections suggests. Moreover, apparently the prediction horizon T plays an important role. As expected, increasing the prediction horizon T leads to a decrease of the stabilization indicators and more importantly the value of objective function J_{T_∞}. Moreover as it can been seen from Figures 3.3, 3.8, and

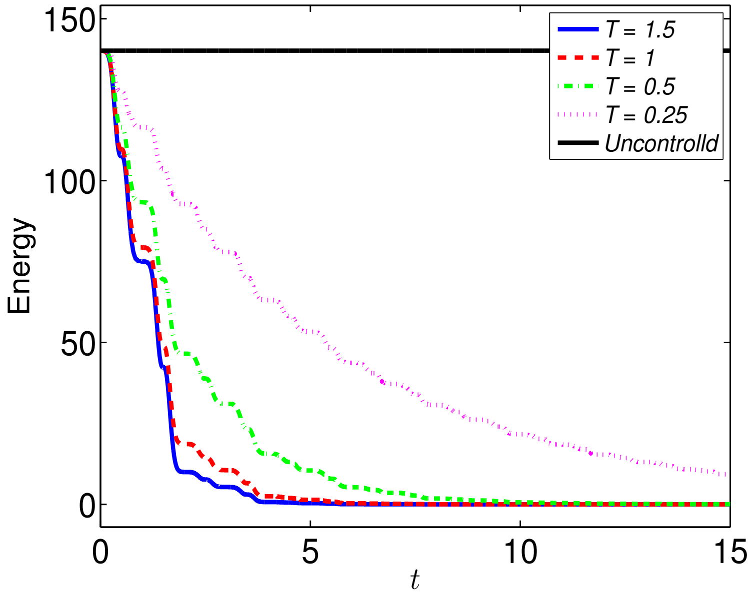


Figure 3.11: Evolution of $\|\mathcal{Y}_{rh}(t)\|_{\mathcal{H}_3}$ corresponding to Example 3.5.3 for different choices of T

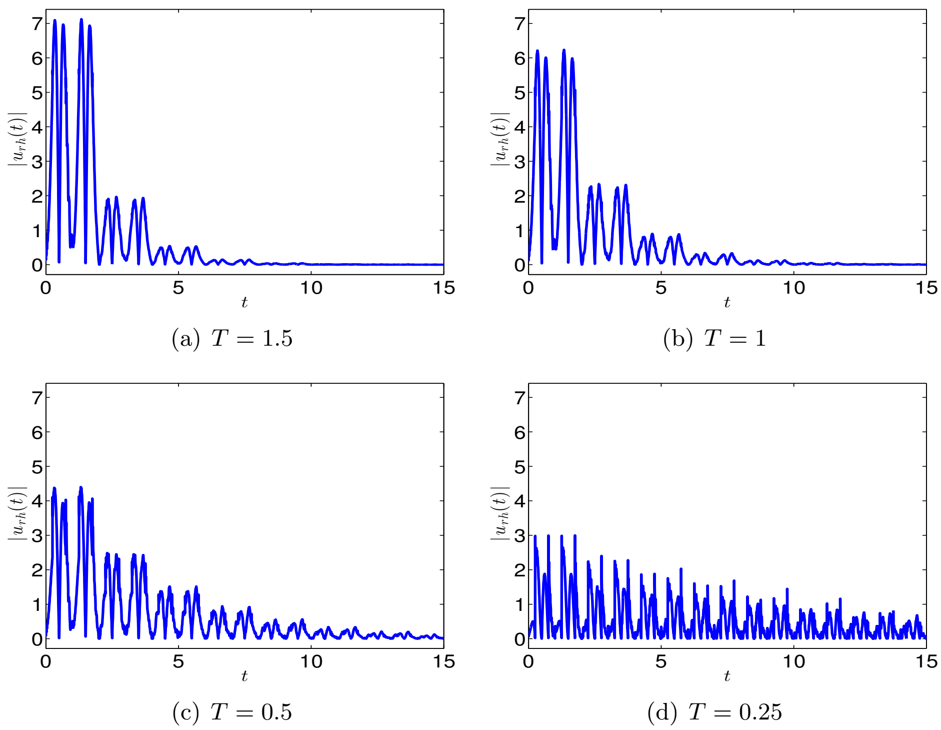
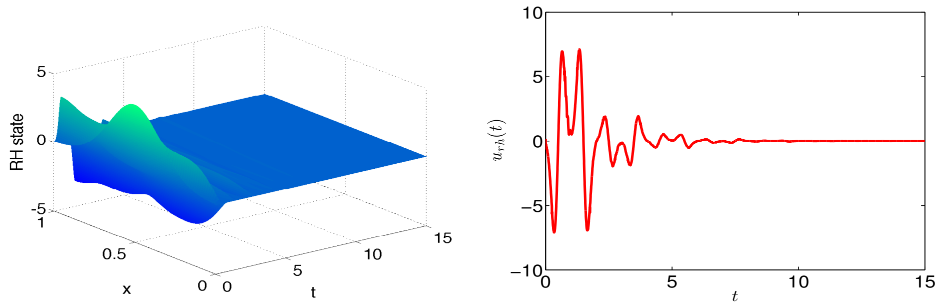


Figure 3.12: Evolution of $|u_{rh}(t)|$ corresponding to Example 3.5.3 for different choices of T



(a) RH state (b) RHC

Figure 3.13: Receding horizon trajectories corresponding to Example 3.5.3 for the choice of $T = 1.5$

3.12, the corresponding RHCs are more regular, if the ratio of prediction horizon T to sampling time δ is large. On the other hand, the shorter prediction horizon T (i.e. the closer to the sampling time δ) is chosen, the fewer overall iterations and computational efforts are required.

Chapter 4

On the Stabilizability of the Burgers Equation by RHC

4.1 Introduction

In this chapter, the applicability of our theoretical work proposed in Chapter 2 will be demonstrated for the Burgers equation

$$\frac{d}{dt}y - \mu y_{xx} + y y_x = 0,$$

where $\mu > 0$ and $y = y(t, x)$ is a real valued function of real variables t and x. This is a nonlinear partial differential equation (PDE) that combines both nonlinear propagation and diffusion effects. It shares some important features with the Navier-Stokes equation. The Burgers equation has the origin as a steady state. It is asymptotically stable in the case of homogeneous Dirichlet boundary conditions. For homogeneous Neumann boundary conditions and periodic boundary conditions the origin is not asymptotically stable. Control theory for the Burgers equation has been investigated, both theoretically and numerically, by many authors. From among them we mention only [8, 9, 16, 32, 33, 34, 36, 74, 86, 89, 101, 135].

To be more precise, we apply Algorithm 2.1 of Chapter 2 for the stabilization of the viscous Burgers equation with periodic and homogeneous Neumann boundary conditions, For these boundary conditions the origin of the uncontrolled system is stable but not asymptotically stable.

The remainder of the chapter is structured as follows. In Sections 4.2 and 4.3 we investigate Assumptions (A1)-(A3) in Chapter 2 for the case of periodic boundary conditions and the case of homogeneous Neumann boundary conditions. We show that in the case of periodic boundary conditions Algorithm 2.1 provides globally stabilizing controls, while for Neumann boundary conditions we obtain locally stabilizing controls. Section 4.4 contains numerical experiments which highlight the effect of the ratio $\frac{T}{\delta}$ on the stabilizing effect of the RHC strategy. Moreover, comparisons are carried out comparing the effect of RHC with and without a terminal control penalty.

4.2 Burgers Equation with Periodic Boundary Conditions

For an arbitrary finite horizon $T > 0$ we consider the controlled Burgers equation with periodic boundary conditions of the form

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu(t) & \text{in } (0,T)\times(0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases} \tag{4.1}$$

Throughout, $\mu > 0$ and $y_0 \in L^2(0,1)$ are fixed, and the control operator B is the extension-by-zero operator given by

$$(Bu)(x) = \begin{cases} u(x) & x \in \hat{\Omega}, \\ 0 & x \in (0,1)\backslash\hat{\Omega}, \end{cases}$$

where the control domain $\hat{\Omega}$ is a nonempty open subset of $(0,1)$.

For the function space setting of (4.1) we introduce the spaces

$$V := \{y \in H^1(0,1) \mid y(0) = y(1)\}, \quad H := L^2(0,1),$$

and

$$W(0,T) := \left\{\phi : \phi \in L^2(0,T;V), \quad \frac{d}{dt}\phi \in L^2(0,T;V^*)\right\},$$

where V^* is the adjoint space of V. The spaces H and V are endowed with the usual norms $\|\cdot\|_H := \|\cdot\|_{L^2(0,1)}$ and $\|\cdot\|_V := \|\cdot\|_{H^1(0,1)}$. Further $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{V^*,V}$ denote the inner product in H and the duality pairing between V and V^*. We recall that $W(0,T)$ is continuously embedded into $C([0,T];H)$; see, e.g., [131]. It will be convenient to define the continuous trilinear form $b : V \times V \times V \to \mathbb{R}$ by

$$b(\varphi,\psi,\phi) = \int_0^1 \varphi\psi_x\phi dx.$$

We shall frequently use the property that

$$b(y,y,y) = \int_0^1 y_x y^2 dx = \frac{1}{3}(y^3(1) - y^3(0)) = 0 \quad \text{for all } y \in V. \tag{4.2}$$

It is well known that for every control $u \in L^2(0,T;L^2(\hat{\Omega}))$, equation (4.1) admits a unique weak solution $y \in W(0,T)$, i.e. y satisfies $y(0) = y_0$ in H, and for almost every $t \in (0,T)$,

$$\frac{d}{dt}\langle y(t),\varphi\rangle_{V^*,V} + \mu\langle y(t),\varphi\rangle_V - \mu\langle y(t),\varphi\rangle_H + b(y(t),y(t),\varphi) = \langle Bu(t),\varphi\rangle_H \tag{4.3}$$

holds for all $\varphi \in V$. Using (4.2) and Gronwall's lemma it can easily be shown that there exists a constant C_T such that

$$|y(\cdot;y_0,u)|_{W(0,T)} \le C_T\left(|y_0|_H + |u|_{L(0,T;L^2(\hat{\Omega}))}\right), \tag{4.4}$$

where $y(\cdot; y_0, u)$ indicates the dependence of the solution on y_0 and u. The running cost will be taken of the form

$$\ell(y,u) := \frac{1}{2}\|y\|_H^2 + \frac{\beta}{2}\|u\|^2_{L^2(\hat{\Omega})}, \tag{4.5}$$

where $\beta > 0$.

We have now specified all items of the finite horizon problem of the form (P_T) which was defined in Chapter 2. Using (4.4) it follows from standard subsequential limit arguments that (P_T) with the control system given by (4.1) admits a solution for each $y_0 \in H$. In particular (A1) holds with $\mathcal{N}_0 = H$. In the following lemma we show that Assumption (A2) holds as well.

Lemma 4.2.1 (Global stabilizability)**.** *For each $T > 0$ and initial state $y_0 \in H$ there exists a control $\hat{u}(\cdot; y_0) \in L^2(0,T; L^2(\hat{\Omega}))$ such that*

$$V_T(y_0) \leq J_T(\hat{u}, y_0) \leq \gamma(T)\|y_0\|_H^2, \tag{4.6}$$

for a continuous, nondecreasing and bounded function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$.

Proof. Set $\hat{u}(t) := -y(t)|_{\hat{\Omega}}$ and consider

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) - By(t)|_{\hat{\Omega}} & \text{in } (0,T)\times(0,1),\\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (0,T),\\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases} \tag{4.7}$$

Taking the inner product of the first equation of (4.7) with $y(t)$, we have for almost every $t \in (0,T)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu(\|y_x(t)\|_H^2 - y_x(t,1)y(t,1) + y_x(t,0)y(t,0))\\ + b(y(t),y(t),y(t)) + \|y(t)\|^2_{L^2(\hat{\Omega})} = 0.\end{aligned} \tag{4.8}$$

Taking into account the boundary conditions and (4.2), one can express (4.8) as

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + \|y(t)\|^2_{L^2(\hat{\Omega})} = 0.$$

One can easily show that $\|y\|_1^2 := \|y_x\|_H^2 + \frac{1}{\mu}\|y\|^2_{L^2(\hat{\Omega})}$ is a norm which is equivalent to the H^1-norm [107, p. 26]. Thus there exist positive constants $c_2 > c_1 > 0$ such that

$$c_1\|y\|_V^2 \leq \|y\|_1^2 \leq c_2\|y\|_V^2 \quad \text{for all } y \in V,$$

and consequently

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu c_1\|y(t)\|_V^2 \leq 0,$$

and therefore,

$$\frac{d}{dt}\|y(t)\|_H^2 + 2\mu c_1\|y(t)\|_H^2 \leq 0 \quad \text{for all } t \in [0,T].$$

Multiplying both sides of the above equation by $e^{2\mu c_1 t}$ and integrating from 0 to t we obtain

$$\|y(t)\|_H^2 \leq \|y_0\|_H^2 e^{-2\mu c_1 t} \quad \text{for all } t \in [0,T].$$

By integrating the above inequality over the interval $[0,T]$, we obtain

$$\int_0^T \|y(t)\|_H^2 dt \leq \frac{1}{2\mu c_1}(1 - e^{-2\mu c_1 T})\|y_0\|_H^2. \tag{4.9}$$

By the definition of the value function $V_T(\cdot)$ and (4.9) we have

$$V_T(y_0) \leq \int_0^T \left(\frac{1}{2}\|y(t)\|_H^2 + \frac{\beta}{2}\| - y(t)\|_{L^2(\hat{\Omega})}^2 \right) dt \leq \frac{1+\beta}{4\mu c_1}(1 - e^{-2\mu c_1 T})\|y_0\|_H^2,$$

and (A2) follows with $\gamma(T) := \frac{1+\beta}{4\mu c_1}(1 - e^{-2\mu c_1 T})$, and $\mathcal{N}_0 = H$. □

From Lemma 4.2.1, we infer that Assumption (A2) holds globally. Thus by Remark 2.2.1 we can directly apply Theorem 2.2.1 without addressing (A3) and conclude that, for any arbitrary sampling time δ, there exists a positive T^* such that for every $T \geq T^*$ the RHC u_{rh} is globally suboptimal (within H) with suboptimality factor $\alpha > 0$, and we have

$$\alpha V_\infty(y_0) \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq V_\infty(y_0) \tag{4.10}$$

for every $y_0 \in H$. Now it remains for us to show that the RHC u_{rh} computed by Algorithm 2.1 is globally stabilizing. This property will be verified by means of the following theorem.

Theorem 4.2.1. *Let $y_0 \in H$ and $\delta > 0$ be arbitrary, and apply Algorithm 2.1 for the stabilization of the Burgers equation (4.1) with a prediction horizon $T \geq T^*$, where T^* is introduced by Proposition 2.2.1. Then the receding horizon trajectory y_{rh} satisfies $\lim_{t\to\infty} \|y_{rh}(t)\|_H = 0$.*

Proof. First, we show that

$$\|y_{rh}\|_{L^\infty(0,\infty;H)} \leq \nu \|y_0\|_H \tag{4.11}$$

for a constant $\nu > 0$.

Due to (4.5), (4.6), and (4.10), we have

$$\frac{\alpha \min\{1,\beta\}}{2} \int_0^\infty \left(\|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2\right) dt \leq \alpha J_\infty(u_{rh}(\cdot), y_0) \leq V_T(y_0) \leq \gamma(T)\|y_0\|_H^2.$$

Therefore by choosing $\sigma_1 := \frac{2\gamma(T)}{\alpha \min\{1,\beta\}}$, we obtain

$$\int_0^\infty \|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt \leq \sigma_1 \|y_0\|_H^2. \tag{4.12}$$

Moreover, the receding horizon state given by Algorithm 2.1 satisfies $y_{rh} \in C([0,\infty);H)$; for every $k \in \mathbb{N}$ we have

$$y_{rh}\,|_{(t_k,t_{k+1})} \in L^2(t_k,t_{k+1};V), \quad \frac{d}{dt}y_{rh}\,|_{(t_k,t_{k+1})} \in L^2(t_k,t_{k+1};V^*); \tag{4.13}$$

and y_{rh} is the solution of

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu_{rh}(t) & \text{in } (t_k,t_{k+1}) \times (0,1), \\ y(t,0) = y(t,1), \quad y_x(t,0) = y_x(t,1) & \text{on } (t_k,t_{k+1}), \\ y(t_k,\cdot) = y_{rh}(t_k) \text{ for } k > 0, \text{ and } y(0,\cdot) = y_0 \text{ for } k = 0 & \text{in } (0,1). \end{cases}$$

By multiplying the above equation by $y_{rh}(\cdot)$ and integrating over the interval $(0,1)$, we have

$$\frac{1}{2}\frac{d}{dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 = \langle Bu_{rh}(t), y_{rh}(t)\rangle_H \text{ for almost every } t \in (t_k,t_{k+1}).$$

From the Cauchy-Schwarz and Young inequalities we infer that

$$\frac{d}{dt}\|y_{rh}(t)\|_H^2 + 2\mu\|(y_{rh})_x(t)\|_H^2 \le \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 + \|y_{rh}(t)\|_H^2 \text{ for almost every } t \in (t_k,t_{k+1}).$$

Integrating from t_k to t, for every $t \in (t_k,t_{k+1})$ we have

$$\begin{aligned} &\|y_{rh}(t)\|_H^2 \\ &\quad \le \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_{t_k}^t \|y_{rh}(s)\|_H^2\, ds \quad \text{for all } t \in (t_k,t_{k+1}). \end{aligned}$$

By the same estimate as above for the interval (t_{k-1},t_k) we have

$$\|y_{rh}(t_k)\|_H^2 \le \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t_k} \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_{t_{k-1}}^{t_k} \|y_{rh}(s)\|_H^2\, ds. \tag{4.14}$$

Moreover, by the above two estimates we have

$$\|y_{rh}(t)\|_H^2 \le \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_{t_{k-1}}^t \|y_{rh}(s)\|_H^2\, ds.$$

By repeating the above argument for $k-2, k-3, \dots, 0$, one can show that

$$\begin{aligned} \|y_{rh}(t)\|_H^2 \le& \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_{t_k}^t \|y_{rh}(s)\|_H^2\, ds \\ \le& \|y_0\|_H^2 + \int_0^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_0^t \|y_{rh}(s)\|_H^2\, ds \\ \le& \|y_0\|_H^2 + \int_0^\infty \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + \int_0^\infty \|y_{rh}(s)\|_H^2\, ds \\ \le& (1+\sigma_1)\|y_0\|_H^2, \end{aligned}$$

where in the last line (4.12) has been used. Choosing $\nu := \sqrt{1+\sigma_1}$, we obtain (4.11).

Next we are in the position to prove

$$\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 = 0.$$

For every $t'' \geq t'$ we have

$$\begin{aligned}
\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 &= \int_{t'}^{t''} \frac{d}{dt}\|y_{rh}(t)\|_H^2\, dt \\
&= 2\int_{t'}^{t''} \langle y_{rh}(t), \mu(y_{rh})_{xx}(t) - (y_{rh})_x(t) y_{rh}(t) + Bu_{rh}(t)\rangle_{V,V^*}\, dt \\
&= -2\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H\, dt \\
&\leq 2\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})} \|y_{rh}(t)\|_H\, dt \\
&\leq 2\Big(\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}},
\end{aligned}$$

and thus

$$\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 \leq 2\sqrt{\sigma_1}\nu \|y_0\|_H^2 (t''-t')^{\frac{1}{2}}. \tag{4.15}$$

For the last inequality, (4.11) and (4.12) have been used. Moreover, from (4.5), (4.6) and (4.10) we have

$$\frac{\alpha}{2}\int_0^\infty \|y_{rh}(t)\|_{L^2(\hat{\Omega})}^2 \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq \gamma(T)\|y_0\|_H^2 < \infty.$$

This estimate implies that

$$\lim_{t\to\infty} \int_{t-L}^{t} \|y_{rh}(s)\|_H^2 ds = 0 \tag{4.16}$$

for all $L > 0$. Suppose to the contrary that

$$\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 \neq 0.$$

Then there exists an $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^\infty$ with $t_n > 0$ and $\lim_{n\to\infty} t_n = \infty$ for which

$$\|y_{rh}(t_n)\|_H^2 > \epsilon \quad \text{for all } n = 1, 2, \ldots. \tag{4.17}$$

It follows from (4.15) and (4.17) that for every $L > 0$ and $n = 1, 2, \ldots$

$$\begin{aligned}
&\int_{t_n-L}^{t_n} \|y_{rh}(t)\|_H^2 dt \\
&\quad = \int_{t_n-L}^{t_n} \|y_{rh}(t_n)\|_H^2 dt - \int_{t_n-L}^{t_n} \big(\|y_{rh}(t_n)\|_H^2 - \|y_{rh}(t)\|_H^2\big) dt, \qquad (4.18) \\
&\quad > L\epsilon - 2\sqrt{\sigma_1}\nu\|y_0\|_H^2 \int_{t_n-L}^{t_n} (t_n - t)^{\frac{1}{2}} dt = L\epsilon - \frac{4}{3}\sqrt{\sigma_1}\nu\|y_0\|_H^2 L^{\frac{3}{2}}.
\end{aligned}$$

Setting $\omega := \frac{4}{3}\sqrt{\sigma_1}\nu\|y_0\|_H^2$, and choosing $L := (\frac{\epsilon}{2\omega})^2$, we obtain

$$\int_{t_n-L}^{t_n} \|y_{rh}(t)\|_H^2 dt > \frac{L\epsilon}{2} \quad \text{for all } n = 1, 2, \ldots.$$

This contradicts (4.16). Hence $\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 = 0$, and the proof is complete.

□

4.3 Burgers Equation with Homogeneous Neumann Boundary Conditions

Here we consider the controlled Burgers equation with homogenous Neumann boundary conditions of the form

$$\begin{cases} \frac{d}{dt}y(t) = \mu y_{xx}(t) - y(t)y_x(t) + Bu(t) & \text{in } (0,T)\times(0,1), \\ y_x(t,0) = y_x(t,1) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y_0 & \text{in } (0,1). \end{cases} \tag{4.19}$$

We can utilize the same notation as in Section 4.2, except for the energy space which is now chosen to be

$$V := \{y \in H^1(0,1) \mid y_x(0) = y_x(1) = 0\}.$$

Again $V \subset H \subset V^*$ is a Gelfand triple and $W(0,T)$ is continuously embedded in $C([0,T];H)$.

The significant difference between (4.19) and (4.1) is given by the fact that in the case of periodic boundary conditions the nonlinearity is conservative; i.e., we have that $b(\phi,\phi,\phi) = 0$ for all $\phi \in V$, which is not the case for Neumann boundary conditions. As a consequence we have to rely on the local version of the results of Chapter 2.

Again we use the weak or variational solution concept of (4.3). Due to the fact that the nonlinearity is not conservative, the verification of a global weak solution is not trivial. We have the following result.

Lemma 4.3.1. *For every $T > 0$, every $y_0 \in H$, and $u \in L^2(0,T;L^2(\hat{\Omega}))$ there exists a unique solution $y(\cdot;y_0,u) \in W(0,T)$ to (4.19). Moreover, there exists a constant C_T such that*

$$|y(\cdot;y_0,u)|_{W(0,T)} \le C_T\left(1 + |y_0|_H + |u|_{L(0,T;L^2(\hat{\Omega}))}\right),$$

for all $y_0 \in H$, and $u \in L^2(0,T;L^2(\hat{\Omega}))$.

For the proof we refer to [137]. For the step that the local solution can be extended to a global one we prefer the argument given in [60], for which it is useful to recall that for a measurable function, which will be u in our case, the function $E \to \int_E |u(t)|_{L^2(\hat{\Omega})}\, dt$, with E a measurable subset of $(0,T)$, is absolutely continuous.

The running cost will again be taken to be of the form (4.5). It is now standard to argue the existence of a solution to (P_T) with the control system given by (4.19). In particular, (A1) holds with $\mathcal{N}_0 = H$. In the following lemmas we show that Assumptions (A2) and (A3) holds as well.

Lemma 4.3.2 (Local stabilizability). *There exists a neighborhood $\mathcal{B}_{\delta_1}(0) \subset H$ such that for every $T > 0$ and every $y_0 \in \mathcal{B}_{\delta_1}(0)$ there exists a control $\hat{u}(\cdot, y_0) \in L^2(0,T;L^2(\hat{\Omega}))$ with*

$$V_T(y_0) \leq J_T(\hat{u}, y_0) \leq \gamma(T)\|y_0\|_H^2,$$

where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing, and bounded function.

Proof. Setting $\hat{u}(t) := -y(t)|_{\hat{\Omega}}$ in the first equation of (4.19), multiplying $y(t)$, and taking the L^2-scalar product, we obtain

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + b(y(t), y(t), y(t)) + \|y(t)\|_{L^2(\hat{\Omega})}^2 = 0. \tag{4.20}$$

As in the case of periodic boundary conditions, one can argue that $\|y\|_1^2 := \|y_x\|_H^2 + \frac{1}{\mu}\|y\|_{L^2(\hat{\Omega})}^2$ is a norm equivalent to the H^1-norm (see, e.g., [107, p. 26]), and hence there exist positive constants $c_2 > c_1 > 0$ such that $c_1\|y\|_V^2 \leq \|y\|_1^2 \leq c_2\|y\|_V^2$ for all $y \in V$. The nonlinearity satisfies the following equality:

$$b(y,y,y) = \int_0^1 y_x y^2 dx \leq \|y\|_{L^\infty(0,1)}\|y_x\|_H\|y\|_H \leq c_a\|y\|_1^2\|y\|_H \quad \text{for all } y \in V,$$

where c_a, depends on the embedding constant of V into $L^\infty(0,1)$ and c_1. From (4.20) we therefore deduce that

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y_x(t)\|_H^2 + \|y(t)\|_{L^2(\hat{\Omega})}^2 \leq c_a\|y(t)\|_1^2\|y(t)\|_H,$$

and consequently

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y(t)\|_1^2 \leq c_a\|y(t)\|_1^2\|y(t)\|_H.$$

Now let us choose $\|y_0\|_H$ sufficiently small, say $\|y_0\|_H \leq \frac{\mu}{4c_a}$. Then by continuity of the solution for a short interval of time $[0, T^*]$, we have $\|y(t)\|_H \leq \frac{\mu}{2c_a}$ for all $t \in [0, T^*]$ and further

$$\frac{d}{dt}\|y(t)\|_H^2 + \mu c_1\|y(t)\|_H^2 \leq 0 \quad \text{for all } t \in [0, T^*].$$

Multiplying both sides of the above equation by $e^{\mu c_1 t}$ and integrating from 0 to t, we obtain

$$\|y(t)\|_H^2 \leq \|y_0\|_H^2 e^{-\mu c_1 t} \leq \left(\frac{\mu}{4c_a}\right)^2 \quad \text{for all } t \in [0, T^*]. \tag{4.21}$$

Repeating the above argument implies that $\|y(t)\|_H \leq \frac{\mu}{4c_a}$ will remain small for all $t \in [0, T]$ and, moreover, we have

$$\|y(t)\|_H^2 \leq \|y_0\|_H^2 e^{-\mu c_1 t} \quad \text{for all } t \in [0, T].$$

Integration over $[0, T]$ implies that

$$\int_0^T \|y(t)\|_H^2 dt \leq \frac{1}{\mu c_1}(1 - e^{-\mu c_1 T})\|y_0\|_H^2. \tag{4.22}$$

By the definition of the value function $V_T(\cdot)$ and (4.22) we have

$$V_T(y_0) \le \int_0^T \left(\frac{1}{2}\|y(t)\|_H^2 + \frac{\beta}{2}\|y(t)\|_{L^2(\hat{\Omega})}^2\right) dt \le \frac{1+\beta}{2\mu c_1}(1-e^{-\mu c_1 T})\|y_0\|_H^2,$$

where $\gamma(T) := \frac{1+\beta}{2\mu c_1}(1-e^{-\mu c_1 T})$ is a nondecreasing, continuous, and bounded function, as desired, and $\delta_1 := \frac{\mu}{4c_a}$. □

Lemma 4.3.3. *Assumption* (A3) *holds for* (4.19) *with* $\mathcal{N}_0 = \mathcal{B}_{\delta_1}(0)$ *defined in Lemma 4.3.2.*

Proof. For every $y_0 \in \mathcal{B}_{\delta_1}(0)$ we have from (4.19) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 + \mu\|y(t)\|_V^2 \\ &\qquad \le \mu\|y(t)\|_H^2 + |b(y(t),y(t),y(t))| + |\langle y(t), Bu(t)\rangle_H| \quad \text{for almost every } t \in [0,T].\end{aligned} \tag{4.23}$$

From Agmon's inequality we recall that there exists a constant c_A such that

$$\|\phi\|_{L^\infty(0,1)} \le c_A\|\phi\|_H^{\frac{1}{2}}\|\phi\|_V^{\frac{1}{2}} \quad \text{for all } \phi \in V,$$

and consequently there exists a constant c_I such that

$$b(\phi,\phi,\phi) \le \|\phi\|_{L^\infty(0,1)}\|\phi\|_H\|\phi\|_V \le c_A\|\phi\|_H^{\frac{3}{2}}\|\phi\|_V^{\frac{3}{2}} \le \mu\|\phi\|_V^2 + c_I\|\phi\|_H^6 \quad \text{for all } \phi \in V.$$

Utilizing the above inequality and (4.23), we obtain

$$\frac{d}{dt}\|y(t)\|_H^2 \le 2\mu\|y(t)\|_H^2 + 2c_I\|y(t)\|_H^6 + 2\|y(t)\|_H\|u(t)\|_{L^2(\hat{\Omega})} \quad \text{for almost every } t \in [0,T].$$

Upon integration we obtain

$$\|y(t)\|_H^2 \le \|y(0)\|_H^2 + (2\mu + 1 + 2c_I\|y\|_{C([0,T];H)}^4)\int_0^t \|y(s)\|_H^2\, ds + \int_0^t \|u(s)\|_{L^2(\hat{\Omega})}^2 ds.$$

By Lemma 4.3.1 the family

$$\left\{\|y(\cdot;y_0,u)\|_{C([0,T];H)} \,|\, y_0 \in \mathcal{B}_{\delta_1}(0), \|u\|_{L^2(0,T;L^2(\hat{\Omega}))} \le \sqrt{2\gamma(T)/\min\{1,\beta\}}\,\|y_0\|_H\right\}$$

is bounded, and hence the desired estimate follows. □

Now we are in the position that we can apply Theorem 2.2.1, and it remains for us to show that the receding horizon control u_{rh} computed by Algorithm 2.1 is stabilizing. This will be accomplished in the following theorem, which uses the quantifier $d_2(T)$ for the size of the neighborhood of the initial data. Recall that $d_2(T)$ depends on $\gamma(T)$, which was given explicitly in the proof of Lemma 4.3.2, and on c_T, the existence of which was provided in the proof of Lemma 4.3.3.

Theorem 4.3.1. *Let a sampling time $\delta > 0$ be given, and apply Algorithm 2.1 for the stabilization of the Burgers equation* (4.19) *with a prediction horizon $T \geq T^*$, where T^* is introduced by Proposition 2.2.1. Then we have* $\lim_{t\to\infty} \|y_{rh}(t)\|_H = 0$, *provided that* $|y_0|_H \leq d_2(T)$.

Proof. We recall that $\delta_1 = \frac{\mu}{4c_a}$ depends on embedding constants and was introduced in the proof of Lemma 4.3.2. Furthermore, we have $d_2(T) \leq d_1(T) \leq \delta_1$. To verify the claim we can follow for the most part the proof of Theorem 4.2.1. Again we first show that there exists some $\nu > 0$ such that

$$\|y_{rh}\|_{L^\infty(0,\infty;H)} \leq \nu \|y_0\|_H, \tag{4.24}$$

for each $y_0 \in \mathcal{B}_{d_2}(0)$. By construction we have that $y_{rh} \in C([0,\infty), H)$, that (4.13) holds, and that

$$\int_0^\infty \|y_{rh}(t)\|_H^2 + \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt \leq \sigma_1 \|y_0\|_H^2, \tag{4.25}$$

where $\sigma_1 := \frac{2\gamma(T)}{\alpha \min\{1,\beta\}}$. For any $k = 0, 1, \ldots$ we have

$$\begin{aligned}\frac{d}{2dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 &+ b(y_{rh}(t), y_{rh}(t), y_{rh}(t))\\ &= \langle Bu_{rh}(t), y_{rh}(t)\rangle_H \quad \text{for almost every } t \in (t_k, t_k + 1).\end{aligned}$$

Furthermore, d_2 in Theorem 2.2.1 has been chosen in such way that for every $t > 0$ the receding horizon trajectory $y_{rh}(t)$ stays in the neighborhood $\mathcal{B}_{\delta_1}(0)$. In other words, we have

$$\|y_{rh}(t)\|_H \leq \delta_1 = \frac{\mu}{4c_a} < \frac{\mu}{2c_a} \quad \text{for all } t > 0, \tag{4.26}$$

where c_a is defined in Lemma 4.3.2. Now by the Cauchy-Schwarz and Young's inequalities, by (4.26), and the definition of $\|\cdot\|_1$, we infer that

$$\begin{aligned}b(y_{rh}(t), &y_{rh}(t), y_{rh}(t))\\ &\leq \|y_{rh}(t)\|_{L^\infty(0,1)}\|y_{rh}(t)\|_H\|(y_{rh})_x(t)\|_H\\ &\leq c_a\|y_{rh}(t)\|_H\|y_{rh}(t)\|_1^2\\ &\leq \frac{\mu}{2}\|y_{rh}(t)\|_1^2 \leq \frac{\mu}{2}\|(y_{rh})_x(t)\|_H^2 + \frac{1}{2}\|y_{rh}(t)\|_H^2 \quad \text{for almost every } t \in (t_k, t_{k+1}).\end{aligned}$$

Thus for every $k \in \mathbb{N}$ we have

$$\frac{d}{dt}\|y_{rh}(t)\|_H^2 + \mu\|(y_{rh})_x(t)\|_H^2 \leq \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 + 2\|y_{rh}(t)\|_H^2 \text{ for almost every } t \in (t_k, t_{k+1}),$$

and therefore for $t \in (t_k, t_{k+1})$

$$\begin{aligned}\|y_{rh}(t)\|_H^2 + \mu\int_{t_k}^t &\|(y_{rh})_x(s)\|_H^2 ds\\ &\leq \|y_{rh}(t_k)\|_H^2 + \int_{t_k}^t \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_k}^t \|y_{rh}(s)\|_H^2 ds.\end{aligned} \tag{4.27}$$

By the same estimate as above for the interval (t_{k-1}, t_k), we have

$$\begin{aligned}\|y_{rh}(t_k)\|_H^2 + &\mu \int_{t_{k-1}}^{t_k} \|(y_{rh})_x(s)\|_H^2 ds \\ &\leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t_k} \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_{k-1}}^{t_k} \|y_{rh}(s)\|_H^2 ds.\end{aligned} \tag{4.28}$$

By summing (4.27) and (4.28), we have

$$\begin{aligned}\|y_{rh}(t)\|_H^2 + &\mu \int_{t_{k-1}}^{t} \|(y_{rh})_x(s)\|_H^2 ds \\ &\leq \|y_{rh}(t_{k-1})\|_H^2 + \int_{t_{k-1}}^{t} \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_{k-1}}^{t} \|y_{rh}(s)\|_H^2 ds.\end{aligned}$$

Repeating the above argument for $k-2, k-3, \ldots, 0$, it follows that

$$\begin{aligned}\|y_{rh}(t)\|_H^2 \leq &\|y_{rh}(t_k)\|_H^2 + \int_{t_k}^{t} \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_{t_k}^{t} \|y_{rh}(s)\|_H^2 ds \\ \leq &\|y_0\|_H^2 + \int_0^\infty \|u_{rh}(s)\|_{L^2(\hat{\Omega})}^2 ds + 2\int_0^\infty \|y_{rh}(s)\|_H^2 ds \\ \leq &(1+2\sigma_1)\|y_0\|_H^2,\end{aligned}$$

where for the last inequality (4.25) has been used. Choosing $\nu := \sqrt{1+2\sigma_1}$, we obtain (4.24).

Now we are in the position to prove

$$\lim_{t\to\infty} \|y_{rh}(t)\|_H^2 = 0.$$

For every $t'' \geq t'$ we have

$$
\begin{aligned}
\|y_{rh}(t'')\|_H^2 - \|y_{rh}(t')\|_H^2 &= \int_{t'}^{t''} \frac{d}{dt}\|y_{rh}(t)\|_H^2 dt \\
&= 2\int_{t'}^{t''} \langle y_{rh}(t), \mu(y_{rh})_{xx}(t) - (y_{rh})_x(t) y_{rh}(t) + Bu_{rh}(t)\rangle_{V,V^*} dt \\
&\leq -2\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + 2\int_{t'}^{t''} c_a \|y_{rh}(t)\|_H \|y_{rh}(t)\|_1^2 dt \\
&\quad + 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H dt \\
&\leq -\mu \int_{t'}^{t''} \|(y_{rh})_x(t)\|_H^2 dt + \int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt + 2\int_{t'}^{t''} \langle Bu_{rh}(t), y_{rh}(t)\rangle_H dt \\
&\leq 2\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})} \|y_{rh}(t)\|_H dt + \int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt \\
&\leq 2\Big(\int_{t'}^{t''} \|u_{rh}(t)\|_{L^2(\hat{\Omega})}^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}} \Big(\int_{t'}^{t''} \|y_{rh}(t)\|_H^2 dt\Big)^{\frac{1}{2}} \\
&\leq 3\sqrt{\sigma_1}\nu \|y_0\|_H^2 (t'' - t')^{\frac{1}{2}},
\end{aligned}
$$

where (4.24) and (4.25) were used to obtain the last inequality. Now the proof can be completed following that for Theorem 4.2.1, except that the factor 2 in (4.18) has to be replaced by the factor 3 which appeared in the last estimate. □

4.4 Numerical Results

We present numerical results to illustrate the theoretical findings of the previous sections. For the Burgers equation with periodic or homogeneous Neumann boundary conditions, every constant function is a steady state of the uncontrolled equation. Hence the origin is stable, but it is not asymptotically stable. Consequently it is of interest to force the state of these equations to the steady state by an external control which is computed on the basis of the RHC.

Our numerical experiments will also include a comparison of the performance of the RHC scheme with and without terminal penalty term. The latter case was investigated in the previous section, the former in [75], where it was shown that the quadratic penalty term $G(y) = \frac{1}{2}\|y\|_{L^2(\Omega)}^2$ can be used as a control Lyapunov function for the Navier-Stokes eqaution.

Our numerical tests utilize the following Algorithm 4.1, where G is chosen as one of the two cost functionals

$$
\textit{Zero: } G(y) = 0, \qquad \text{or} \qquad \textit{Quadratic: } G(y) = \frac{1}{2}\|y\|_{L^2(0,1)}^2. \tag{4.29}
$$

For $G(y) = 0$, Algorithm 4.1 essentially coincides with Algorithm 2.1, except for the fact that we need to terminate our computations at some $T_\infty < \infty$.

Algorithm 4.1

Input: Let a final computational time horizon T_∞, and an initial state $y_0 \in L^2(0,1)$ be given.

1: Choose a prediction horizon $T < T_\infty$, and a sampling time $\delta \in (0, T]$.

2: Consider a grid $0 = t_0 < t_1 \cdots < t_r = T_\infty$ on the interval $[0, T_\infty]$, where $t_i := i\delta$ for $i = 0, \ldots, r$.

3: Solve successively the open-loop subproblem on $[t_i, t_i + T]$:

$$\min \frac{1}{2} \int_{t_i}^{t_i+T} \|y(t)\|^2_{L^2(0,1)} dt + \frac{\beta}{2} \int_{t_i}^{t_i+T} \|u(t)\|^2_{L^2(\hat{\Omega})} dt + G(y(t_i + T)), \qquad (4.30)$$

subject to Burgers equations (4.19) (or (4.1)) for the initial condition

$$y(t_i) = y^*_T(t_i) \text{ if } i \geq 1 \text{ and } y(t_i) = y_0 \text{ if } i = 0,$$

where $y^*_T(\cdot)$ is the solution to the preceding subproblem on $[t_{i-1}, t_{i-1} + T]$.

4: The receding horizon pair $(y_{rh}(\cdot), u_{rh}(\cdot))$ is obtained by concatenation of the optimal pairs $(y^*_T(t), u^*_T(t))$ of the finite horizon subproblems on $[t_i, t_{i+1}]$ for $i = 0, \ldots, r-1$.

The numerical simulations were carried out on the MATLAB platform. Throughout, the spatial discretization was done by the standard Galerkin method based on piecewise linear and continuous basis functions with mesh-size $h = 0.0125$. The ordinary differential equations resulting after spatial discretization were solved by the implicit Euler method with step-size $\Delta t = 0.0125$, where the nonlinear systems of equations within the implicit Euler method were solved by Newton's method. Every open-loop problem was solved by applying the Barzilai-Borwein (BB) gradient steps [21] with a nonmonotone line search [49] on the reduced problem in the "first optimize, then discretize" manner. For every open-loop problem, the optimization algorithm was terminated when L^2-norm of the gradient for the reduced objective function was less than the tolerance 10^{-6}. Furthermore in all examples, we set $\delta = 0.25$ and $\beta = 10^{-3}$.

For every example, we implemented the receding horizon strategy for different choices of the prediction horizon T and the two terminal costs G in (4.29). Furthermore, in order to have a measure for the performance of the receding horizon strategy, we consider

1. $J_{T_\infty}(u_{rh}, y_0) := \frac{1}{2} \int_0^{T_\infty} \|y_{rh}(t)\|^2_{L^2(0,1)} dt + \frac{\beta}{2} \int_0^{T_\infty} \|u_{rh}(t)\|^2_{L^2(\hat{\Omega})} dt$,
2. $\|y_{rh}\|_{L^2(Q)}$ with $Q := (0, T_\infty) \times (0, 1)$,
3. $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$,

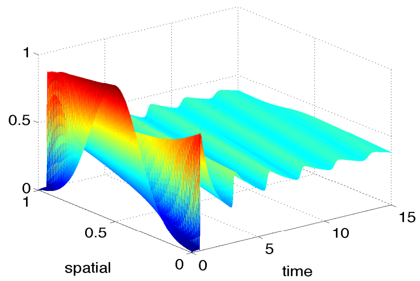
4. iter : the *total* number of iterations (BB-gradient steps) that the optimizer needs for all open-loop problems on the intervals $(t_i, t_i + T)$ for $i = 0, \dots, r-1$.

Example 4.4.1. We considered the Burgers equation (4.1) with periodic boundary conditions. We chose $y_0(x) = \exp(-20(x-0.5)^2)$ as the initial function, $\mu = 10^{-3}$ as a viscosity parameter, and $T_\infty = 15$. Further, the RHC acts only on the set

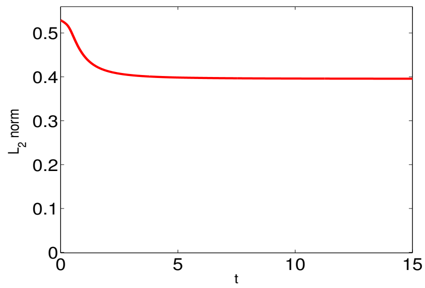
$$\hat{\Omega} = (0.1, 0.2) \cup (0.4, 0.6) \cup (0.8, 0.9) \subset (0, 1).$$

Figures 4.1(a) and 4.1(b) depict, respectively, the solution and the evolution of the $L^2(0,1)$-norm for the state of the uncontrolled Burgers equation (4.1). For the uncontrolled solution y^u we have

$$\|y^u\|_{L^2(Q)} = 1.5768, \qquad \|y^u(T_\infty)\|_{L^2(0,1)} = 0.3958.$$



(a) State



(b) Evolution of $L^2(0,1)$-norm of state

Figure 4.1: Uncontrolled solution for Example 4.4.1

The results of Algorithm 4.1 for different choices of T and G and the fixed sampling time $\delta = 0.25$ are summarized in Table 4.1. Figure 4.2 shows the results for the receding horizon pairs (y_{rh}, u_{rh}) if $G = 0$ and $T = 1$.

G	Prediction horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
Quadratic	$T = 1$	0.021891	0.1799	2.62×10^{-5}	11861
	$T = 0.5$	0.023196	0.1820	3.95×10^{-5}	8352
	$T = 0.25$	0.027547	0.1828	1.32×10^{-4}	6041
Zero	$T = 1$	0.021886	0.1805	8.41×10^{-5}	6220
	$T = 0.5$	0.021893	0.1818	2.10×10^{-4}	3139
	$T = 0.25$	0.021943	0.1856	4.26×10^{-4}	1467

Table 4.1: Numerical results for Example 4.4.1

As expected, increasing the prediction horizon T results in a decrease of the stabilization measures $\|y_{rh}\|_{L^2(Q)}$ and $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$, for both quadratic and zero terminal

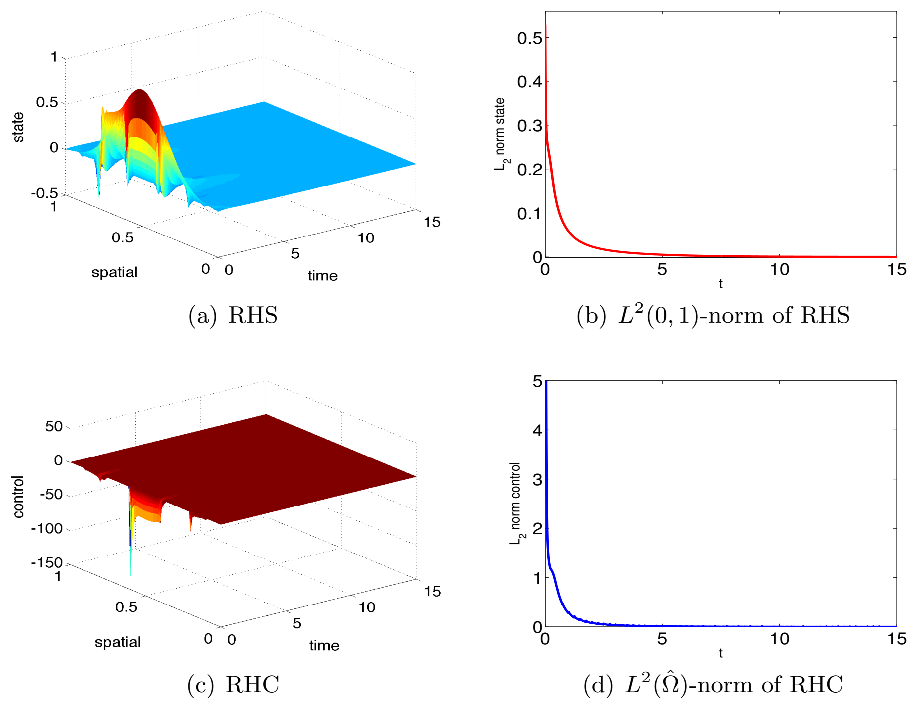


Figure 4.2: Receding horizon trajectories for Example 4.4.1

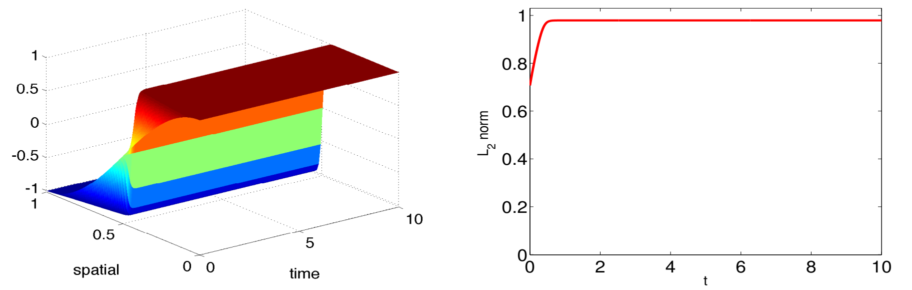
penalties. The quadratic terminal penalty term results in smaller values of the stabilization measures, with the difference in the $\|y_{rh}\|_{L^2(Q)}$-norm less pronounced than in the $\|y_{rh}(T_\infty)\|_{L^2(0,1)}$-norm. Using a nontrivial terminal penalty results in higher iteration numbers for the optimizer. In view of the fact that the choice of T has only little effect on the stabilization measures, but significant effect on the number of iterations in the optimization algorithm, small T is preferable for this class of problems. It should also be of interest to search for methods which adaptively tune the prediction horizon.

Example 4.4.2. Here we considered the stabilization of the Burgers equation (4.19) with homogeneous Neumann boundary conditions. We set $y_0(x) = \cos(\pi x)$ as the initial function and chose $T_\infty = 10$. The spatial support for the controls is

$$\hat{\Omega} = (0, 0.15) \cup (0.85, 1) \subset (0, 1).$$

Furthermore, $\mu = 0.01$. Note that for this small viscosity parameter and the above antisymmetric initial function, the uncontrolled numerical solution of (4.19) approaches a nonconstant, time independent steady state; see [31]. The uncontrolled solution y^u is illustrated in Figure 4.3, and we have

$$\|y^u\|_{L^2(Q)} = 3.08, \qquad \|y^u(T_\infty)\|_{L^2(0,1)} = 0.9791.$$



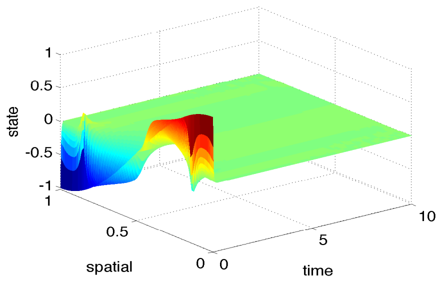
(a) State (b) Evolution of $L^2(0,1)$-norm of state

Figure 4.3: Uncontrolled solution for Example 4.4.2

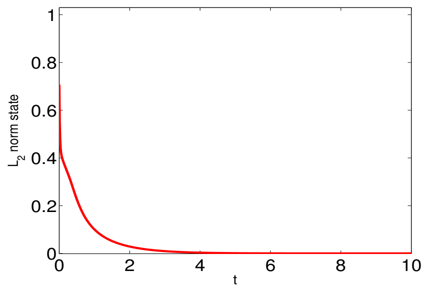
Table 4.2 reveals the numerical results of Algorithm 4.1 for different choices of the prediction horizon T and the terminal cost G. Figure 4.4 shows the results for the receding horizon pairs (y_{rh}, u_{rh}) in the case that zero terminal cost and $T = 1$ were chosen.

Concerning the effect of different choices of T and G, the same observations as in Example 4.4.1 apply.

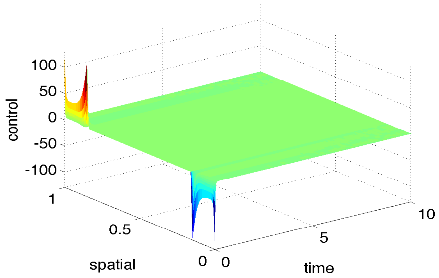
Example 4.4.3. In this example, we dealt with the stabilization of a *noisy* Burgers equation with homogeneous Neumann boundary conditions. We chose $y_0(x) = \exp(-20(x - 0.5)^2)$ as the initial function, $\mu = 0.01$ as a viscosity parameter, and



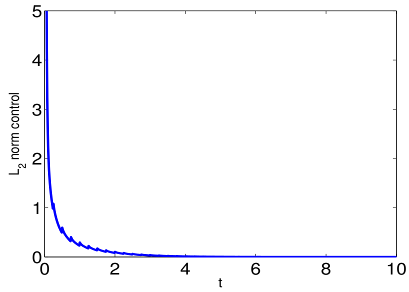
(a) RH State



(b) $L^2(0,1)$-norm of RHS



(c) RH Control



(d) $L^2(\hat{\Omega})$-norm of RHC

Figure 4.4: Receding horizon trajectories for Example 4.4.2

G	Prediction horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
Quadratic	$T = 1$	0.053394	0.2820	3.38×10^{-6}	5890
	$T = 0.5$	0.056004	0.2792	9.99×10^{-6}	3957
	$T = 0.25$	0.060580	0.2788	9.74×10^{-6}	3285
Zero	$T = 1$	0.053058	0.2835	7.23×10^{-6}	3382
	$T = 0.5$	0.052961	0.2873	1.46×10^{-5}	1698
	$T = 0.25$	0.053717	0.2977	4.00×10^{-5}	903

Table 4.2: Numerical results for Example 4.4.2.

$T_\infty = 10$. Furthermore, the noise was simulated by generating uniformly distributed random numbers within the range $[-4, 4]$; it was added to the right-hand side of (4.19) at the spatial-temporal grid points. The results corresponding to uncontrolled solutions are reported in Table 4.3.

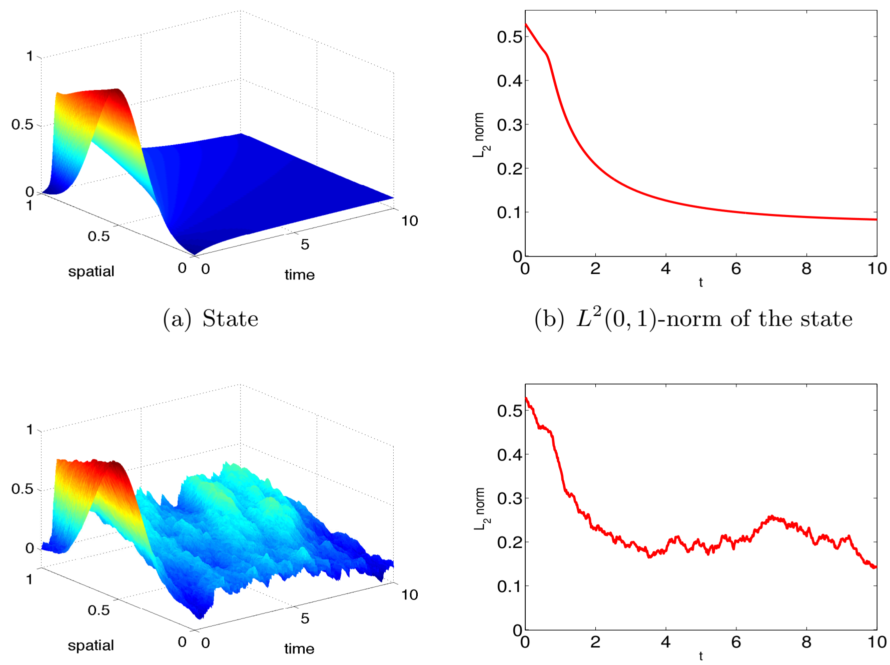
Problem types	$\|y^u\|_{L^2(Q)}$	$\|y^u(T_\infty)\|_{L^2(0,1)}$
Uncontrolled state without noise	0.6291	0.0829
Uncontrolled state with noise	0.7945	0.1436

Table 4.3: Uncontrolled solutions for Example 4.4.3

In Figure 4.5, we show the results for uncontrolled solutions with noise and without noise. The control acts only on the set

$$\hat{\Omega} = (0.1, 0.3) \cup (0.7, 0.9) \subset (0, 1).$$

In implementations of Algorithm 4.1 on every interval $[t_i, t_i + T]$, first an open-loop



(a) State (b) $L^2(0,1)$-norm of the state

(c) Noisy state (d) $L^2(0,1)$-norm of the noisy state

Figure 4.5: Uncontrolled solutions for Example 4.4.3

optimal control u_T^* was computed for every subproblem without noise. Then the optimal control u_T^* is used to steer the noisy Burgers equation. This process was repeated for every interval $[t_i, t_i + T]$ with $i = 0, \ldots, r - 1$. Table 4.4 (resp., Table 4.5) represents the results of Algorithm 4.1 applied on the Burgers equation (4.19) with noise (resp., without noise) for different choices of the prediction horizon T and the terminal cost G.

In Figures 4.6 and 4.7, we show the results for the receding horizon pairs (y_{rh}, u_{rh}) in

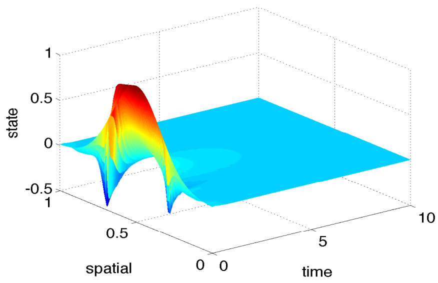
G	Prediction horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
Quadratic	$T=1$	0.049456	0.2996	0.0426	4327
	$T=0.5$	0.052292	0.3031	0.0418	3489
	$T=0.25$	0.060291	0.3063	0.0441	2739
Zero	$T=1$	0.049545	0.3009	0.0438	2463
	$T=0.5$	0.050112	0.3046	0.0462	1479
	$T=0.25$	0.053008	0.3163	0.0512	880

Table 4.4: Numerical results corresponding to the noisy equation for Example 4.4.3

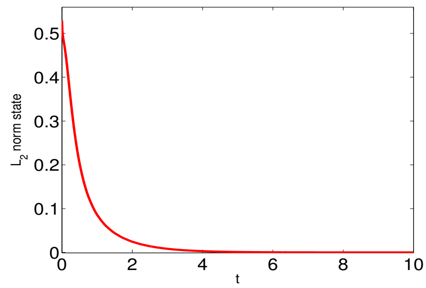
G	Prediction horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(0,1)}$	iter
Quadratic	$T=1$	0.044212	0.2828	2.56×10^{-6}	4215
	$T=0.5$	0.046700	0.2855	9.70×10^{-6}	3346
	$T=0.25$	0.054818	0.2892	5.58×10^{-6}	2636
Zero	$T=1$	0.044212	0.2839	9.26×10^{-6}	2532
	$T=0.5$	0.044626	0.2870	2.79×10^{-5}	1450
	$T=0.25$	0.047149	0.2983	1.51×10^{-4}	835

Table 4.5: Numerical results corresponding to the equation without noise for Example 4.4.3

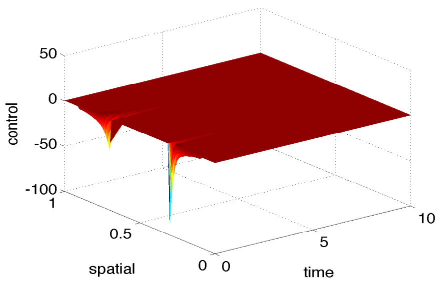
the case that zero terminal cost and $T = 1$ were chosen.



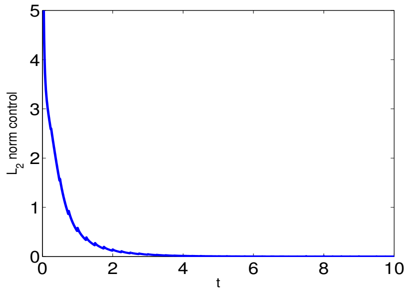
(a) RH State



(b) $L^2(0,1)$-norm of RHS

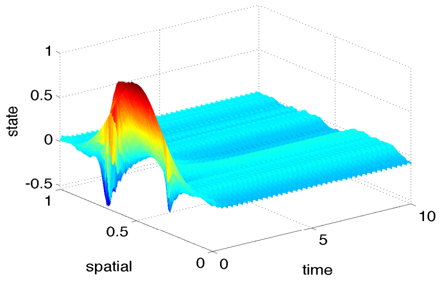


(c) RH Control

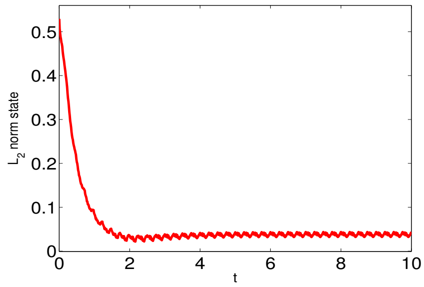


(d) $L^2(\hat{\Omega})$-norm of RHC

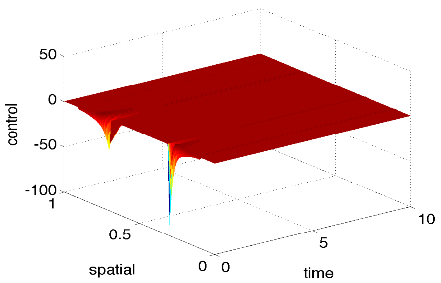
Figure 4.6: Receding horizon trajectories for Example 4.4.3 without noise



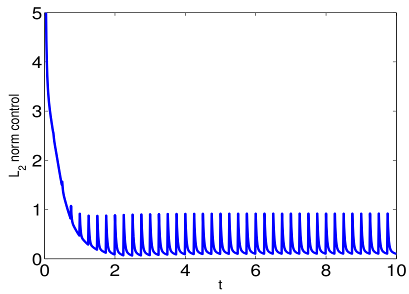
(a) RH State



(b) $L^2(0,1)$-norm of RHS



(c) RH Control



(d) $L^2(\hat{\Omega})$-norm of RHC

Figure 4.7: Receding horizon trajectories for Example 4.4.3 with presence of noise

From Tables 4.4 and 4.5 we note that the stabilization quantifiers for the quadratic and zero terminal penalties differ less in the case with noise than without noise. Comparing Figures 4.6(d) and 4.7(d) we note the effect on the required control action due to noise in the equation.

Consistently over all numerical results it can be observed that a longer prediction horizon leads to smaller values of J_{T_∞}. Concerning the total number of iterations, it can be seen that, for the problems under consideration, Algorithm 4.1 with zero terminal cost requires significantly fewer iterations than in the case with quadratic terminal cost.

Chapter 5

On the semi-global Stabilizability of the KdV Equation via RHC

5.1 Introduction

This chapter is devoted to applicability of RHC poposed in Chapter 2 in the context of stabilization of the nonlinear Korteweg-de Vries equation

$$\partial_t y + \partial_x y + y\partial_x y + \partial_x^3 y = 0, \tag{KdV}$$

where $y = y(t, x)$ is a real valued function of real variables t and x. The KdV equation was first derived by Boussinesq [28] and rediscovered by Korteweg and de Vries [82] as a model for the propagation of water waves along a channel. This equation serves also as a very useful approximation in studies aiming to include and balance a weak nonlinearity and weak dispersive effects. Particularly, the equation is now commonly used as a mathematical model for the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems. In the past decades, many authors studied the KdV equation from various aspects of mathematics, including the well-posedness, existence and stability of solitary waves, the long-time behavior, stabilization, and the controllability. Among all of them we can point out the works [25, 26, 43, 54, 55, 73, 84] for well-posedness and [37, 39, 46, 59, 79, 96, 105, 114, 116, 122, 123, 124, 125, 143] for stabilization and control theory.

Here we consider the following optimal control problem which consists in minimizing the performance index

$$J_\infty(u, y_0) := \int_0^\infty \ell(y(t), u(t))dt \tag{5.1}$$

subject to the Korteweg-de Vries (KdV) equation posed on the space-time cylinder $(0, L) \times [0, \infty)$

$$\begin{cases} \partial_t y + \partial_x y + y\partial_x y + \partial_x^3 y = Bu & x \in (0, L), \quad t > 0, \\ y(t, 0) = y(t, L) = \partial_x y(t, L) = 0 & t > 0, \\ y(0, \cdot) = y_0 & x \in (0, L), \end{cases} \tag{5.2}$$

where the external control $u(t) = u(t,x)$ is real valued function, and $y_0 \in L^2(0,L)$. The control operator B is the extension-by-zero operator given by

$$(Bu)(x) = \begin{cases} u(x) & x \in \hat{\Omega}, \\ 0 & x \in (0,L)\backslash\hat{\Omega}, \end{cases}$$

where the control domain $\hat{\Omega}$ is a nonempty open subset of $(0,L)$. Further, the incremental function $\ell : L^2(0,L) \times L^2(\hat{\Omega}) \to \mathbb{R}_+$ is defined by

$$\ell(y,u) := \frac{1}{2}\|y\|^2_{L^2(0,L)} + \frac{\beta}{2}\|u\|^2_{L^2(\hat{\Omega})}. \tag{5.3}$$

In this chapter we continue our study on the analysis of the unconstrained receding horizon strategy for infinite-dimensional controlled systems. Based on the semi-global stabilizability result from [114] we first show that RHC for (5.2) is suboptimal. Then by an observability type estimate, we prove that the resulting receding horizon controlled system is semi-globally exponentially stable. This requires techniques which differ from those which were employed in Chapters 2, 3, and 4. For the sake of consistency in presentation, we reformulate Algorithm 2.1 for the problem (5.1)-(5.2) and summarize the corresponding steps in Algorithm 5.1.

The remainder of this chapter is organized as follows: Section 5.2 deals with the global well-posedness of the nonlinear KdV equation in the weak sense. In Section 5.3, existence of the finite horizon optimal control is investigated. Section 5.4 analyzes the suboptimality and semi-global exponential stability of RHC obtained by Algorithm 5.1. Finally, Section 5.5 is devoted to numerical simulations.

5.2 Well-posedness of the KdV equation

In this section we deal with the existence of global solution of the nonlinear KdV equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = f & x \in (0,L), \quad t \in (0,T), \\ y(t,0) = y(t,L) = \partial_x y(t,L) = 0 & t \in (0,T), \\ y(0,\cdot) = y_0 & x \in (0,L), \end{cases} \tag{5.4}$$

with an arbitrary finite time horizon T, forcing function $f \in L^2(0,T;L^2(0,L))$, and initial function $y_0 \in L^2(0,L)$. Throughout we shall refer to the following function spaces:

$$\mathcal{B}_{0,T} := C([0,T];L^2(0,T)) \cap L^2(0,T;H^1_0(0,L))$$

equipped with the norm

$$\|v\|_{\mathcal{B}_{0,T}} := \sup_{t\in[0,T]} \|v(t)\|_{L^2(0,L)} + \|v\|_{L^2(0,T;H^1_0(0,L))},$$

the space in which solutions will be sought

$$\mathcal{W}_{0,T} := L^2(0,T;H^1_0(0,L)) \cap C([0,T];L^2(0,L)) \cap H^1(0,T;H^{-2}(0,L)),$$

Algorithm 5.1 Receding Horizon Control Algorithm

Input: Let the prediction horizon T, the sampling time $\delta < T$, and the initial state $y_0 \in L^2(0, L)$ be given.

1: Set $k := 0$, $t_0 := 0$, and $y_{rh}(t_0) := y_0$.

2: Find the optimal pair $(y_T^*(\cdot; y_{rh}(t_k), t_k), u_T^*(\cdot; y_{rh}(t_k), t_k))$ over the time horizon $[t_k, t_k + T]$ by solving the finite horizon open-loop problem

$$\min_{u \in L^2(t_k, t_k+T; L^2(\hat{\Omega}))} J_T(u, y_{rh}(t_k)) := \min_{u \in L^2(t_k, t_k+T; L^2(\hat{\Omega})))} \int_{t_k}^{t_k+T} \ell(y(t), u(t)) dt,$$

$$\text{subject to} \begin{cases} \partial_t y + \partial_x y + y \partial_x y + \partial_x^3 y = Bu & x \in (0, L), \quad t \in (t_k, t_k + T), \\ y(t, 0) = y(t, L) = \partial_x y(t, L) = 0 & t \in (t_k, t_k + T), \\ y(t_k, \cdot) = y_{rh}(t_k) & x \in (0, L). \end{cases}$$

3: Set

$$\begin{aligned} u_{rh}(\tau) &:= u_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta), \\ y_{rh}(\tau) &:= y_T^*(\tau; y_{rh}(t_k), t_k) \quad \text{for all } \tau \in [t_k, t_k + \delta], \\ t_{k+1} &:= t_k + \delta, \\ k &:= k + 1. \end{aligned}$$

4: Go to Step 2.

and the space of test functions

$$\mathcal{X} := \{\omega \in H^2(0,L) \mid \omega(0) = \omega(L) = \omega'(0) = 0\}. \tag{5.5}$$

First of all we show that, for every forcing function $f \in L^1(0,T;L^2(0,L))$ and initial function $y_0 \in L^2(0,L)$, the linear KdV equation

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = f & x \in (0,L), \quad t \in (0,T), \\ y(t,0) = y(t,L) = \partial_x y(t,L) = 0 & t \in (0,T), \\ y(0,\cdot) = y_0 & x \in (0,L), \end{cases} \tag{5.6}$$

is well-posed. We give a proof to explicit the dependence of the solution of f. This proof is largely inspired from [122].

Theorem 5.2.1. *Let T, $L > 0$ be given. For any $y_0 \in L^2(0,L)$ and any $f \in L^1(0,T;L^2(0,L))$, the Cauchy problem (5.6) admits a unique mild solution which belongs to the space $\mathcal{B}_{0,T}$. Furthermore, for the mild solution y we have the estimate*

$$|y|_{L^2(0,T;H_0^1(0,T))} + \sup_{0\le t\le T} |y(t)|_{L^2(0,L)} \le C(|y_0|_{L^2(0,L)} + |f|_{L^1(0,T;L^2(0,L))}), \tag{5.7}$$

where the constant $C > 0$ depends on L and T.

Proof. First, we consider the operator $\mathcal{A} := -\partial_x - \partial_x^3$ on the dense domain $\mathcal{D}(\mathcal{A}) \subset L^2(0,L)$ which is defined by

$$\mathcal{D}(\mathcal{A}) := \{\phi \in H^3(0,L) \mid \phi(0) = \phi(L) = \phi'(L) = 0\}.$$

It has been shown [122] that the operator $\mathcal{A}$ and its adjoint $\mathcal{A}^*$ with domain

$$\mathcal{D}(\mathcal{A}^*) := \{\phi \in H^3(0,L) \mid \phi(0) = \phi(L) = \phi'(0) = 0\}.$$

are dissipative. Therefore, due to [115, cor. 4.4, Chapter 1, Page 15] the operator $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup of contractions $\{W(t)\}_{t\ge 0}$ defined on $L^2(0,L)$ and we have the mild form of the solution to (5.6) given by

$$y(t) = W(t)y_0 + \int_0^t W(t-s)f(s)ds \quad \text{for all } t \in [0,T].$$

Moreover the following estimate holds:

$$\|y\|_{C([0,T];L^2(0,L))} \le \|y_0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}. \tag{5.8}$$

To show that $y \in L^2(0,T;H_0^1(0,L))$, we first assume that $y_0 \in \mathcal{D}(\mathcal{A})$, $f \in C^1([0,T],L^2(0,L))$, and $q \in C^\infty([0,T]\times[0,L])$. Under these assumption on y_0 and f we have that $y \in C([0,T];\mathcal{D}(\mathcal{A})) \cap C^1([0,T];L^2(0,L))$ (see, e.g., [115]). Multiplying (5.6) by qy and integrating over $(0,T)\times(0,L)$, we obtain

$$\int_0^T \int_0^L qy(\partial_t y + \partial_x y + \partial_x^3 y - f)dxdt = 0.$$

Integrating by parts we have

$$\begin{aligned}
&-\int_0^T\int_0^L(\partial_t q+\partial_x q+\partial_x^3 q)\frac{y^2}{2}dxdt+\int_0^L(q\frac{y^2}{2})(T,x)dx\\
&-\int_0^L q(0,x)\frac{y_0^2(x)}{2}dx+\frac{3}{2}\int_0^T\int_0^L\partial_x q(\partial_x y)^2dxdt\\
&+\int_0^T(q\frac{(\partial_x y)^2}{2})(t,0)dt-\int_0^T\int_0^L yqf\,dxdt=0.
\end{aligned}$$

Choosing $q(t,x)=x$ leads to

$$\begin{aligned}
-\int_0^T\int_0^L y^2dxdt+&\int_0^L xy(T,x)^2dx-\int_0^L xy_0^2(x)dx\\
&+3\int_0^T\int_0^L(\partial_x y)^2dxdt-\int_0^T\int_0^L xfydxdt=0.
\end{aligned}$$

Now, by using (5.8) we obtain

$$\begin{aligned}
&\int_0^T\int_0^L(\partial_x y)^2dxdt\\
&\leq\frac{1}{3}\Big(\int_0^T\int_0^L y^2dxdt+L\int_0^L y_0^2(x)dx+L\int_0^T\int_0^L|fy|dxdt\Big)\\
&\leq\frac{1}{3}\Big(T\|y\|^2_{C([0,T];L^2(0,L))}+L\|y_0\|^2_{L^2(0,L)}\\
&+L(\|y\|_{C([0,T];L^2(0,L))}\|f\|_{L^1(0,T;L^2(0,L))})\Big)\\
&\leq\frac{1}{3}\Big(T(\|y_0\|_{L^2(0,L)}+\|f\|_{L^1(0,T;L^2(0,L))})^2\\
&+L\|y_0\|^2_{L^2(0,L)}+L(\|y_0\|_{L^2(0,L)}+\|f\|_{L^1(0,T;L^2(0,L))})\|f\|_{L^1(0,T;L^2(0,L))})\Big)\\
&\leq\frac{(T+L)}{3}\Big(\|y_0\|_{L^2(0,L)}+\|f\|_{L^1(0,T;L^2(0,L))}\Big)^2.
\end{aligned}\tag{5.9}$$

Therefore we have

$$\|y\|_{L^2(0,T;H_0^1(0,L))}\leq C(\|y_0\|_{L^2(0,L)}+\|f\|_{L^1(0,T;L^2(0,L))}),$$

where C depends on L and T. By density of $\mathcal{D}(\mathcal{A})$ and $C^1([0,T];L^2(0,L))$ in $L^2(0,L)$ and $L^1(0,T;L^2(0,L))$, respectively, we can extend the estimate for the mild solutions with arbitrary $y_0\in L^2(0,L)$ and $f\in L^1(0,T;L^2(0,L))$. □

Lemma 5.2.1. *Let $T>0$ and $y\in\mathcal{B}_{0,T}$. Then $y\partial_x y\in L^1(0,T;L^2(0,L))$. Moreover the mapping $y\in\mathcal{B}_{0,T}\to y\partial_x y\in L^1(0,T;L^2(0,L))$ is continuous, and for every $y,z\in\mathcal{B}_{0,T}$ we have the following estimate*

$$\|y\partial_x y - z\partial_x z\|_{L^1(0,T;L^2(0,L))} = C_a T^{\frac{1}{4}}(\|y\|_{\mathcal{B}_{0,T}} + \|z\|_{\mathcal{B}_{0,T}})\|y-z\|_{\mathcal{B}_{0,T}}, \tag{5.10}$$

where C_a is a positive constant independent of T.

Proof. Assume that y and $z \in \mathcal{B}_{0,T}$ are arbitrary. Then we have

$$\begin{aligned}
\|y\partial_x y - z\partial_x z\|_{L^1(0,T;L^2(0,L))}& \\
&\leq \|y(\partial_x y - \partial_x z)\|_{L^1(0,T;L^2(0,L))} + \|\partial_x z(y-z)\|_{L^1(0,T;L^2(0,L))} \\
&\leq (\|y\|_{L^2(0,T;L^\infty(0,L))}\|\partial_x y - \partial_x z\|_{L^2(0,T;L^2(0,L))} \\
&+ \|\partial_x z\|_{L^2(0,T;L^2(0,L))}\|y-z\|_{L^2(0,T;L^\infty(0,L))}) \\
&\leq C_a\|y\|^{\frac{1}{2}}_{L^2(0,T;H^1_0(0,L))}\|y\|^{\frac{1}{2}}_{L^2(0,T;L^2(0,L))}\|y-z\|_{L^2(0,T;H^1_0(0,L))} \\
&+ C_a\|z\|_{L^2(0,T;H^1_0(0,L))}\|y-z\|^{\frac{1}{2}}_{L^2(0,T;H^1_0(0,L))}\|y-z\|^{\frac{1}{2}}_{L^2(0,T;L^2(0,L))} \\
&\leq C_a T^{\frac{1}{4}}\|y\|^{\frac{1}{2}}_{L^2(0,T;H^1_0(0,L))}\|y\|^{\frac{1}{2}}_{C([0,T];L^2(0,L))}\|y-z\|_{L^2(0,T;H^1_0(0,L))} \\
&+ C_a T^{\frac{1}{4}}\|z\|_{L^2(0,T;H^1_0(0,L))}\|y-z\|^{\frac{1}{2}}_{L^2(0,T;H^1_0(0,L))}\|y-z\|^{\frac{1}{2}}_{C([0,T];L^2(0,L))} \\
&= C_a T^{\frac{1}{4}}(\|y\|_{\mathcal{B}_{0,T}} + \|z\|_{\mathcal{B}_{0,T}})\|y-z\|_{\mathcal{B}_{0,T}},
\end{aligned} \tag{5.11}$$

where the constant C_a stands for the Agmon's inequality. By taking $z=0$ in (5.10), we see that $y\partial_x y \in L^1(0,T;L^2(0,L))$. □

We now turn to the nonlinear equation.

Definition 5.2.1 (Mild solution)**.** *Suppose that $T > 0$ is arbitrary, and we are given $f \in L^1(0,T;L^2(0,L))$ and $y_0 \in L^2(0,L)$. Then $y \in \mathcal{B}_{0,T}$ is referred to as a mild solution to* (5.4) *if the following integral equation is satisfied*

$$y(t) = W(t)y_0 - \int_0^t W(t-s)(y\partial_x y)(s)ds + \int_0^t W(t-s)f(s)ds \quad \text{for all } t \in [0,T],$$

where the C_0-semigroup of contractions $\{W(t)\}_{t\geq 0}$ defined in the proof of Theorem 5.2.1.

Theorem 5.2.2. *Let T, $L > 0$ be given. For any initial function $y_0 \in L^2(0,L)$ and forcing function $f \in L^1(0,T;L^2(0,L))$, there exists a $T^* \in [0,T]$ depending on $|y_0|_{L^2(0,L)}$ and $|f|_{L^1(0,T;L^2(0,L))}$ such that* (5.4) *admits a unique solution in the space $\mathcal{B}_{0,T^*}$.*

Proof. We express problem (5.4) as a fixed point equation $y = \Psi(y)$. For this purpose we write (5.4) in integral form as

$$y(t) = W(t)y_0 - \int_0^t W(t-s)(y\partial_x y)(s)ds + \int_0^t W(t-s)f(s)ds.$$

For any $r > 0$ and time horizon θ, we define the ball $S_{\theta,r}$ centered at zero by $S_{\theta,r} := \{x \in \mathcal{B}_{0,\theta}, |x|_{\mathcal{B}_{0,\theta}} \le r\}$. This is a closed, convex, and bounded subset of $\mathcal{B}_{0,\theta}$. We define the mapping Ψ on $S_{\theta,r}$ by

$$\Psi(y) := W(t)y_0 - \int_0^t W(t-s)(y\partial_x y)(s)ds + \int_0^t W(t-s)f(s)ds \quad \text{for } y \in S_{\theta,r}.$$

Then by (5.7) and (5.10), we have

$$\begin{aligned}|\Psi(y)|_{B_{0,\theta}} \le& C\Big(|y_0|_{L^2(0,L)} + |f|_{L^1(0,T;L^2(0,L))} + |y\partial_x y|_{L^1(0,T;L^2(0,L))}\Big)\\ \le& C\Big(|y_0|_{L^2(0,L)} + |f|_{L^1(0,T;L^2(0,L))}\Big) + CC_a\theta^{\frac{1}{4}}|y|^2_{B_{0,\theta}}.\end{aligned}$$

Choosing the r and θ such that

$$\begin{cases} r = \frac{4}{3}C\Big(|y_0|_{L^2(0,L)} + |f|_{L^1(0,T;L^2(0,L))}\Big), \\ CC_a\theta^{\frac{1}{4}}r \le \frac{1}{4}. \end{cases} \tag{5.12}$$

we obtain

$$|\Psi(y)|_{B_{0,\theta}} \le r \quad \text{for all } y \in S_{\theta,r},$$

and

$$|\Psi(y_1) - \Psi(y_2)|_{B_{0,\theta}} \le \frac{1}{2}|y_1 - y_2|_{B_{0,\theta}} \quad \text{for all } y_1, y_2 \in S_{\theta,r}.$$

The existence of a unique solution to the Cauchy problem (5.4) follows by Banach's fixed point theorem. Note that by (5.12) we have

$$T^* \le \frac{1}{\Big(\frac{16C_aC^2}{3}(|y_0|_{L^2(0,L)} + |f|_{L^1(0,T;L^2(0,L))})\Big)^4}.$$

Therefore for any $y_0 \in L^2(0,L)$ and $f \in L^1(0,T;L^2(0,L))$, there exists $T^* \in [0,T]$ depending on $|y_0|_{L^2(0,L)}$ and $|f|_{L^1(0,T;L^2(0,L))}$ such that (5.4) admits a unique solution in the space $\mathcal{B}_{0,T^*}$. □

To show global well-posedness we need an a-priori estimate for solutions of (5.4). This is attained next.

Lemma 5.2.2. *Let $T > 0$ be arbitrary. Then for every $y_0 \in L^2(0,L)$ and $f \in L^2(0,T;L^2(0,L))$, the solution $y \in \mathcal{B}_{0,T'}$ to* (5.4) *with $T' \in (0,T]$ satisfies the following estimate*

$$|y|_{\mathcal{B}_{0,T'}} \le K_1(T,L,y_0,f). \tag{5.13}$$

Moreover, the solution y belongs to the space $\mathcal{W}_{0,T'}$ and we have that following estimate

$$|y|_{\mathcal{B}_{0,T'}} + |\partial_t y|_{L^2(0,T';H^{-2}(0,L))} \le K_2(T,L,y_0,f), \tag{5.14}$$

where the constants K_1 and K_2 depend on the quantities T, L, $|y_0|_{L^2(0,L)}$, and $|f|_{L^2(0,T;L^2(0,L))}$. Further these constants will grow unboundedly as at least one of the above quantities tends to infinity.

Proof. Multiplying (5.4) with y and integrating on $(0, L)$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(0,L)} + \frac{1}{2}(\partial_x y(t,0))^2 &= \langle y(t), f(t)\rangle_{L^2} \\ &\leq \|y(t)\|_{L^2(0,L)}\|f(t)\|_{L^2(0,L)} \leq \frac{1}{2}\|y(t)\|^2_{L^2(0,L)} + \frac{1}{2}\|f(t)\|^2_{L^2(0,L)},\end{aligned} \tag{5.15}$$

where we assume that the solution y is smooth enough to allow the calculations. Integrating on $(0, T')$ and using Gronwall's inequality we have

$$\|y(t)\|^2_{L^2(0,L)} \leq \exp(T)(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,T;L^2(0,L))}), \quad \text{for all } t \in [0, T']. \tag{5.16}$$

Now by a density argument and considering the fact that for $y_0 \in \mathcal{D}(\mathcal{A})$ and $f \in C^1([0,T]; L^2(0,L))$, the solution y of (5.4) belongs to the space $L^2(0,T'; H^4(0,L)) \cap C([0,T']; H^3(0,L))$ (see, [55]), we can write

$$|y|_{L^\infty(0,T';L^2(0,L))} \leq C_1(T, y_0, f), \tag{5.17}$$

with

$$C_1(T, y_0, f) := \Big(\exp(T)(\|y_0\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,T;L^2(0,L))})\Big)^{\frac{1}{2}}, \tag{5.18}$$

for every $y_0 \in L^2(0,L)$ and $f \in L^2(0,T,L^2(0,L))$.

It remains to find an estimate for the term $\|\partial_x y\|_{L^2(0,T;L^2(0,T))}$. As in Lemma 5.2.1 we assume that the solution is smooth enough. Then by multiplying equation (5.4) by xy and integrating over $(0,L)$, for every $t \in (0,T')$ we have

$$\begin{aligned}\frac{d}{dt}\int_0^L |x^{\frac{1}{2}} y(t,\cdot)|^2 dx + 3\int_0^L (\partial_x y)^2(t,\cdot)dx + x(\partial_x y)^2(t,0) \\ = \int_0^L y^2(t,\cdot)dx + \frac{2}{3}\int_0^L y^3(t,\cdot)dx + 2\int_0^L xfydx.\end{aligned} \tag{5.19}$$

Moreover, we have for almost every $t \in (0, T')$

$$\int_0^L |y(t,\cdot)|^2 dx \leq \|y\|^2_{L^\infty(0,T';L^2(0,L))}, \tag{5.20}$$

and

$$\begin{aligned}\frac{2}{3}\int_0^L |y(t,\cdot)|^3 dx &\leq \frac{2}{3}\|y(t)\|_{L^\infty(0,L)}\|y\|^2_{L^\infty(0,T';L^2(0,L))} \\ &\leq \frac{2c'}{3}\|\partial_x y(t)\|_{L^2(0,L)}\|y\|^2_{L^\infty(0,T';L^2(0,L))} \\ &\leq \frac{\epsilon c'}{3}\|\partial_x y(t)\|^2_{L^2(0,L)} + \frac{c'}{3\epsilon}\|y\|^4_{L^\infty(0,T';L^2(0,L))},\end{aligned} \tag{5.21}$$

where the constant c' is the embedding constant of $H^1_0(0,L)$ into $L^\infty(0,L)$, and the positive number ϵ will be chosen later. Furthermore, we have

$$\begin{aligned}2\int_0^L |xfy|dx &\leq L\|f(t)\|^2_{L^2(0,L)} + L\|y(t)\|^2_{L^2(0,L)} \\ &\leq L\|f(t)\|^2_{L^2(0,L)} + L\|y\|^2_{L^\infty(0,T';L^2(0,L))}.\end{aligned} \tag{5.22}$$

Now by choosing $\epsilon := \frac{6}{c'}$, and combining inequalities (5.20), (5.21), and (5.22) with (5.19), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^L |x^{\frac{1}{2}}y(t,\cdot)|^2 dx + \int_0^L (\partial_x y)^2(t,\cdot)dx \\ &\quad \leq (1+L)\|y\|^2_{L^\infty(0,T';L^2(0,L))} + \frac{c'^2}{18}\|y\|^4_{L^\infty(0,T';L^2(0,L))} + L\|f(t)\|^2_{L^2(0,L)}. \end{aligned} \tag{5.23}$$

Integration with respect to t over interval $(0,T')$ implies that

$$\begin{aligned} \int_0^L |x^{\frac{1}{2}}y(T',\cdot)|^2 dx + &\int_0^{T'}\int_0^L (\partial_x y)^2 dxdt \\ &\leq L\|y_0\|^2_{L^2(0,L)} + T(1+L)\|y\|^2_{L^\infty(0,T';L^2(0,L))} \\ &+ T\frac{c'^2}{18}\|y\|^4_{L^\infty(0,T';L^2(0,L))} + L\|f\|^2_{L^2(0,T;L^2(0,L))} \\ &\leq L\|y_0\|^2_{L^2(0,L)} + T(1+L)C_1^2(T,y_0,f) \\ &+ T\frac{c'^2}{18}C_1^4(T,y_0,f) + L\|f\|^2_{L^2(0,T;L^2(0,L))}, \end{aligned} \tag{5.24}$$

and as consequence of (5.17) and (5.18), we can conclude that

$$|y|_{\mathcal{B}_{0,T'}} \leq K_1(T,L,|y_0|_{L^2(0,L)},|f|_{L^2(0,T;L^2(0,L))}). \tag{5.25}$$

Turing to inequality (5.14), we obtain from (5.25) that

$$\begin{aligned} &\|\partial_t y\|_{L^2(0,T';H^{-2}(0,L))} = \sup_{\|\phi\|_{L^2(0,T;H_0^2(0,L))}=1}\int_0^{T'}\langle\partial_t y,\phi\rangle_{H^{-2},H_0^2} \\ &= \sup_{\|\phi\|_{L^2(0,T';H_0^2(0,L)))}=1}\int_0^L\int_0^{T'}(-\partial_x y\phi - \partial_x y\partial_x^2\phi - y\partial_x y\phi + f\phi)dxdt \\ &\leq (2 + c_1\|y\|_{L^\infty(0,T';L^2(0,L))})\|y\|_{L^2(0,T';H_0^1(0,L))} + \|f\|_{L^2(0,T;L^2(0,L))} \\ &\leq (2 + c_1K_1(T,L,|y_0|_{L^2(0,L)},|f|_{L^2(0,T;L^2(0,L))})K_1(T,L,|y_0|_{L^2(0,L)},|f|_{L^2(0,T;L^2(0,L))}) \\ &+ \|f\|_{L^2(0,T;L^2(0,L))}, \end{aligned} \tag{5.26}$$

where c_1 stands for the continuous embedding from $H^2(0,L)$ to $L^\infty(0,L)$. Combining (5.25) and (5.17), we conclude (5.14).

□

Theorem 5.2.3. *Let an arbitrary $T > 0$ be given. Then for every $y_0 \in L^2(0,L)$ and $f \in L^2(0,T;L^2(0,L))$, there exist a unique mild solution $y \in \mathcal{W}_{0,T}$ for the nonlinear KdV equation* (5.4).

Proof. Local existence due to Theorem 5.2.2 together with the a-priori bound (5.13) of Lemma 5.2.2 imply global existence by the standard continuation argument. Uniqueness follows from Theorem 5.2.2 as well. □

We will later use the following useful lemma from [30, Page. 45].

Lemma 5.2.3. *Let E and F be two Banach spaces and $\mathcal{A} : E \supset \mathcal{D}(A) \to F$ be a densely defined unbounded linear operator, then the adjoint operator $\mathcal{A}^*$ is closed. That is, the graph of this operator $\mathcal{G}(\mathcal{A}^*)$ is closed in $F^* \times E^*$. Moreover we have*

$$\mathcal{I}(\mathcal{G}(\mathcal{A}^*)) = \mathcal{G}(\mathcal{A})^{\perp},$$

where the isomorphism $\mathcal{I} : F^* \times E^* \to E^* \times F^*$ is defined by

$$\mathcal{I}(X, Y) = (-Y, X) \quad \text{for all } (X, Y) \in F^* \times E^*.$$

Definition 5.2.2 (Weak solution)**.** *Suppose that $T > 0$ is arbitrary, and we are given $f \in L^2(0, T; L^2(0, L))$ and $y_0 \in L^2(0, L)$. Then $y \in \mathcal{W}_{0,T}$ is referred to as a weak solution to* (5.4) *if $y(0) = y_0$ in $L^2(0, L)$ and the following equality holds*

$$\langle \partial_t y(t), \phi \rangle_{H^{-2},H^2} + \langle \partial_x y(t), \phi \rangle_{L^2} + \langle y(t)\partial_x y(t), \phi \rangle_{L^2} + \langle \partial_x y(t), \partial_x^2 \phi \rangle_{L^2} = \langle f(t), \phi \rangle_{L^2} \quad (5.27)$$

for almost every $t \in (0, T)$ and every $\phi \in \mathcal{X}$.

Theorem 5.2.4. *For every $T > 0$, $f \in L^2(0, T; L^2(0, L))$, and $y_0 \in L^2(0, L)$, problem* (5.4) *admits a unique weak solution.*

Proof. Inspired by [15, 23], we first show that any mild solution of (5.4) is a weak solution. Let $y \in \mathcal{W}_{0,T}$ be a mild solution of (5.4). Then for every $t \in [0, T]$ we have

$$y(t) = W(t)y_0 - \int_0^t W(t-s)(y\partial_x y)(s)ds + \int_0^t W(t-s)f(s)ds,$$

where $y \in C([0, T]; L^2(0, L)) \subset L^1(0, T; L^2(0, L))$. For every $\phi \in D(\mathcal{A}^*)$ and $\sigma \in \mathcal{D}(0, T)$, the vectorial distributional derivative of y is obtained by

$$\begin{aligned} &- \int_0^T \langle y(t), \phi \rangle \sigma'(t) dt \\ &= - \int_0^T \Big[\Big\langle W(t)y_0 + \int_0^t W(t-s)\big(f(s) - y(s)\partial_x y(s) ds, \phi \Big\rangle \Big] \sigma'(t)\, dt \\ &= - \int_0^T \langle W(t)y_0, \phi \rangle \sigma'(t) dt - \int_0^T \int_s^T \langle W(t-s)\big(f(s) - y(s)\partial_x y(s)\big), \phi \rangle \sigma'(t) dt ds. \end{aligned} \quad (5.28)$$

For every $\psi \in D(\mathcal{A})$ and $\phi \in D(\mathcal{A}^*)$, we can write for all most every $t > 0$

$$\frac{d}{dt} \langle W(t)\psi, \phi \rangle = \langle \mathcal{A}W(t)\psi, \phi \rangle = \langle W(t)\psi, \mathcal{A}^*\phi \rangle. \quad (5.29)$$

Since $\mathcal{D}(\mathcal{A})$ is dense in $L^2(0,L)$, this equality can be extended for every $\psi \in L^2(0,T)$. Moreover, by integrating by parts we have

$$-\int_0^T \langle W(t)y_0, \phi\rangle \sigma'(t)dt = \int_0^T \langle W(t)y_0, \mathcal{A}^*\phi\rangle \sigma(t)dt, \tag{5.30}$$

and,

$$\begin{aligned} &-\int_s^T \langle W(t-s)(f(s)-y(s)\partial_x y(s), \phi\rangle \sigma'(t)dt \\ &= \langle f(s)-y(s)\partial_x y(s), \phi\rangle \sigma(s) + \int_s^T \langle W(t-s)(f(s)-y(s)\partial_x y(s)), \mathcal{A}^*\phi\rangle \sigma(t)dt. \end{aligned} \tag{5.31}$$

Substituting (5.30)-(5.31) into (5.28), we obtain

$$\begin{aligned} &-\int_0^T \langle y(t), \phi\rangle \sigma'(t)dt \\ &\qquad = \int_0^T \langle y(t), \mathcal{A}^*\phi\rangle \sigma(t)dt + \int_0^T \langle f(t)-y(t)\partial_x y(s), \phi\rangle \sigma(t)dt. \end{aligned} \tag{5.32}$$

Due to Lemma 5.2.1, we have $y\partial_x y \in L^1(0,T;L^2(0,L))$. Furthermore, $y \in C([0,T];L^2(0,L)) \subset L^1(0,T;L^2(0,L))$, and $f \in L^2(0,T;L^2(0,L))$. Therefore $\langle y(\cdot), \phi\rangle \in W^{1,1}(0,T;\mathbb{R})$ and for almost every $t \in [0,T]$ we have by (5.32)

$$\frac{d}{dt}\langle y(t), \phi\rangle = \langle y(t), \mathcal{A}^*\phi\rangle - \langle y(t)\partial_x y(t), \phi\rangle + \langle f(t), \phi\rangle \quad \text{for all } \phi \in D(\mathcal{A}^*). \tag{5.33}$$

By Lemma 5.2.2, we recall that $y \in \mathcal{W}_{0,T}$. Hence, we can rewrite (5.33) as

$$\frac{d}{dt}\langle y(t), \phi\rangle_{H^{-2},H^2} + \langle \partial_x y(t), \phi\rangle_{L^2} + \langle y(t)\partial_x y(t), \phi\rangle_{L^2} + \langle \partial_x y(t), \partial_x^2\phi\rangle_{L^2} = \langle f(t), \phi\rangle_{L^2}.$$

Since $\mathcal{D}(\mathcal{A}^*)$ is dense in $\mathcal{X}$, the above equality holds for every $\phi \in \mathcal{X}$, and hence y is a weak solution.

Now we show that every weak solution (5.27) is a mild solution of (5.4). By using the fact that $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{X}$ and integrating by parts in (5.27), we have for almost every $t \in [0,T]$

$$\frac{d}{dt}\langle y(t), \phi\rangle = \langle y(t), \mathcal{A}^*\phi\rangle - \langle y(t)\partial_x y(t), \phi\rangle + \langle f(t), \phi\rangle \quad \text{for all } \phi \in D(\mathcal{A}^*).$$

Integrating on $(0,t)$ for an arbitrary $t \in [0,T]$, we obtain

$$\langle y(t) - y_0 + \int_0^t \big(y(s)\partial_x y(s) - f(s)\big)ds, \phi\rangle = \langle \int_0^t y(s)ds, \mathcal{A}^*\phi\rangle \quad \text{for all } \phi \in D(\mathcal{A}^*).$$

This equality implies that

$$\Big(\int_0^t y(s)ds, y(t) - y_0 + \int_0^t \big(y(s)\partial_x y(s) - f(s)\big)ds\Big) \in (\mathcal{I}(\mathcal{G}(A^*))^\perp \quad \text{for all } t \in [0,T].$$

By Lemma 5.2.3, we can conclude that

$$(\mathcal{I}(\mathcal{G}(A^*))^\perp = \big((\mathcal{G}(A))^\perp\big)^\perp = \overline{\mathcal{G}(A)} = \mathcal{G}(A).$$

Therefore, for all $t \in [0,T]$ we have $\int_0^t y(s)ds \in \mathcal{D}(\mathcal{A})$, and

$$\mathcal{A}\int_0^t y(s)ds = y(t) - y_0 + \int_0^t \big(y(s)\partial_x y(s) - f(s)\big)ds.$$

Now by defining $z(t) := \int_0^t y(s)ds$ for all $t \in [0,T]$, we have

$$\begin{cases} \dot z(t) = \mathcal{A}z(t) + y_0 - \int_0^t \big(y(s)\partial_x y(s) - f(s)\big)ds, \\ z(0) = 0. \end{cases} \tag{5.34}$$

We set $\mathcal{A}_\lambda = \lambda\mathcal{A}(\lambda I - \mathcal{A})^{-1}$ for $\lambda > 0$ as the Yosida approximations of the operator $\mathcal{A}$. Then by (5.34) we can write

$$\begin{aligned} \frac{d}{dt}\big(e^{-\mathcal{A}_\lambda t}z(t)\big) &= e^{-\mathcal{A}_\lambda t}\dot z(t) - \mathcal{A}_\lambda e^{-\mathcal{A}_\lambda t}z(t) \\ &= (\mathcal{A} - \mathcal{A}_\lambda)e^{-\mathcal{A}_\lambda t}z(t) + e^{-\mathcal{A}_\lambda t}\big[y_0 - \int_0^t \big(y(s)\partial_x y(s) - f(s)\big)ds\big]. \end{aligned}$$

It follows that

$$\begin{aligned} z(t) &= e^{\mathcal{A}_\lambda t}\int_0^t \Big((\mathcal{A} - \mathcal{A}_\lambda)e^{-\mathcal{A}_\lambda s}z(s) + e^{-\mathcal{A}_\lambda s}\big[y_0 - \int_0^s \big(y(r)\partial_x y(r) - f(r)\big)dr\big]\Big)ds \\ &= \int_0^t (\mathcal{A} - \mathcal{A}_\lambda)e^{\mathcal{A}_\lambda (t-s)}z(s)ds + \int_0^t e^{\mathcal{A}_\lambda(t-s)}\big[y_0 - \int_0^s \big(y(r)\partial_x y(r) - f(r)\big)dr\big]ds. \end{aligned} \tag{5.35}$$

For every $\lambda > 0$ and $s \in [0,T]$, by using (5.34) and Lemma 5.2.1 we have that

$$\begin{aligned} \|\mathcal{A}z(s)\|_{L^2(0,L)} &\le \|y(s)\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)} + T^{\frac{1}{2}}\|f\|_{L^2(0,T;L^2(0,L))} \\ &\quad + \|y\partial_x y\|_{L^1(0,T;L^2(0,L)} \le C, \\ \|\mathcal{A}_\lambda z(s)\|_{L^2(0,L)} &\le \|\lambda(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(L^2(0,L))}\|\mathcal{A}z(s)\|_{L^2(0,L)} \\ &\le \|\mathcal{A}z(s)\|_{L^2(0,L)} \le C, \end{aligned} \tag{5.36}$$

where, using that $y \in \mathcal{W}_{0,T}$, the constant C is independent of $s \in [0,T]$. In addition,

$$\begin{cases} \lim_{\lambda\to\infty}(\mathcal{A} - \mathcal{A}_\lambda)z(s) = 0 & \text{for all } s \in [0,T], \\ \lim_{\lambda\to\infty} e^{\mathcal{A}_\lambda t}y = W(t)y & \text{for all } y \in L^2(0,L), \quad t \in [0,T]. \end{cases}$$

Now by using the dominated convergence theorem and (5.36), we obtain from (5.35) for $\lambda \to \infty$

$$\begin{aligned} z(t) &= \int_0^t W(t-s)\big[y_0 + \int_0^s \big(-y(r)\partial_x y(r) + f(r)\big)\,dr\big]ds \\ &= \int_0^t W(s)y_0 ds + \int_0^t W(s)\big[\int_0^{t-s} \big(-y(r)\partial_x y(r) + f(r)\big)\,dr\big]\,ds. \end{aligned}$$

Therefore,

$$y(t) = \dot{z}(t) = W(t)y_0 + \int_0^t W(t-s)(-y(s)\partial_x y(s) + f(s))ds,$$

and thus y is a mild solution. Finally uniqueness of the weak solution follows from the uniqueness of the mild solution. □

5.3 Existence of an optimal control

In Step 2 of any iteration of Algorithm 5.1, we need to solve a finite horizon optimal control problem consisting in minimizing

$$J_T(u, y_0) := \int_0^T \ell(y(t), u(t))dt,$$

over all $u \in L^2(0, T; L^2(\hat{\Omega}))$ subject to the nonlinear KdV equation

$$\begin{cases} \partial_t y + \partial_x y + y\partial_x y + \partial_x^3 y = Bu & x \in (0, L), \quad t \in (0, T), \\ y(t, 0) = y(t, L) = \partial_x y(t, L) = 0 & t \in (0, T), \\ y(0, \cdot) = y_0 & x \in (0, L), \end{cases} \tag{5.37}$$

where $y_0 \in L^2(0, L)$. Therefore we need to verify that the above optimal control problem has a solution. This question will be addressed by the following theorem. We denote the above optimal control problem by (OP) and write it as

$$\min\{J_T(u, y_0) \mid (y, u) \text{ satisfies } (5.37), u \in L^2(0, T; L^2(\hat{\Omega}))\}. \tag{OP}$$

Theorem 5.3.1. *For every finite horizon $T > 0$ and $y_0 \in L^2(0, L)$, the optimal control problem* (OP) *admits a solution.*

Proof. According to Theorem 5.2.4, for every control $u \in L^2(0, T; L^2(\hat{\Omega}))$ there exist a unique weak solution $y \in \mathcal{W}_{0,T}$ to (5.37). As a result, the set of admissible controls is nonempty and by (5.3) we have

$$J_T(u, y_0) \geq \frac{\beta}{2}\|u\|^2_{L^2(0,T;L^2(\hat{\Omega}))}. \tag{5.38}$$

Let $(y^n, u^n) \in \mathcal{W}_{0,T} \times L^2(0, T; L^2(\hat{\Omega}))$ be a minimizing sequence such that

$$\lim_{n\to\infty} J_T(u^n, y_0) = \sigma.$$

By (5.14), (5.38), and due to the structure of ℓ, the set $\{(y^n, u^n)\}_n$ is bounded in $\mathcal{W}_{0,T} \times L^2(0, T; L^2(\hat{\Omega}))$. Therefore there exist subsequences y^n and u^n such that

$$\begin{aligned} &y^n \rightharpoonup^* y^* \text{ in } L^2(0, T; H_0^1(0, L)) \cap L^\infty(0, T; L^2(0, L)) \cap H^1(0, T; H^{-2}(0, L)), \\ &u^n \rightharpoonup u^* \text{ in } L^2(0, T; L^2(\hat{\Omega})), \end{aligned} \tag{5.39}$$

where

$$y^* \in L^2(0,T;H_0^1(0,L)) \cap L^\infty(0,T;L^2(0,L)) \cap H^1(0,T;H^{-2}(0,L)),$$
$$u^* \in L^2(0,T;L^2(\hat{\Omega})).$$

It remains to show that y^* is the weak solution to (5.37) corresponding to control u^*. By definition of weak convergence we have

$$\int_0^T \langle \partial_t y^n - \partial_t y^*, \phi \rangle_{H^{-2},H^2}\, dt \to 0 \qquad \text{for every } \phi \in L^2(0,T;\mathcal{X}).$$

By the compact embedding [128] of the space $L^2(0,T;H_0^1(0,L)) \cap H^1(0,T;H^{-2}(0,L))$ into the space $L^2(0,T;L^2(0,L))$, we obtain for every $\phi \in L^2(0,T;\mathcal{X})$

$$\begin{aligned} &\int_0^T \int_0^L (y^n \partial_x y^n - y^* \partial_x y^*)\phi\, dxdt = -\frac{1}{2}\int_0^T \int_0^L ((y^n)^2 - (y^*)^2)\partial_x \phi\, dxdt \\ &\leq \frac{1}{2}\int_0^T (\|y^n(t)\|_{L^2(0,L)} \\ &+ \|y^*(t)\|_{L^2(0,L)})\|y^n(t) - y^*(t)\|_{L^2(0,L)}\|\partial_x \phi(t)\|_{L^\infty(0,L)}\, dt \\ &\leq \frac{1}{2}c_5(\|y^n\|_{C([0,T];L^2(0,L))} \\ &+ \|y^*\|_{C([0,T];L^2(0,L))})\|y^n - y^*\|_{L^2(0,T;L^2(0,L))}\|\phi\|_{L^2(0,T,H^2(0,T))} \to 0, \end{aligned} \tag{5.40}$$

where the constant c_5 stands for the continuous embedding of $H^2(0,L)$ into $W^{1,\infty}(0,L)$. By (5.39) we obtain

$$\int_0^T \langle Bu^n - Bu^*, \phi \rangle\, dt \to 0 \quad \text{for all } \phi \in L^2(0,T;\mathcal{X}). \tag{5.41}$$

Due to the fact that $y^*(0) \in L^2(0,L)$ and using (5.40), (5.41), and (5.27) with $f = Bu$, we conclude that $y^* \in \mathcal{W}_{0,T}$ is the weak solution to (5.37) corresponding to u^*. Since $y^n \to y^*$ strongly in $L^2(0,T;L^2(0,L))$ and $u^n \to u^*$ weakly in $L^2(0,T;L^2(\hat{\Omega}))$ we have

$$0 \leq J_T(u^*, y_0) \leq \liminf_{n\to\infty} J_T(u^n, y_0) = \sigma,$$

and as a consequence the pair (y^*, u^*) is optimal. □

5.4 Semi-global stabilizability of KdV

In this section, we review some results about the stablizability of the nonlinear KdV equation by feedback. We consider

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = F(y) & x \in (0,L), \quad t \in (0,T), \\ y(t,0) = y(t,L) = \partial_x y(t,L) = 0 & t \in (0,T), \\ y(0,\cdot) = y_0 & x \in (0,L), \end{cases} \tag{5.42}$$

where F is a linear feedback control which acts only on a subdomain of $[0, L]$. Our objective is to find a control which dissipates enough energy to force the decay of the solution with respect to the L^2-norm. The control is of the form $F(y) = -\omega y$, where ω is defined by

$$\begin{cases} \omega \in L^\infty(0, L) \text{ and } \omega(x) \geq \omega_0 > 0 \text{ for a.e. in } \hat{\Omega}, \\ \text{where } \hat{\Omega} \text{ is any nonempty open subset of } [0, L]. \end{cases} \tag{5.43}$$

In [122] Rosier studied the controllability of the linear KdV equation and he found the set of critical points which given by

$$\Upsilon := \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}} \mid k, l \in \mathbb{N} \right\}.$$

Moreover, he discovered that, if the length L of the spatial domain belongs to set Υ, the uncontrolled ($\omega = 0$) linear KdV equation has solutions for which the L^2-norm stays constant as $t \to \infty$. In this case, i.e., $L \in \Upsilon$, one can show that the linear KdV equation is globally exponentially stabilizable by a linear feedback law of the form $F(y) = -\omega y$ acting on an open subset $\hat{\Omega}$ of $[0, L]$, see, e.g., [116].

For the nonlinear KdV equation, the situation is more delicate and it is not clear whether the solutions goes to zero. In [116] by using a perturbation argument it has been shown that the nonlinear KdV equation is locally stabilizable for small initial functions. Alternative approaches [114, 116] are directly dealing with the semi-global stabilizability of the nonlinear KdV equation.

Theorem 5.4.1 (see [114])**.** *Let $L > 0$ and $\omega = \omega(x)$ be defined by* (5.43)*. Then by setting $F(y) = -\omega y$ as a feedback control in* (5.42)*, the resulting closed loop system is semi-globally exponentially stable. That is, for every $r > 0$ there exist $c = c(r)$ and $\mu = \mu(r)$ such that*

$$\|y(t)\|^2_{L^2(0,L)} \leq c\|y_0\|^2_{L^2(0,L)} e^{-\mu t}$$

holds for all $t > 0$ and any initial function $y_0 \in L^2(0, L)$ with $\|y_0\|_{L^2(0,L)} \leq r$.

The following estimates will be used later.

Lemma 5.4.1. *Consider the controlled system* (5.2)*. Then for every control $u \in L^2(0, T; L^2(\hat{\Omega}))$ and $y_0 \in L^2(0, L)$ we have the following estimate*

$$\begin{aligned} \|y(t)\|^2_{L^2(0,L)} &\leq \|y_0\|^2_{L^2(0,L)} \\ &\quad + \int_0^t \|y(s)\|^2_{L^2(0,L)} ds + \int_0^t \|u(s)\|^2_{L^2(\hat{\Omega})} ds \quad \text{for all } t \in [0, T]. \end{aligned} \tag{5.44}$$

Moreover for every $\delta \in (0, T]$ we have

$$\|y(\delta)\|^2_{L^2(0,L)} \leq c_\delta \int_0^\delta \left(\|y(s)\|^2_{L^2(0,L)} + \|u(s)\|^2_{L^2(\hat{\Omega})} \right) ds, \tag{5.45}$$

where the constant c_δ depends only on δ.

Proof. Assume that $q \in C^\infty([0,T]\times[0,L])$ and that the solution y of (5.2) is regular enough to justify the following computations. Multiplying both sides of the equation by yq and integrating over $(0,t)\times(0,L)$ for an arbitrary $t\in[0,T]$ we obtain

$$\int_0^t\int_0^L qy(\partial_t y+\partial_x y+\partial_x^3 y+y\partial_x y-Bu)dxds=0.$$

Integration by parts and use of the boundary conditions implies that

$$\begin{aligned}&-\int_0^t\int_0^L(\partial_t q+\partial_x q+\partial_x^3 q)\frac{y^2}{2}dxds-\frac{1}{3}\int_0^t\int_0^L y^3\partial_x q\,dxds+\int_0^L(q\frac{y^2}{2})(t,x)\,dx\\&-\int_0^L q(0,x)\frac{y_0^2(x)}{2}\,dx+\frac{3}{2}\int_0^t\int_0^L\partial_x q(\partial_x y)^2\,dxds+\int_0^t(q\frac{(\partial_x y)^2}{2})(s,0)\,ds\\&-\int_0^t\int_0^L yqBu\,dxds=0.\end{aligned}\tag{5.46}$$

For the choice $q:=1$, we obtain

$$\begin{aligned}\|y(t)\|^2_{L^2(0,L)}+&\int_0^t(\partial_x y)^2(s,0)\,ds\\&=\|y_0\|^2_{L^2(0,L)}+2\int_0^t\langle y(s),u(s)\rangle_{L^2(0,L)}\,ds\\&\le\|y_0\|^2_{L^2(0,L)}+2\int_0^t\|y(t)\|_{L^2(0,L)}\|u(t)\|_{L^2(\hat\Omega)}\,ds\\&\le\|y_0\|^2_{L^2(0,L)}+\int_0^t\|y(s)\|^2_{L^2(0,L)}\,ds+\int_0^t\|u(s)\|^2_{L^2(\hat\Omega)}ds.\end{aligned}\tag{5.47}$$

By a density argument we obtain (5.44).

Turning to inequality (5.45), by choosing $t=\delta$, $q:=\delta-s$ with $s\in(0,\delta)$ for a fixed $\delta\in(0,T]$ in (5.46) we obtain

$$\begin{aligned}&\frac{1}{2}\delta\|y_0\|^2_{L^2(0,L)}\\&=\frac{1}{2}\int_0^\delta\|y(s)\|^2_{L^2(0,L)}\,ds+\frac{1}{2}\int_0^\delta(\delta-s)(\partial_x y)^2(s,0)\,ds-\int_0^\delta\int_0^L(\delta-s)yBu\,dxds\\&\le\frac{1}{2}\int_0^\delta\|y(s)\|^2_{L^2(0,L)}\,ds+\frac{\delta}{2}\int_0^\delta(\partial_x y)^2(s,0)\,ds+\delta\int_0^\delta|\langle y(s),Bu(s)\rangle_{L^2}|\,ds\\&\le\frac{1}{2}\int_0^\delta\|y(s)\|^2_{L^2(0,L)}\,ds+\frac{\delta}{2}\int_0^\delta(\partial_x y)^2(s,0)\,ds\\&+\frac{\delta}{2}\int_0^\delta\left(\|y(s)\|^2_{L^2(0,L)}+\|u(s)\|^2_{L^2(\hat\Omega)}\right)ds\\&\le\frac{\delta}{2}\int_0^\delta(\partial_x y)^2(s,0)\,ds+\frac{\delta+1}{2}\int_0^\delta\left(\|y(s)\|^2_{L^2(0,L)}+\|u(s)\|^2_{L^2(\hat\Omega)}\right)ds,\end{aligned}$$

and as consequence we can write

$$-\int_0^\delta (\partial_x y(s,0))^2\,ds \\ \le -\|y_0\|^2_{L^2(0,L)} + \frac{\delta+1}{\delta}\int_0^\delta \left(\|y(s)\|^2_{L^2(0,L)} + \|u(s)\|^2_{L^2(\hat{\Omega})}\right) ds. \tag{5.48}$$

Moreover, by using (5.47) for $t=\delta$ we infer that

$$\|y(\delta)\|^2_{L^2(0,L)} \le \|y_0\|^2_{L^2(0,L)} - \int_0^\delta (\partial_x y(s,0))^2\,ds \\ + \int_0^\delta \left(\|y(s)\|^2_{L^2(0,L)}\,ds + \|u(s)\|^2_{L^2(\hat{\Omega})}\right) ds. \tag{5.49}$$

By combining (5.48) and (5.49), we have

$$\|y(\delta)\|^2_{L^2(0,L)} \le \frac{2\delta+1}{\delta}\int_0^\delta \left(\|y(s)\|^2_{L^2(0,L)} + \|u(s)\|^2_{L^2(\hat{\Omega})}\right) ds,$$

and with $c_\delta := \frac{2\delta+1}{\delta}$, we conclude the proof. □

Definition 5.4.1. *For any $y_0 \in L^2(0,L)$ the infinite horizon value function $V_\infty(\cdot)$ is defined as the extended real valued function*

$$V_\infty(y_0) := \inf_{u\in L^2(0,\infty;L^2(\hat{\Omega}))} \{J_\infty(u,y_0) \text{ subject to } (5.2)\}.$$

Similarly, the finite horizon value function $V_T(\cdot)$ is defined by

$$V_T(y_0) := \min_{u\in L^2(0,T;L^2(\hat{\Omega}))} \{J_T(u,y_0) \text{ subject to } (5.2)\}.$$

From this point forward, $\mathcal{B}_r(0)$ denotes a ball in $L^2(0,L)$ centered at 0 with radius r and we define $\alpha_\ell := \frac{\min\{\beta,1\}}{2}$. Furthermore, the pair $(y^*_T(\cdot;y_0,t_0), u^*_T(\cdot;y_0,t_0))$ stands for an optimal solution to the problem (OP) with finite time horizon T, and initial function y_0 at initial time t_0. In the following the function

$$\gamma(T,r) := \frac{(1+\beta)c(r)}{2\mu(r)}(1-e^{-\mu(r)T})$$

with $c(r)$ and $\mu(r)$ from Theorem 5.4.1 will be of significance. For every $r>0$, it is nondecreasing, continuous, and bounded function in the variable T.

Lemma 5.4.2. *Let a positive number r be given. Then for every $y_0 \in \mathcal{B}_r(0) \subset L^2(0,L)$ and $T>0$, there exists a control $\hat{u}(\cdot;y_0) \in L^2(0,T;L^2(\hat{\Omega}))$ such that*

$$V_T(y_0) \le J_T(\hat{u},y_0) \le \gamma(T,r)\|y_0\|^2_{L^2(0,L)}. \tag{5.50}$$

Proof. Assume that positive numbers r, T, and $y_0 \in \mathcal{B}_r(0)$ are given. By setting $u(t) := -y(t)|_{\hat{\Omega}}$ in the controlled system (5.2), and using Theorem 5.4.1 for the choice

$$\omega(x) := \begin{cases} 1 & x \in \hat{\Omega}, \\ 0 & \textit{otherwise}. \end{cases}$$

we obtain

$$\|y(t)\|^2_{L^2(0,L)} \leq c(r)\|y_0\|^2_{L^2(0,L)} e^{-\mu(r)t} \quad \text{for all } t \in [0,T].$$

Here the constants $c(r)$ and $\mu(r)$ were defined in Theorem 5.4.1. By integrating from 0 to T we have

$$\int_0^T \|y(t)\|^2_{L^2(0,L)} dt \leq \frac{c(r)}{\mu(r)}(1 - e^{-\mu(r)T})\|y_0\|^2_{L^2(0,L)}. \tag{5.51}$$

By the definition of value function $V_T(\cdot)$ and (5.3) we have

$$\begin{aligned} V_T(y_0) &\leq \int_0^T \frac{1}{2}\|y(t)\|^2_{L^2(0,L)} + \frac{\beta}{2}\|y(t)\|^2_{L^2(\hat{\Omega})} dt \\ &\leq \frac{(1+\beta)c(r)}{2\mu(r)}(1 - e^{-\mu(r)T})\|y_0\|^2_{L^2(0,L)} \\ &= \gamma(T, r)\|y_0\|^2_{L^2(0,L)}. \end{aligned}$$

□

Lemma 5.4.3. *Let $r_0 > 0$, $\delta > 0$, and $T > \delta$ be given. Then there exists a radius d_1 depending on r_0 such that for every $r \geq d_1(r_0)$ and $y_0 \in \mathcal{B}_{r_0}(0)$ the following inequities are satisfied*

$$\begin{aligned} V_T(y_T^*(\delta; y_0, 0)) \leq &\int_\delta^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \\ &+ \gamma(T + \delta - \tilde{t}, r)\|y_T^*(\tilde{t}; y_0, 0)\|^2_{L^2(0,L)} \quad \textit{for all } \tilde{t} \in [\delta, T], \end{aligned} \tag{5.52}$$

and

$$\int_{\tilde{t}}^T \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \leq \gamma(T - \tilde{t}, r)\|y_T^*(\tilde{t}; y_0, 0)\|^2_{L^2(0,L)} \quad \textit{for all } \tilde{t} \in [0, T]. \tag{5.53}$$

Proof. For every $y_0 \in L^2(0, L)$ and $\tilde{t} \in [0, T]$, due to (5.3) and Bellman's optimality principle we have

$$\begin{aligned} \alpha_\ell \int_0^{\tilde{t}} (\|y_T^*(t; y_0, 0)\|^2_{L^2(0,L)} + &\|u_T^*(t; y_0, 0)\|^2_{L^2(\hat{\Omega})})dt \\ &\leq \int_0^{\tilde{t}} \ell(y_T^*(t; y_0, 0), u_T^*(t; y_0, 0))dt \\ &= V_T(y_0) - V_{T-\tilde{t}}(y_T^*(\tilde{t}; y_0, 0)). \end{aligned}$$

Now by (5.44), (5.50) and the above inequality we have for $y_0 \in \mathcal{B}_{r_0}(0)$

$$\begin{aligned}\|y_T^*(\tilde t; y_0,0)\|^2_{L^2(0,L)} &\le \|y_0\|^2_{L^2(0,L)} + \int_0^{\tilde t} \|y_T^*(t;y_0,0)\|^2_{L^2(0,L)}dt + \int_0^{\tilde t} \|u_T^*(t;y_0,0)\|^2_{L^2(\hat\Omega)}dt \\ &\le \|y_0\|^2_{L^2(0,L)} + \frac{1}{\alpha_\ell}(V_T(y_0) - V_{T-\tilde t}(y_T^*(\tilde t; y_0,0)) \\ &\le \|y_0\|^2_{L^2(0,L)} + \frac{1}{\alpha_\ell}V_T(y_0) \le \big(1 + \frac{\gamma(T,r_0)}{\alpha_\ell}\big)r_0^2 \\ &\le \big(1 + \frac{(1+\beta)c(r_0)}{2\alpha_\ell \mu(r_0)}\big)r_0^2 =: d_1^2(r_0).\end{aligned}$$

Hence for the radius d_1 defined in the above inequality, we have

$$y_T^*(\tilde t; y_0, 0) \in \mathcal{B}_{d_1}(0) \qquad \text{for all } \tilde t \in [0,T].$$

We turn to the verification of (5.52). For simplicity of notation, we denote $y_T^*(\delta; y_0, 0)$ by $y^*(\delta)$. Then for every fixed $r \ge d_1$ we have $y_0 \in \mathcal{B}_r(0)$. Due to Bellman's optimality principle, we have for every $\tilde t \in [\delta, T]$

$$\begin{aligned}V_T(y^*(\delta)) &= \int_\delta^{T+\delta} \ell(y_T^*(t;y^*(\delta),\delta), u_T^*(t;y^*(\delta),\delta))dt \\ &= \int_\delta^{\tilde t} \ell(y_T^*(t;y^*(\delta),\delta), u_T^*(t;y^*(\delta),\delta))dt + V_{T+\delta-\tilde t}(y_T^*(\tilde t; y^*(\delta),\delta)).\end{aligned}$$

By optimality of $y_T^*(\cdot; y^*(\delta),\delta)$ as a solution on $[\delta, T+\delta]$ with initial state $y^*(\delta) \in \mathcal{B}_{d_1}(0) \subseteq \mathcal{B}_r(0)$ at $t=\delta$ we obtain

$$\begin{aligned}V_T(y^*(\delta)) &\le \int_\delta^{\tilde t} \ell(y_T^*(t;y_0,0), u_T^*(t;y_0,0))dt + V_{T+\delta-\tilde t}(y_T^*(\tilde t; y_0,0)) \\ &\le \int_\delta^{\tilde t} \ell(y_T^*(t;y_0,0), u_T^*(t;y_0,0))dt + \gamma(T+\delta-\tilde t, r)\|y_T^*(\tilde t;y_0,0)\|^2_{L^2(0,L)},\end{aligned}$$

where for the last inequality we used (5.50).

To prove the second inequality, let $\tilde t \in [0,T]$ be arbitrary. By Bellman's principle and (5.50), we have

$$\begin{aligned}&V_T(y_0) \\ &\quad= \int_0^{\tilde t} \ell(y_T^*(t;y_0,0), u_T^*(t;y_0,0))dt + \int_{\tilde t}^T \ell(y_T^*(t;y_0,0), u_T^*(t;y_0,0))dt \\ &\quad= \int_0^{\tilde t} \ell(y_T^*(t;y_0,0), u_T^*(t;y_0,0))dt + V_{T-\tilde t}(y_T^*(\tilde t;y_0,0)) \\ &\quad\le \int_0^{\tilde t} \ell(y_T^*(t;y_0,0), u_T^*(t;y_0,0))dt + \gamma(T-\tilde t, r)\|y_T^*(\tilde t;y_0,0)\|^2_{L^2(0,L)}.\end{aligned} \tag{5.54}$$

Therefore,

$$\int_{\tilde{t}}^{T} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \leq \gamma(T-\tilde{t},r)\|y_T^*(\tilde{t};y_0,0)\|_{L^2(0,L)}^2 \quad \text{for all } \tilde{t}\in[0,T],$$

as desired. □

Lemma 5.4.4. *Suppose that $r_0>0$, $\delta>0$, and $T>\delta$ are given. Then for every $r\geq d_1(r_0)$ with d_1 defined in Lemma 5.4.3, and the choices*

$$\theta_1(\delta,T,r):=1+\frac{\gamma(T,r)}{\alpha_\ell(T-\delta)}, \qquad \theta_2(\delta,T,r):=\frac{\gamma(T,r)}{\alpha_\ell\delta},$$

the estimates

$$V_T(y_T^*(\delta;y_0,0))\leq \theta_1\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt, \tag{5.55}$$

and

$$\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt\leq \theta_2\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \tag{5.56}$$

hold for every $y_0\in\mathcal{B}_{r_0}(0)$.

Proof. According to Lemma 5.4.3, the estimates (5.52) and (5.53) are satisfied for every $y_0\in\mathcal{B}_{r_0}(0)$ and $r\geq d_1(r_0)$.

We first verify inequality (5.55) for arbitrary initial function $y_0\in\mathcal{B}_{r_0}(0)$ and $r\geq d_1(r_0)$. Recall that for the solution of the KdV equation we have $y_T^*(\cdot;y_0,0)\in C([0,T];L^2(0,L))$. Hence there is a $\bar{t}\in[\delta,T]$ such that

$$\bar{t}=\arg\min_{t\in[\delta,T]}\|y_T^*(t;y_0,0)\|_{L^2(0,L)}^2.$$

By (5.52) we have

$$\begin{aligned}
&V_T(y_T^*(\delta;y_0,0))\\
&\leq \int_\delta^{\bar{t}} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt+\gamma(T+\delta-\bar{t},r)\|y_T^*(\bar{t};y_0,0)\|_{L^2(0,L)}^2\\
&\leq \int_\delta^{T} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt+\gamma(T,r)\|y_T^*(\bar{t};y_0,0)\|_{L^2(0,L)}^2\\
&\leq \int_\delta^{T} \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt+\frac{\gamma(T,r)}{T-\delta}\int_\delta^T \|y_T^*(t;y_0,0)\|_{L^2(0,L)}^2dt.
\end{aligned} \tag{5.57}$$

Furthermore, by (5.3)

$$\int_\delta^T \|y_T^*(t;y_0,0)\|_{L^2(0,L)}^2dt\leq\frac{1}{\alpha_\ell}\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt. \tag{5.58}$$

By (5.57) and (5.58) we have

$$V_T(y_T^*(\delta;y_0,0)) \leq (1+\frac{\gamma(T,r)}{\alpha_\ell(T-\delta)})\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt.$$

Turning to (5.56) we define

$$\hat{t} = \arg\min_{t\in[0,\delta]} \|y_T^*(t;y_0,0)\|^2_{L^2(0,L)}.$$

Then by (5.53) we have

$$\begin{aligned}
&\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\
&\leq \int_{\hat{t}}^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \leq \gamma(T-\hat{t},r)\|y_T^*(\hat{t};y_0,0)\|^2_{L^2(0,L)} \\
&\leq \gamma(T,r)\|y_T^*(\hat{t};y_0,0)\|^2_{L^2(0,L)} \leq \frac{\gamma(T,r)}{\delta}\int_0^\delta \|y_T^*(t;y_0,0)\|^2_{L^2(0,L)}dt,
\end{aligned} \tag{5.59}$$

and further

$$\frac{\gamma(T,r)}{\delta}\int_0^\delta \|y_T^*(t;y_0,0)\|^2_{L^2(0,L)}dt \leq \frac{\gamma(T,r)}{\alpha_\ell\delta}\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt. \tag{5.60}$$

By (5.59) and (5.60) we obtain the desired estimate

$$\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \leq \frac{\gamma(T,r)}{\alpha_\ell\delta}\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt.$$

□

Proposition 5.4.1. *Suppose that $r_0>0$ and $\delta>0$ are given. Then for every $r\geq d_1(r_0)$ with d_1 defined in Lemma 5.4.3, there exist positive numbers $T^*=T^*(r,\delta)>\delta$, and $\alpha=\alpha(r,\delta)\in(0,1)$ such that*

$$V_T(y_T^*(\delta;y_0,0)) \leq V_T(y_0) - \alpha\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt, \tag{5.61}$$

for every $T\geq T^*$ and $y_0\in\mathcal{B}_{r_0}(0)$.

Proof. From the definition of $V_T(y_0)$ and Lemma 5.4.4, we have for every $r\geq d_1$

$$\begin{aligned}
V_T(y_T^*(\delta;y_0,0)) - V_T(y_0) &= V_T(y_T^*(\delta;y_0,0)) - \int_0^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\
&\leq (\theta_1-1)\int_\delta^T \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt - \int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt \\
&\leq (\theta_2(\theta_1-1)-1)\int_0^\delta \ell(y_T^*(t;y_0,0),u_T^*(t;y_0,0))dt,
\end{aligned}$$

where θ_1 and θ_2 are defined in Lemma 5.4.4. Since

$$1 - \theta_2(\theta_1 - 1) = 1 - \frac{\gamma^2(T, r)}{\alpha_\ell^2 \delta(T - \delta)},$$

and

$$\frac{\gamma^2(T, r)}{\alpha_\ell^2 \delta(T - \delta)} \to 0 \text{ as } T \to \infty,$$

there exist $T^* > \delta$ and $\alpha \in (0, 1)$ such that $1 - \theta_2(\theta_1 - 1) \geq \alpha$ for all $T \geq T^*$. This implies (5.61). □

Theorem 5.4.2 (Suboptimality). *Let $y_0 \in L^2(0, L)$ and a sampling time $\delta > 0$ be given. Then there exist numbers, $T^* = T^*(\|y_0\|_{L^2(0,L)}, \delta) > \delta$, and $\alpha = \alpha(\|y_0\|_{L^2(0,L)}, \delta) \in (0, 1)$, such that for every fixed prediction horizon $T \geq T^*$, the Receding horizon control u_{rh} obtained from Algorithm 5.1 satisfies*

$$\alpha V_\infty(y_0) \leq \alpha J_\infty(u_{rh}, y_0) \leq V_T(y_0) \leq V_\infty(y_0). \tag{5.62}$$

Proof. The right and left inequalities are obvious, therefore we only need to verify the middle one.

First we show that $V_T(y_0)$ is bounded by a constant r_{y_0} independent of T. We reconsider the proof of Lemma 5.4.2 to find

$$V_T(y_0) \leq \gamma(T, \|y_0\|_{L^2(0,L)})\|y_0\|^2_{L^2(0,L)} \leq \frac{(1+\beta)c(\|y_0\|_{L^2(0,L)})}{2\alpha_\ell \mu(\|y_0\|_{L^2(0,L)})}\|y_0\|^2_{L^2(0,L)} =: r^2_{y_0}. \tag{5.63}$$

Next we define the radius

$$r_0 := \max\{\|y_0\|_{L^2(0,L)}, \sqrt{\frac{c_\delta}{\alpha_\ell} r^2_{y_0}}\}, \tag{5.64}$$

where the constant c_δ, defined in Lemma 5.4.1, depends only on δ.

For $d_1(r_0)$ defined as in Lemma 5.4.3, due to Proposition 5.4.1, there exist positive numbers $T^* = T^*(d_1, \delta) > \delta$, and $\alpha = \alpha(d_1, \delta) \in (0, 1)$ such that the inequality (5.61) holds for every $T \geq T^*$ and $y_0 \in \mathcal{B}_{r_0}(0)$. Therefore, in order to use the dissipative inequality (5.61) for every optimal solution pair $(y^*_T(\cdot; y_{rh}(t_k), t_k), u^*_T(\cdot; y_{rh}(t_k), t_k))$ of Algorithm 5.1, we need to be sure, a priori, that

$$y_{rh}(t_k) \in \mathcal{B}_{r_0}(0) \quad \text{for every } k \in \mathbb{N}_0. \tag{5.65}$$

We proceed by induction with respect to the sampling index k. For every $k \in \mathbb{N}_0$ we will show that the inequality

$$V_T(y_{rh}(t_k)) \leq V_T(y_0) - \alpha \int_0^{t_k} \ell(y_{rh}(t), u_{rh}(t))dt,$$

and condition (5.65) hold true.

First, since $\|y_0\|_{L^2(0,L)} \le r_0$, by Proposition 5.4.1 for every fixed $T \ge T^*(d_1,\delta)$ we have

$$V_T(y_{rh}(t_1)) \le V_T(y_0) - \alpha \int_0^{t_1} \ell(y_{rh}(t), u_{rh}(t))dt, \tag{5.66}$$

with an $\alpha = \alpha(d_1,\delta) \in (0,1)$. Moreover by using estimate (5.45) we can infer that

$$\begin{aligned} \|y_{rh}(t_1)\|^2_{L^2(0,L)} &\overset{(5.45)}{\le} c_\delta\Big(\int_0^{t_1} \|y_{rh}(t)\|^2_{L^2(0,L)} + \|u_{rh}(t)\|^2_{L^2(\Omega)}\Big)\,dt \\ &\overset{(5.3)}{\le} \frac{c_\delta}{\alpha_\ell}\int_0^{t_1} \ell(y_{rh}(t), u_{rh}(t))\,dt \le \frac{c_\delta}{\alpha_\ell} V_T(y_0) \overset{(5.63)}{\le} \frac{c_\delta}{\alpha_\ell} r^2_{y_0} \overset{(5.64)}{\le} r_0^2. \end{aligned} \tag{5.67}$$

Now to carry out the induction step, we assume that

$$y_{rh}(t_k) \in \mathcal{B}_{r_0}(0) \quad \text{for all } k = 0,\dots,k', \tag{5.68}$$

and that

$$V_T(y_{rh}(t_{k'})) \le V_T(y_0) - \alpha \int_0^{t_{k'}} \ell(y_{rh}(t), u_{rh}(t))dt \tag{5.69}$$

for $k' \in \mathbb{N}$.

Since $y_{rh}(t_{k'}) \in \mathcal{B}_{r_0}(0)$, by Proposition 5.4.1 we have

$$V_T(y_{rh}(t_{k'+1})) \le V_T(y_{rh}(t_{k'})) - \alpha \int_{t_{k'}}^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt.$$

Combined with (5.69) this gives

$$V_T(y_{rh}(t_{k'+1})) \le V_T(y_0) - \alpha \int_0^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt.$$

Moreover, by the same argument as in (5.67) we obtain

$$\begin{aligned} \|y_{rh}(t_{k'+1})\|^2_{L^2(0,L)} &\overset{(5.45)}{\le} c_\delta \int_{t_{k'}}^{t_{k'+1}} (\|y_{rh}(t)\|^2_{L^2(0,L)} + \|u_{rh}(t)\|^2_{L^2(\hat{\Omega})})dt \\ &\overset{(5.3)}{\le} \frac{c_\delta}{\alpha_\ell} \int_{t_{k'}}^{t_{k'+1}} \ell(y_{rh}(t), u_{rh}(t))dt \\ &\le \frac{c_\delta}{\alpha_\ell} V_T(y_{rh}(t_{k'})) \overset{(5.69)}{\le} \frac{c_\delta}{\alpha_\ell} V_T(y_0) \le \frac{c_\delta}{\alpha_\ell} r^2_{y_0} \le r_0^2. \end{aligned}$$

Hence $y_{rh}(t_{k'+1}) \in \mathcal{B}_{r_0}(0)$, which concludes the induction step. Taking the limit $k' \to \infty$ we find

$$\alpha J_\infty(u_{rh}(\cdot), y_0) = \alpha \int_0^\infty \ell(y_{rh}(t), u_{rh}(t))dt \le V_T(y_0),$$

which concludes the proof. Note that the constants α and T^* depend only on δ and $\|y_0\|_{L^2(0,L)}$. □

Theorem 5.4.3 (Exponential decay)**.** *Suppose that $y_0 \in L^2(0,L)$ and let a sampling time $\delta > 0$ be given. Then there exist numbers $T^*(\|y_0\|_{L^2(0,L)}, \delta) > \delta$, $\alpha(\|y_0\|_{L^2(0,L)}, \delta) \in (0,1)$ such that for every prediction horizon $T \geq T^*$, the receding horizon trajectory $y_{rh}(\cdot)$ satisfies*

$$V_T(y_{rh}(t_k)) \leq e^{-\zeta t_k} V_T(y_0), \tag{5.70}$$

where ζ is a positive number depending on y_0, δ, and T. Moreover, for every positive t we have

$$\|y_{rh}(t)\|^2_{L^2(0,L)} \leq c e^{-\zeta t} \|y_0\|^2_{L^2(0,L)} \tag{5.71}$$

with a positive constant c depending on y_0, δ, and T.

Proof. Let $y_0 \in L^2(0,L)$ and $\delta > 0$ be given. Then according to Theorem 5.4.2, there exist positive numbers $T^*(\|y_0\|_{L^2(0,L)}, \delta)$ and $\alpha(\|y_0\|_{L^2(0,L)}, \delta)$ such that for every $T \geq T^*$, we have

$$y_{rh}(t_k) \in \mathcal{B}_{r_0}(0) \quad \text{for all } k \in \mathbb{N}_0,$$

where $r_0(\|y_0\|_{L^2(0,L)})$ has been defined in Theorem 5.4.2 by (5.64), and

$$V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \leq -\alpha \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))dt \quad \text{for every } k \in \mathbb{N}. \tag{5.72}$$

By using (5.55) and (5.56) we have

$$\begin{aligned} V_T(y_{rh}(t_{k+1})) \leq& \theta_1 \int_{t_{k+1}}^{t_k+T} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))\, dt \\ \leq& \theta_1\theta_2 \int_{t_k}^{t_{k+1}} \ell(y_T^*(t; y_{rh}(t_k), t_k), u_T^*(t; y_{rh}(t_k), t_k))\, dt \\ =& \theta_1\theta_2 \int_{t_k}^{t_{k+1}} \ell(y_{rh}(t), u_{rh}(t))\, dt, \end{aligned} \tag{5.73}$$

where $\theta_1 = \theta_1(\delta, T, d_1(r_0)) > 0$ and $\theta_2 = \theta_2(\delta, T, d_1(r_0)) > 0$ are defined in the statement of Lemma 5.4.4 and $d_1 = d_1(r_0)$ is introduced by Lemma 5.4.3. Now by combining (5.72) and (5.73) we obtain

$$V_T(y_{rh}(t_{k+1})) - V_T(y_{rh}(t_k)) \leq \frac{-\alpha}{\theta_1\theta_2} V_T(y_{rh}(t_{k+1})) \quad \text{for every } k \in \mathbb{N}.$$

Therefore, by defining $\eta := (1 + \frac{\alpha}{\theta_1\theta_2})^{-1}$ for every $k \in \mathbb{N}$ we can write

$$V_T(y_{rh}(t_k)) \leq \eta V_T(y_{rh}(t_{k-1})) \leq \eta^2 V_T(y_{rh}(t_{k-2})) \leq \cdots \leq \eta^k V_T(y_0). \tag{5.74}$$

Defining $\zeta := \frac{|\ln \eta|}{\delta}$, we obtain inequality (5.70).

Turning to inequality (5.71), let $t > 0$ be arbitrary. Then there exists an index k such that $t \in [t_k, t_{k+1}]$. By using estimate (5.44) for the initial function $y_{rh}(t_k)$, we have for $t \in [t_k, t_{k+1}]$

$$\begin{aligned} \|y_{rh}(t)\|^2_{L^2(0,L)} &\le \|y_{rh}(t_k)\|^2_{L^2(0,L)} + \int_{t_k}^{t} \left(\|y_{rh}(s)\|^2_{L^2(0,L)} + \|u_{rh}(s)\|^2_{L^2(\hat{\Omega})}\right) ds \\ &\le \|y_{rh}(t_k)\|^2_{L^2(0,L)} + \frac{1}{\alpha_\ell}\int_{t_k}^{t} \ell(y_{rh}(s), u_{rh}(s))\, ds \\ &\le \|y_{rh}(t_k)\|^2_{L^2(0,L)} + \frac{1}{\alpha_\ell} V_T(y_{rh}(t_k)). \end{aligned} \tag{5.75}$$

Moreover, by using estimate (5.45) we infer that

$$\begin{aligned} \|y_{rh}(t_k)\|^2_{L^2(0,L)} &\le c_\delta \int_{t_{k-1}}^{t_k} (\|y_{rh}(t)\|^2_{L^2(0,L)} + \|u_{rh}(t)\|^2_{L^2(\hat{\Omega})}) dt \\ &\le \frac{c_\delta}{\alpha_\ell}\int_{t_{k-1}}^{t_k} \ell(y_{rh}(t), u_{rh}(t)) dt \le \frac{c_\delta}{\alpha_\ell} V_T(y_{rh}(t_{k-1})). \end{aligned} \tag{5.76}$$

By using (5.74), (5.75) and (5.76) we obtain for $t \in [t_k, t_{k+1}]$

$$\begin{aligned} \|y_{rh}(t)\|^2_{L^2(0,L)} &\le \|y_{rh}(t_k)\|^2_{L^2(0,L)} + \frac{1}{\alpha_\ell} V_T(y_{rh}(t_k)) \le \frac{1+c_\delta}{\alpha_\ell} V_T(y_{rh}(t_{k-1})) \\ &\le \frac{(1+c_\delta)\eta^{k+1}}{\alpha_\ell \eta^2} V_T(y_0) \le \frac{1+c_\delta}{\alpha_\ell\eta^2} e^{-\zeta t_{k+1}} V_T(y_0) \\ &\le \frac{1+c_\delta}{\alpha_\ell\eta^2} e^{-\zeta t} V_T(y_0) \le \frac{(1+c_\delta)\gamma(T, \|y_0\|_{L^2(0,L)})}{\alpha_\ell\eta^2} e^{-\zeta t}\|y_0\|^2_{L^2(0,L)}. \end{aligned}$$

Setting $c := \frac{(1+c_\delta)\gamma(T,\|y_0\|_{L^2(0,L)})}{\alpha_\ell\eta^2}$, we conclude the proof. □

5.5 Discretization and numerical results

This section is devoted to illustrating the receding horizon technique for stabilizing the KdV equaiton. We describe the discretization of the optimization problem (5.1)-(5.2), as well as the numerical optimization process we use. In the case of bounded domains numerous schemes for solving the nonlinear KdV equation are available including finite differences [51, 141], finite elements [6, 138], finite volumes [52], discontinuous Galerkin schemes [24, 140], or polynomial spectral methods [102, 103, 127]. Spectral discretizations present interesting advantages regarding precision and simulation speed compared to any finite difference or finite element method [29].

5.5.1 Discretization

One of the most recent and efficient numerical methods for solving the Korteweg-de Vries equation with Dirichlet boundary conditions is proposed in [102]. The linear term is treated by a Petrov-Galerkin method based on Legendre polynomials, while the nonlinear term is treated pseudospectrally on the Chebyschev collocation points. Shortly after, Shen [127] proposed an improvement of this Petrov-Galerkin method with nearly optimal computational complexity. This will be our method of choice and we briefly recall it here.

The dual Petrov-Galerkin method

The test and trial function bases are chosen as a compact combination of Legendre polynomials in such a way that the trial functions satisfy the underlying boundary conditions of the primal equation and the test functions satisfy the boundary conditions as defined in (5.5). As a consequence, all matrices involved in the resolution of the problem are sparse [127]. We present the method for the reference domain $\Omega := (-1, 1)$, but it can be extended to any other domain of the type (a, b) by scaling the Legendre polynomials and the integrals. We denote by P_N the space of polynomials of degree $\leq N$ and set

$$V_N = \{y \in P_N : y(1) = y(-1) = \partial_x y(1) = 0\},$$
$$V_N^* = \{y \in P_N : y(1) = y(-1) = \partial_x y(-1) = 0\}.$$

Then for $T > 0$, we consider the semi-discrete problem: find

$$y_N : [0, T] \to V_N, \quad t \mapsto y_N(t, \cdot),$$

such that for almost every $t \in [0, T]$

$$\langle \partial_t y_N, \varphi_N \rangle + (\partial_x y_N, \varphi_N) + (\partial_x y_N, \partial_{xx} \varphi_N) - \left(\frac{y_N^2}{2}, \partial_x \varphi_N \right) = \left(\chi_{\hat{\Omega}} u, \varphi_N \right) \quad \forall v_N \in V_N^*, \tag{5.77}$$

where $(\cdot, \cdot)$ denotes the usual $L^2(\Omega)$ spatial inner product, $\langle \cdot, \cdot \rangle$ is the spatial duality pairing between $H^{-2}(\Omega)$ and $H_0^2(\Omega)$, and $\hat{\Omega} \subseteq \Omega$ is the control domain, as in the continuous case.

Denoting by L_k the kth Legendre polynomial, the basis functions are defined as follows (see Figure 5.1)

$$\phi_k(x) = L_k(x) - \frac{2k+3}{2k+5} L_{k+1}(x) - L_{k+2}(x) + \frac{2k+3}{2k+5} L_{k+3}(x),$$
$$\psi_k(x) = L_k(x) + \frac{2k+3}{2k+5} L_{k+1}(x) - L_{k+2}(x) - \frac{2k+3}{2k+5} L_{k+3}(x).$$

Thus for $N \geq 3$, we have

$$V_N = \operatorname{span}\{\phi_0, \phi_1, \dots, \phi_{N-3}\}, \quad V_N^* = \operatorname{span}\{\psi_0, \psi_1, \dots, \psi_{N-3}\}.$$

The semi-discrete state variable $y_N(t, \cdot)$ on the spectral space is given in vector representation as

$$y_N(t, \cdot) = \sum_{k=0}^{N-3} \hat{y}_k(t) \phi_k(\cdot), \quad \mathbf{y}(t) = (\hat{y}_0(t), \hat{y}_1(t), \dots, \hat{y}_{N-3}(t))^T.$$

Analogously the vector representation of the control is given by:

$$\mathbf{u}(t) = \left((u_N(t, \cdot), \psi_0(\cdot)), (u_N(t, \cdot), \psi_1(\cdot)), \dots, (u_N(t, \cdot), \psi_{N-3}(\cdot)) \right)^T, \tag{5.78}$$

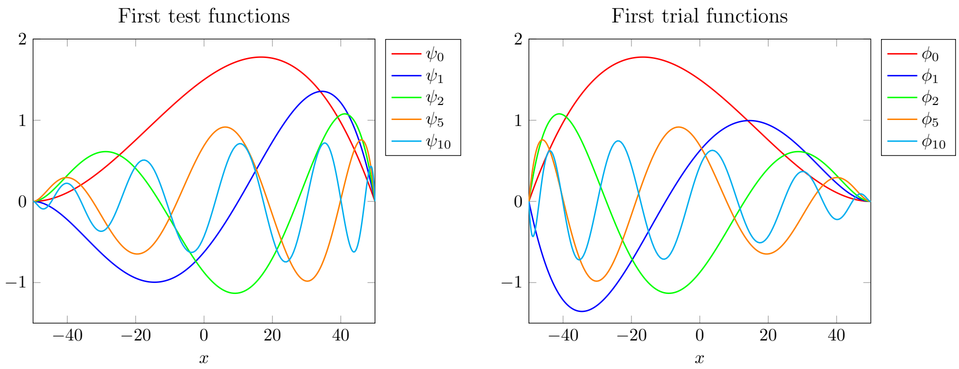


Figure 5.1: First test and trial functions.

where the expression for the semi-discrete control $u_N(t,\cdot)$ is given in Section 5.5.1. Afterwards, one builds the matrices $\mathbf{M}$, $\mathbf{P}$, and $\mathbf{S}$ of size $(N-2)\times(N-2)$ with coefficients m_{ij}, p_{ij}, q_{ij}, and s_{ij} defined as follows:

$$m_{ij} = (\phi_j, \psi_i), \quad p_{ij} = (\partial_x \phi_j, \psi_i), \quad s_{ij} = (\partial_x \phi_j, \partial_{xx}\psi_i). \tag{5.79}$$

The variational formulation (5.77) thus yields

$$\mathbf{M}\frac{d\mathbf{y}}{dt} + (\mathbf{P}+\mathbf{S})\,\mathbf{y} + F(\mathbf{y}) = \mathbf{B}\mathbf{u}, \tag{5.80}$$

where $\mathbf{B}$ is the matrix representing the characteristic function $\chi_{\hat{\Omega}}$ in (5.77) and $F(\mathbf{y})$ represents the nonlinear term. It is approximated as suggested in [127] using the pseudospectral approach. Thus the nonlinearity is evaluated at the chosen Chebyshev-Gauss-Lobatto (CGL) points in the spatial domain and then it is transformed back to the Legendre spectral space in the efficient manner.

Discretization of the control

The control is discretized in space with piecewise linear, continuous finite elements on a grid whose nodes are the Chebyschev-Gauss-Lobatto points $(x_n), n = 0,\dots,N$ as previously mentioned. The various norms involved in the optimization problem are computed using the trapezoidal rule for the evaluation of the spatial integrals for each cell. Thus, $u_N = \sum_{j=1}^{N_T} \chi_{I_j} \sum_{n=0}^{N} \hat{\mathbf{u}}_{jn} e_n$, where $I_j = (\delta_{j-1}^t, \delta_j^t]$ is the j^{th} time interval corresponding to the grids $0 = \delta_0^t < \delta_1^t < \cdots < \delta_{N_T}^t = T$. Moreover e_n is the basis vector for piecewise linear, continuous finite elements centered at the grid point x_n. Then for u_N it holds that

$$\|u_N\|^2_{L^2(0,T;L^2(\hat{\Omega}))} = \sum_{j=1}^{N_T} \Delta t \left(\sum_{n=0}^{N} d_n \hat{\mathbf{u}}_{nj}^2 \right), \quad (u_N, \psi) = \sum_{j=1}^{N_T} \chi_{I_j} \sum_{n=0}^{N} d_n \hat{\mathbf{u}}_{jn} \psi_n \tag{5.81}$$

for all spectral basis test functions ψ where we have denoted $\psi_n = \psi(x_n)$, and $d_n = \int_{\hat{\Omega}} e_n \,\mathrm{d}x$.

Time-stepping scheme

Following the idea in [27, 102, 103, 127], we use the multistep Crank-Nicolson Leap Frog scheme. In this setting, the third derivative is treated implicitely and the nonlinear term is treated explicitely. This allows to circumvent possible step size restrictions due to the third order derivative. In addtion, since the nonlinear term is treated explicitly, there is no need to solve a nonlinear system of equations at every time step. A proper derivation of the discrete adjoint and gradient is available in [27].

5.5.2 Numerical examples

In this section we present numerical experiments. They are based on Algorithm 5.2 that takes as initial input the time horizon T_∞ and an initial condition $y_0 \in L^2(\Omega)$.

Algorithm 5.2 Receding Horizon Control(y_0, T_∞)

1: Choose a prediction horizon $T < T_\infty$ and a sampling time $\delta \in (0, T]$.

2: Consider a grid $0 = t_0 < t_1 < \cdots < t_r = T_\infty$ on the interval $[0, T_\infty]$ where $t_i = i\delta$ for $i = 0, \ldots, r$.

3: **for** $i = 0, \ldots, r-1$ **do**

Solve the open-loop subproblem on $[t_i, t_i + T]$

$$\min \frac{1}{2} \int_{t_i}^{t_i+T} \|y(t)\|^2_{L^2(\Omega)} dt + \frac{\beta}{2} \int_{t_i}^{t_i+T} \|u(t)\|^2_{L^2(\hat{\Omega})} dt$$

subject to the Korteweg-de Vries equation (5.2) for the initial condition

$$y(t_i) = y_T^*(t_i) \text{ if } i \geq 1 \text{ or } y(t_i) = y_0 \text{ if } i = 0,$$

where $y_T^*(\cdot)$ is the solution to previous subproblem on $[t_{i-1}, t_{i-1} + T]$.

4: The receding horizon pair $(y_{rh}^*(\cdot), u_{rh}^*(\cdot))$ is the concatenation of the optimal pairs $(y_T^*(\cdot), u_T^*(\cdot))$ on the finite horizon intervals $[t_i, t_{i+1}]$ with $i = 0, \ldots, r-1$.

Each open-loop problem is solved with the help of Barzilai-Browein gradient steps [21] improved by a nonmonotonous line-search method [49]. Moreover we consider the following quantities in order to interpret the results of the stabilization problem for different settings:

1. $J_{T_\infty}(u_{rh}, y_0) := \frac{1}{2} \int_0^{T_\infty} \|y_{rh}(t)\|^2_{L^2(\Omega)} dt + \frac{\beta}{2} \int_0^{T_\infty} \|u_{rh}(t)\|^2_{L^2(\hat{\Omega})} dt,$

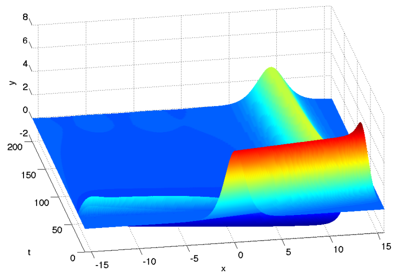
2. $\|y_{rh}\|_{L^2(Q)}$ with $Q := (0, T_\infty) \times \Omega$,

3. $\|y_{rh}(T_\infty)\|_{L^2(\Omega)}$,

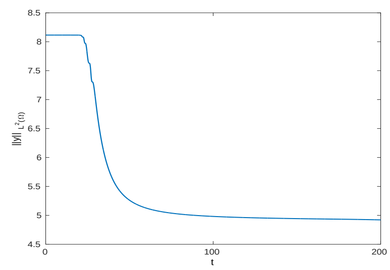
4. iter : the *total* number of iterations (BB-gradient steps) that the optimizer needs for all open-loop problems on the intervals $(t_i, t_i + T)$ for $i = 0, \ldots, r-1$.

Turning to the description of the numerical experiment that we carried out, we first recall that it is not known whether the system can be stabilized to zero without control, due to the fact that $\partial_x y(t,0)$ might be zero for a domain of the critical length [46, 122].

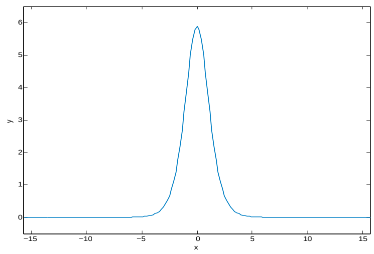
Here, we propose a situation where a soliton starts travelling at time $t=0$ (its initial shape is given in Figure 5.2(c)). On an infinite domain, a soliton is a solitary wave that travels at constant speed without losing its shape. This phenomenon is a result of the balance between nonlinearity and dispersion which typically occurs for the Korteweg-de Vries equation [44, 83]. In our case though, the initial soliton encounters the right boundary. Then its balance is broken and due to the dispersive effect, it is decomposed into several smaller reflected waves. See Figure 5.2(a). One of them evolves almost into a stationary one, while the other one travels at constant speed without hitting the boundaries. This is depicted in Figure 5.2(b) over a long period of time. In this case, i.e. without any control, the objective functional has the value $J_{T_\infty} = 3152.8$, whilst $\|y\|_{L^2(Q)} = 79.4$ and more importantly at the final time, $\|y(T_\infty)\|_{L^2(\Omega)} = 4.9$.



(a) State



(b) Evolution of $L^2(0,1)$-norm of state



(c) State at $t=0$

Figure 5.2: Uncontrolled solution

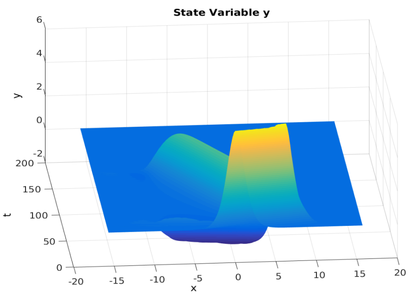
As a very large time horizon is considered, this would be prohibitive to apply the classical open-loop control on problem (5.1)-(5.2). Hence, the use of RHC is key for stabilization. Our simulations are carried out with the choice of: $\Omega = (-10\pi, 10\pi)$, $N = 256$, $\beta = 10^{-1}$, $\delta = 1$, $T_\infty = 200$, and various prediction horizons $T = 1, 1.5, 2$, $y_0 = 12\kappa^2 \operatorname{sech}^2(\kappa(x - x_0))$ with $\kappa = 0.7$, and $x_0 = 0.0$. Finally, the control domain consists of two components and is given by

$$\hat{\Omega} := (-15.24, -8.00) \cup (7.74, 15.14).$$

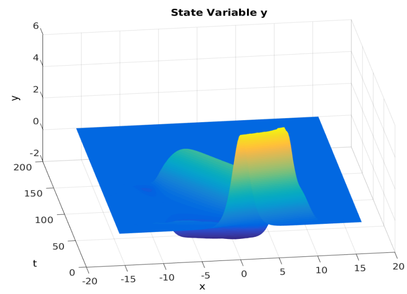
The results are gathered in Table 5.1 and Figure 5.3 - 5.6. In all three cases, the stabilizing measures are satisfying. As expected, the prediction horizon T plays an important role. The smaller it is (i.e. the closer to the sampling time δ), the fewer iterations are required (1098 for $T = 1$ versus 1598 for $T = 2$). However, one can observe from Figure 5.3 and Figure 5.4 - Figure 5.6, and it is verified by Table 5.1, that a smaller time horizon leads to a less efficient, and slower stabilization.

Prediction Horizon	J_{T_∞}	$\|y_{rh}\|_{L^2(Q)}$	$\|y_{rh}(T_\infty)\|_{L^2(\Omega)}$	iter
$T = 1.0$	1366	52.2	1.4×10^{-5}	1098
$T = 1.5$	1051	45.7	6.4×10^{-6}	1386
$T = 2.0$	728	37.8	4.1×10^{-6}	1598

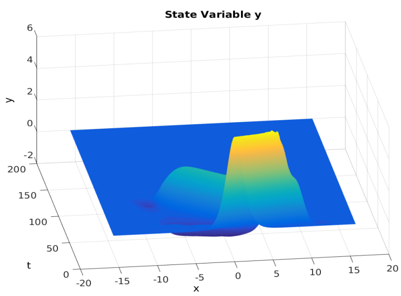
Table 5.1: Various indicators of the efficiency of the receding horizon control process for different prediction horizons.



(a) $T = 1$

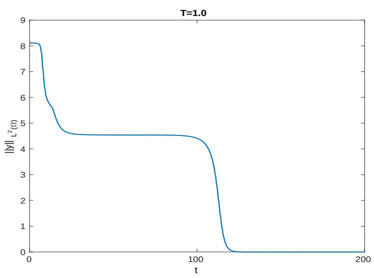


(b) $T = 1.5$

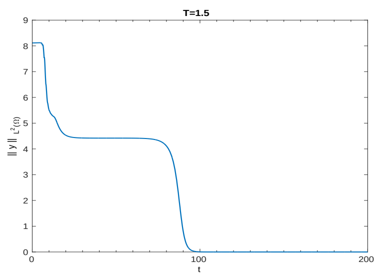


(c) $T = 2$

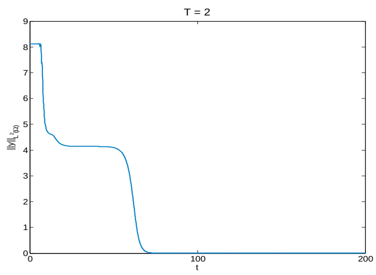
Figure 5.3: Evolution of the state during the receding horizon control process for the prediction horizons (from left to right): $T = 1$, 1.5, 2.



(a) $T = 1$

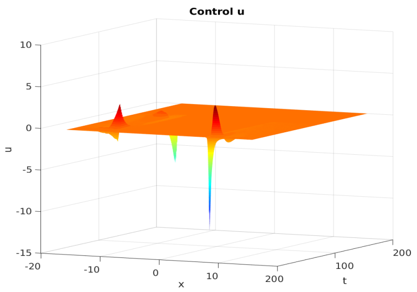


(b) $T = 1.5$

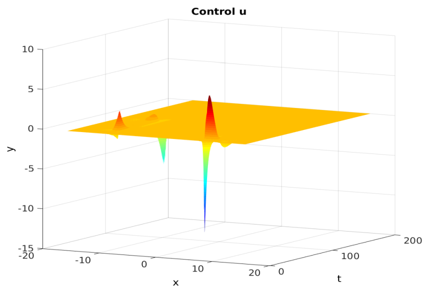


(c) $T = 2$

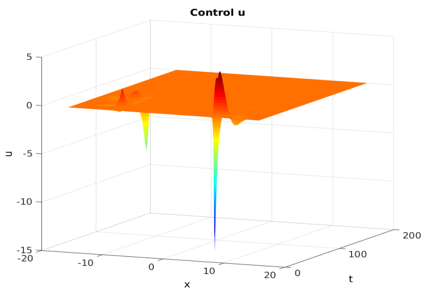
Figure 5.4: Evolution of the L^2-norm of the state during the receding horizon control process for the prediction horizons: $T = 1,\ 1.5,\ 2$.



(a) $T = 1$

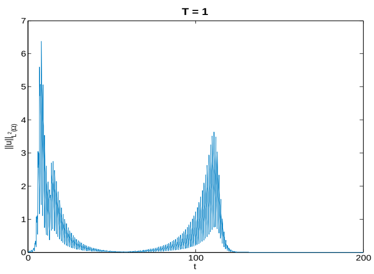


(b) $T = 1.5$

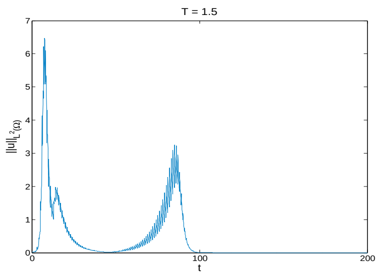


(c) $T = 2$

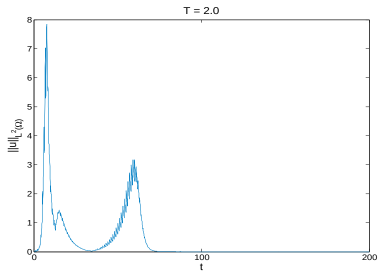
Figure 5.5: Evolution of the control during the receding horizon control process for the prediction horizons: $T = 1$, 1.5, 2.



(a) $T = 1$



(b) $T = 1.5$



(c) $T = 2$

Figure 5.6: Evolution of the L^2-norm of the control during the receding horizon control process for the prediction horizons: $T = 1,\ 1.5,\ 2.$

Chapter 6

Conclusion and Future Work

In this thesis, we dealt with the stabilization of a class of continuous-time infinite-dimensional controlled systems within the scope of the Receding Horizon Control (RHC) framework. The proposed framework does not need any terminal cost and terminal constraint to ensure the stability of RHC. The stability of RHC is rather obtained by generating an appropriate sequence of overlapping temporal intervals and applying a suitable concatenation scheme. The applicability of this framework relies on the well-posedness of the finite horizon open-loop problems and the stabilizability of the controlled system. Based on these conditions the suboptimality and asymptotic stability of RHC were investigated. Later this framework was applied and analysed for the stabilization of controlled systems governed by partial differential equations including the linear wave equation, the viscous Burgers equation, and the nonlinear KdV. For each partial differential equation, depending to the regularity of the solution and the structure of the equation, the stability and suboptimality of RHC were investigated. In addition, for each case, we reported numerical experiments which confirm the theoretical results presented in the thesis.

Although the receding horizon framework for finite-dimensional controlled systems has been studied extensively over the last decades and there is a rich literatures on the theoretical and computational aspects of this frameworks, there is very little research dealing with infinite-dimensional controlled systems.

As topics for future research, we can name the stabilization of time-delay controlled systems governed by partial differential equations. These problems have a wide range of applications. Therefore it is really demanding to design robust stabilizing RHC laws to deal with these problems. For the case of ordinary differential equations, the stability of RHC for delay-time nonlinear systems has been studied by many researchers. For instance, see, e.g., [119, 121] and the references therein. Another interesting topic would be to study the stability and performance of RHC subject to state and control constraints for infinite-dimensional controlled systems. These problems are theoretically, and also computationally very challenging.

RHC is obtained by solving a sequence of finite horizon open-loop problems. In the case of infinite-dimensional controlled system, the discretization of these open-loop problems leads to very large scale optimization problems. Therefore solving these problems and as consequence, computing RHC, require considerably computational effort. This computational effort can be reduced significantly, by studying the following question:

1. Whether we need to solve each open-loop problem exactly or an approximation of its solution is enough to ensure the stability and suboptimality of RHC.

2. How to efficiently compute the solution of an open-loop problem on an interval by utilizing the information of open-loop problems defined on the previous intervals.

3. How to estimate the prediction horizon $T > \delta$ adaptively at every step of the receding horizon algorithm in order to reduce the overall computational effort and to gain a better performance.

Another important point in the context of PDE-constrained optimization is to analyse the structural properties of the resulting finite-dimensional systems with respect to commutativity of discretizing before or after optimizing, and with respect to uniform closed-loop dissipativity.

In this thesis, we addressed only the stabilization problem around a steady state. It would be interesting to also consider other control objective within the scope of the receding horizon framework. Recently, there has been a widespread interest in the so called *economic* receding horizon frameworks. In these frameworks, the control objective is to minimize of some performance index function which is not necessary related to the stabilization any particular steady state. Here the trajectory controlled by economic RHC can have more complex behaviour, for instance, periodic behaviour. The turnpike property [38, 108, 142] is essential condition for the stability of economic RHC. For the case of finite-dimensional controlled systems, we can mention the works [50, 56, 63, 64, 65, 67, 111] and the references therein. But as far as we know, there are only very few results concerning the stability of the economic RHC for infinite-dimensional controlled system. As a research outlook, we suggest to study these receding horizon frameworks for controlled systems governed by partial differential equations. The first step in this directions, is to study the turnpike property, qualitative and theoretically, for infinite-dimensional systems.

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