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RECEDING HORIZON CONTROL FOR THE STABILIZATION OF THE WAVE EQUATION

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Abstract. Stabilization of the wave equation by the receding horizon framework is investigated. Distributed control, Dirichlet boundary control, and Neumann boundary control are considered. Moreover for each of these control actions, the well-posedness of the control system and the exponential stability of Receding Horizon Control (RHC) with respect to a proper functional analytic setting are investigated. Observability conditions are necessary to show the suboptimality and exponential stability of RHC. Numerical experiments are given to illustrate the theoretical results.

1. Introduction.

In this work we deal with the stabilization of the wave equation within the scope of Receding Horizon Control (RHC)

$$\ddot{y} - \Delta y = 0,$$

where $y = y(t, x)$ is a real valued function of real variables t and x, and $\ddot{y}$ stands for the second derivative with respect to time. Our RHC acts on either a part of the domain or within Dirichlet or Neumann boundary conditions. The stabilization problem for the wave equation has been studied extensively by many authors, see for instance [2, 23, 26, 34, 38, 47, 50] and the references cited therein. In these contributions the stabilization problem is obtained by means of a proper choice of a feedback control law, and only few of them provide numerical results. In this work, we use a control law which rests on the solutions of a sequence of open-loop

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optimal control problems governed by the wave equation on finite intervals. To study the open-loop optimal control problems for the wave equation, numerically and analytically, we refer to [14, 24, 25, 32, 33, 45, 46].

To be more precise, we are concerned with minimizing an infinite horizon performance index

$$J_\infty(u;(y_0^1,y_0^2)) := \int_0^\infty \ell((y(t),\dot y(t)),u(t))dt \tag{1}$$

over all control functions $u \in L^2(0,\infty;\mathcal{U})$ with an appropriate control space $\mathcal{U}$ and subject to the following cases:

1. Distributed control: In this case we consider the following controlled system

$$\begin{cases} \ddot y - \Delta y = Bu & \text{in } (0,\infty)\times\Omega,\\ y = 0 & \text{on } (0,\infty)\times\partial\Omega,\\ (y(0),\dot y(0)) = (y_0^1,y_0^2) & \text{on } \Omega, \end{cases} \tag{2}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, the control u is active on a nonempty open subset of Ω, and the control operator B is an extension-by-zero operator from a subdomain ω.

2. Dirichlet control: In this case the control acts on a part of Dirichlet boundary conditions

$$\begin{cases} \ddot y - \Delta y = 0 & \text{in } (0,\infty)\times\Omega,\\ y = u & \text{on } (0,\infty)\times\Gamma_c,\\ y = 0 & \text{on } (0,\infty)\times\Gamma_0,\\ (y(0),\dot y(0)) = (y_0^1,y_0^2) & \text{on } \Omega, \end{cases} \tag{3}$$

where, similar to the above case, $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Moreover, the two disjoint components Γ_c, Γ_0 are relatively open in $\partial\Omega$ and $int(\Gamma_c) \neq \emptyset$.

3. Neumann control: In this case, we are dealing with the following one-dimensional wave equation with a Neumann control action at one side of boundary

$$\begin{cases} \ddot y - y_{xx} = 0 & \text{in } (0,\infty)\times(0,L),\\ y(\cdot,0) = 0 & \text{in } (0,\infty),\\ y_x(\cdot,L) = u & \text{in } (0,\infty),\\ (y(0,\cdot),\dot y(0,\cdot)) = (y_0^1,y_0^2) & \text{in } (0,L), \end{cases} \tag{4}$$

where $L > 0$.

By denoting $\mathcal{Y}(t) := (y(t),\dot y(t))$, and choosing an appropriate control space $\mathcal{U}$, each controlled system in the above cases can be rewritten as a first order controlled system in an abstract Hilbert space $\mathcal{H}$:

$$\begin{cases} \dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{B}u, \quad t > 0,\\ \mathcal{Y}(0) = \mathcal{Y}_0 := (y_0^1,y_0^2), \end{cases} \tag{AP}$$

where for each case, the state space $\mathcal{H}$, the unbounded operator $\mathcal{A}$, and the control operator $\mathcal{B}$ will be specified appropriately below, compare also e.g., [36, 49, 51]. In particular it will be guaranteed that for every $T > 0$ and $u \in L^2(0,T;\mathcal{U})$, there exists a unique solution $\mathcal{Y} \in C^0([0,T];\mathcal{H})$ to (AP) which satisfies the estimate

$$\|\mathcal{Y}(t)\|_{\mathcal{H}} \le c_{est}\left(\|\mathcal{Y}_0\|_{\mathcal{H}} + \|u\|_{L^2(0,T;\mathcal{U})}\right) \qquad \text{for every } t \in [0,T], \tag{Est}$$

where the constant c_{est} is independent of $\mathcal{Y}_0$ and u. Now we can reformulate our infinite horizon problem as the following problem

$$\inf\{J_\infty(u;\mathcal{Y}_0) \mid (\mathcal{Y},u) \text{ satisfies (AP)}, u \in L^2(0,\infty;\mathcal{U})\}, \tag{OP_∞}$$

where the incremental function $\ell : \mathcal{H} \times \mathcal{U} \to \mathbb{R}_+$ is given by

$$\ell(\mathcal{Y}, u) := \frac{1}{2}\|\mathcal{Y}\|^2_{\mathcal{H}} + \frac{\beta}{2}\|u\|^2_{\mathcal{U}}, \tag{5}$$

and β is a positive constant. To deal with the infinite horizon problem (OP_∞), one can employ the algebraic Riccati equation, see, e.g., [27, 37, 39]. But for the case of infinite-dimensional controlled systems, discretization leads to finite-dimensional Riccati equations of very large order and ultimately one is confronted with the curse of dimensionality. Model reduction techniques do not offer an efficient alternative either. In fact, the transfer function corresponding to the controlled system (2)-(4) has infinitely many unstable poles and thus, the model reduction based on balanced truncation will not produce finite H_∞−error bounds, see, e.g., [16].

An alternative approach to deal with (OP_∞) is the receding horizon framework. In this framework, the stabilizing control, namely, RHC is obtained by concatenation of a sequence of open-loop optimal controls on a sequence of overlapping temporal intervals. Further, the process of generating the sequence of intervals and concatenation are carried out in such way that the resulting control has a feedback mechanism and is defined on the whole of the interval $[0, \infty)$. Indeed, the receding horizon framework bridges to a certain degree the gap between the open- and closed-loop control. In the past three decades, numerous results have been published on RHC for finite-dimensional systems, among them we can mention [13, 20, 22, 29, 44, 48] and the references therein. More recently, some authors have addressed the case of infinite-dimensional systems as well [3, 21, 28]. Here we employ the receding horizon framework which was proposed in [48] for finite-dimensional controlled systems, and in [3] for infinite-dimensional controlled systems. In this framework, neither terminal costs nor terminal constraints are imposed to the subproblems in order to guarantee the stability of RHC. But rather, by defining an appropriate sequence of overlapping temporal intervals and applying a suitable concatenation scheme, one can ensure the stability and also suboptimality of RHC. In the previous work [3], this RHC was applied for the stabilization of the Burgers equation with different boundary conditions. In addition, based on a stabilizability condition, the asymptotic stability and suboptimality of RHC were investigated. In the present work, we investigate the suboptimality and *exponential stability* of RHC for all the cases 1-3 of the wave equation with respect to an appropriate functional analytic setting. The key properties are the observability conditions which were not available for the Burgers equation in [3]. By help of these conditions, we obtain not just asymptotic stability but also *exponential stability* of RHC.

Turning to the receding horizon approach, we choose a sampling time $\delta > 0$ and an appropriate prediction horizon $T > \delta$. Then, we define sampling instances $t_k := k\delta$ for $k = 0 \dots$. At every sampling instance t_k, an open-loop optimal control problem is solved over a finite prediction horizon $[t_k, t_k + T]$. Then the optimal control is applied to steer the system from time t_k with the initial state $\mathcal{Y}_{rh}(t_k)$ until time $t_{k+1} := t_k + \delta$ at which point, a new measurement of state is assumed to be available. The process is repeated starting from the new measured state: we obtain a new optimal control and a new predicted state trajectory by shifting the prediction horizon forward in time. The sampling time δ is the time period between two sample instances. Throughout, we denote the receding horizon state- and control variables by $\mathcal{Y}_{rh}(\cdot)$ and $u_{rh}(\cdot)$, respectively. Also, $(\mathcal{Y}^*_T(\cdot; \mathcal{Y}_0, t_0), u^*_T(\cdot; \mathcal{Y}_0, t_0))$ stands for the optimal state and control of the optimal control problem with finite time horizon

T, and initial function $\mathcal{Y}_0$ at initial time t_0. Next, we summarize these steps in Algorithm 1.

Algorithm 1 Receding Horizon Algorithm

Require: Let the prediction horizon T, the sampling time $\delta < T$, and the initial point $(y_0^1, y_0^2) \in \mathcal{H}$ be given. Then we proceed through the following steps:

1: $k := 0, \quad t_0 := 0$ and $\mathcal{Y}_{rh}(t_0) := (y_0^1, y_0^2)$.

2: Find the optimal pair $(\mathcal{Y}_T^*(\cdot; \mathcal{Y}_{rh}(t_k), t_k), u_T^*(\cdot; \mathcal{Y}_{rh}(t_k), t_k))$ over the time horizon $[t_k, t_k + T]$ by solving the finite horizon open-loop problem

$$\min_{u \in L^2(t_k, t_k+T; \mathcal{U})} J_T(u; \mathcal{Y}_{rh}(t_k)) := \min_{u \in L^2(t_k, t_k+T; \mathcal{U})} \int_{t_k}^{t_k+T} \ell(\mathcal{Y}(t), u(t)) dt,$$

$$\text{s.t} \begin{cases} \dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{B}u & t \in (t_k, t_k + T) \\ \mathcal{Y}(t_k) = \mathcal{Y}_{rh}(t_k) \end{cases}$$

3: Set

$$\begin{aligned} u_{rh}(\tau) &:= u_T^*(\tau; y_{rh}(t_k), t_k) && \text{for all } \tau \in [t_k, t_k + \delta), \\ \mathcal{Y}_{rh}(\tau) &:= \mathcal{Y}_T^*(\tau; y_{rh}(t_k), t_k) && \text{for all } \tau \in [t_k, t_k + \delta], \\ t_{k+1} &:= t_k + \delta, \\ k &:= k + 1. \end{aligned}$$

4: Go to step 2.

1.1. **Stability and Suboptimality of RHC.** Throughout this paper, we use the following definitions:

Definition 1.1 (Value function)**.** For every pair $(y_0^1, y_0^2) =: \mathcal{Y}_0 \in \mathcal{H}$, the infinite horizon value function $V_\infty : \mathcal{H} \to \mathbb{R}_+$ is defined as

$$V_\infty(\mathcal{Y}_0) := \inf_{u \in L^2(0,\infty; \mathcal{U})} \{J_\infty(u, \mathcal{Y}_0) \text{ subject to (AP)}\}.$$

Similarly, the finite horizon value function $V_T : \mathcal{H} \to \mathbb{R}_+$ is defined by

$$V_T(\mathcal{Y}_0) := \min_{u \in L^2(0,T; \mathcal{U})} \{J_T(u, \mathcal{Y}_0) \text{ subject to (AP)}\}. \tag{6}$$

In order to show the exponential stability and suboptimality of the receding horizon control obtained by Algorithm 1, we need to verify the following properties: Since, in Algorithm 1, the solution of (OP_∞) is approximated by solving a sequence of the finite horizon open-loop optimal controls, one needs, apriori, to be sure that any of these optimal control problems in Step 2 of Algorithm 1 is well-defined:

P1: For every $\mathcal{Y}_0 \in \mathcal{H}$ and $T > 0$, every finite horizon optimal control problems of the form

$$\min \left\{ J_T(u; \mathcal{Y}_0) \mid (\mathcal{Y}, u) \text{ satisfies (AP)}, u \in L^2(0, T; \mathcal{U}) \right\} \tag{OP_T}$$

admits a solution.

Moreover, we require the following properties for the finite horizon value function V_T:

P2: For every $T > 0$, V_T has a *quadratic growth rate* with respect to the $\mathcal{H}$-norm. That is, there exists a continuous, non-decreasing, and bounded function $\gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that we have

$$V_T(\mathcal{Y}_0) \le \gamma_2(T) \|\mathcal{Y}_0\|_{\mathcal{H}}^2 \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}. \tag{7}$$

P3: For every $T>0$, V_T is *uniformly positive* with respect to the $\mathcal{H}$-norm. In other words, for every $T>0$ there exists a constant $\gamma_1(T)>0$ such that we have

$$V_T(\mathcal{Y}_0) \geq \gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}} \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}. \tag{8}$$

Remark 1.2. Property P3 is equivalent to the injectivity of the differential Riccati operator corresponding to (OP_T) which in turn is equivalent to the exact controllability condition for the system $(\mathcal{A}^*, \mathcal{I}^*)$ where $\mathcal{I} \in \mathcal{L}(\mathcal{H})$ is the identity operator and the superscript $*$ stands for the adjoint operator, see, [19, Theorem 3.3].

Now by setting $\alpha_\ell := \frac{\min\{1,\beta\}}{2}$, we recall the following results from [3].

Lemma 1.3. *Suppose that P1-P2 hold and let $\mathcal{Y}_0 \in \mathcal{H}$ be given. Then for the choice*

$$\theta_1 := 1 + \frac{\gamma_2(T)}{\alpha_\ell(T-\delta)}, \qquad \theta_2 := \frac{\gamma_2(T)}{\alpha_\ell \delta},$$

we have the following estimates

$$V_T(\mathcal{Y}^*_T(\delta;\mathcal{Y}_0,0)) \leq \theta_1 \int_\delta^T \ell(\mathcal{Y}^*_T(t;\mathcal{Y}_0,0), u^*_T(t;\mathcal{Y}_0,0))dt, \tag{9}$$

and

$$\int_\delta^T \ell(\mathcal{Y}^*_T(t;\mathcal{Y}_0,0), u^*_T(t;\mathcal{Y}_0,0))dt \leq \theta_2 \int_0^\delta \ell(\mathcal{Y}^*_T(t;\mathcal{Y}_0,0), u^*_T(t;\mathcal{Y}_0,0))dt. \tag{10}$$

Proof. The proof is given in [3]. □

Proposition 1.4. *Suppose that P1-P2 hold and let $\delta>0$ be given. Then there exist $T^*>\delta$ and $\alpha \in (0,1)$ such that the following inequality is satisfied*

$$V_T(\mathcal{Y}^*_T(\delta;\mathcal{Y}_0,0)) \leq V_T(\mathcal{Y}_0) - \alpha \int_0^\delta \ell(\mathcal{Y}^*_T(t;\mathcal{Y}_0,0), u^*_T(t;\mathcal{Y}_0,0))dt \tag{11}$$

for every $T \geq T^*$ and $\mathcal{Y}_0 \in \mathcal{H}$.

Proof. The proof is given in [3]. □

Theorem 1.5 (Suboptimality and exponential decay)**.** *Suppose that P1-P2 hold and let a sampling time $\delta>0$ be given. Then there exist numbers $T^*>\delta$, and $\alpha \in (0,1)$, such that for every fixed prediction horizon $T \geq T^*$, and every $\mathcal{Y}_0 \in \mathcal{H}$ the receding horizon control u_{rh} obtained from Algorithm 1 satisfies the* ***suboptimality*** *inequality*

$$\alpha V_\infty(\mathcal{Y}_0) \leq \alpha J_\infty(u_{rh},\mathcal{Y}_0) \leq V_T(\mathcal{Y}_0) \leq V_\infty(\mathcal{Y}_0). \tag{12}$$

Moreover if, additionally, P3 holds we have the ***exponential stability*** *inequality*

$$\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}} \leq c' e^{-\zeta t}\|\mathcal{Y}_0\|^2_{\mathcal{H}} \quad \text{for } t \geq 0, \tag{13}$$

where the positive numbers ζ and c' depend on α, δ, and T, but are independent of $\mathcal{Y}_0$.

Proof. To show the suboptimality inequality, we refer to [3, Theorem 6]. Now we turn to inequality (13). By Proposition 1.4, there exist a $T^*>0$ and $\alpha \in (0,1)$ such that for every $T \geq T^*$, $\mathcal{Y}_0 \in \mathcal{H}$, and $k \in \mathbb{N}$ with $k \geq 1$, we have

$$V_T(\mathcal{Y}_{rh}(t_k)) - V_T(\mathcal{Y}_{rh}(t_{k-1})) \leq -\alpha \int_{t_{k-1}}^{t_k} \ell(\mathcal{Y}_{rh}(t), u_{rh}(t))dt \leq -\alpha V_\delta(\mathcal{Y}_{rh}(t_{k-1})), \tag{14}$$

where $t_k = k\delta$ for $k=0,1,2,\dots$ and we use that $\delta<T$. Moreover, due to P2 and P3, for every $\mathcal{Y}_0 \in \mathcal{H}$ we obtain

$$V_\delta(\mathcal{Y}_0) \geq \gamma_1(\delta)\|\mathcal{Y}_0\|^2_{\mathcal{H}} \geq \frac{\gamma_1(\delta)}{\gamma_2(T)} V_T(\mathcal{Y}_0). \tag{15}$$

Using (14) and (15) we can write

$$V_T(\mathcal{Y}_{rh}(t_k)) \le \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right) V_T(\mathcal{Y}_{rh}(t_{k-1})) \text{ for every } k \ge 1.$$

Since $0 < \gamma_1(\delta) \le \gamma_2(\delta) \le \gamma_2(T)$ and $\alpha \in (0,1)$, we have $\eta := \left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right) \in (0,1)$. Furthermore, by defining $\zeta := \frac{|\ln \eta|}{\delta}$, using property P2 for $V_T(\mathcal{Y}_0)$, and property P3 for $V_T(\mathcal{Y}_{rh}(t_k))$, we can infer that

$$\gamma_1(T)\|\mathcal{Y}_{rh}(t_k)\|^2_{\mathcal{H}} \le V_T(\mathcal{Y}_{rh}(t_k)) \le e^{-\zeta t_k} V_T(\mathcal{Y}_0) \le e^{-\zeta t_k}\gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}}$$

for every $k \ge 1$. Hence, by setting $c'' := \frac{\gamma_2(T)}{\gamma_1(T)}$ we can write

$$\|\mathcal{Y}_{rh}(t_k)\|^2_{\mathcal{H}} \le c'' e^{-\zeta k\delta}\|\mathcal{Y}_0\|^2_{\mathcal{H}} \text{ for every } k \ge 1. \tag{16}$$

Moreover, for every $t > 0$ there exists a $k \in \mathbb{N}$ such that $t \in [k\delta, (k+1)\delta]$. Using (Est), (5), and (7), we have for $t \in [k\delta, (k+1)\delta]$,

$$\begin{aligned}
\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}} &\le 2c^2_{est}\left(\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}} + \int_{k\delta}^{(k+1)\delta} \|u_{rh}(t)\|^2_{\mathcal{U}} dt\right) \\
&\overset{(5)}{\le} 2c^2_{est}\left(\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}} + \frac{2}{\beta}V_T(\mathcal{Y}_{rh}(k\delta))\right) \\
&\overset{(7)}{\le} 2c^2_{est}\left(\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}} + \frac{2\gamma_2(T)}{\beta}\|\mathcal{Y}_{rh}(k\delta)\|^2_{\mathcal{H}_2}\right) \\
&\overset{(16)}{\le} 2c^2_{est}c''(1 + \frac{2\gamma_2(T)}{\beta})e^{-\zeta k\delta}\|\mathcal{Y}_0\|^2_{\mathcal{H}} \\
&\le 2c^2_{est}c''(1 + \frac{2\gamma_2(T)}{\beta})\left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right)^{-1} e^{-\zeta(k+1)\delta}\|\mathcal{Y}_0\|^2_{\mathcal{H}} \\
&\le 2c^2_{est}c''(1 + \frac{2\gamma_2(T)}{\beta})\left(1 - \frac{\alpha\gamma_1(\delta)}{\gamma_2(T)}\right)^{-1} e^{-\zeta t}\|\mathcal{Y}_0\|^2_{\mathcal{H}},
\end{aligned}$$

and the proof is complete. □

Remark 1.6. It is of interest to derive the exponential decay inequality (13) in an alternative way as in the above. In particular, the constants ζ and c' can be estimated in a different manner. Namely, due to [3, Theorem 7], there exists a $T^* > 0$ such that for every $T \ge T^*$ we have

$$V_{T-\delta}(\mathcal{Y}_{rh}(t)) \le ce^{-\zeta t}V_T(\mathcal{Y}_0) \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}, \quad t > 0, \tag{17}$$

where the constants c and ζ are given by

$$\zeta := \frac{\ln(1 + \frac{\alpha}{1+\theta_1\theta_2})}{\delta}, \qquad c := (1 + \frac{\alpha}{1+\theta_1\theta_2}).$$

Here $\theta_1(T,\delta), \theta_2(T,\delta)$ are defined as in Lemma 1.3. Using properties P2 and P3 we obtain

$$\gamma_1(T-\delta)\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}} \le V_{T-\delta}(\mathcal{Y}_{rh}(t)) \le ce^{-\zeta t}V_T(\mathcal{Y}_0) \le c\gamma_2(T)e^{-\zeta t}\|\mathcal{Y}_0\|^2_{\mathcal{H}}$$

for every $\mathcal{Y}_0 \in \mathcal{H}$. Setting $c' := \frac{c\gamma_2(T)}{\gamma_1(T-\delta)}$ we have

$$\|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}} \le c' e^{-\zeta t}\|\mathcal{Y}_0\|^2_{\mathcal{H}} \quad \text{for every } \mathcal{Y}_0 \in \mathcal{H}, \quad t > 0,$$

and thus (13) holds.

Remark 1.7. Note that, if we had $\alpha = 1$, the inequality (12) would imply the optimality of the receding horizon control u_{rh}. Since here we have global stabilizability of RHC, similarly to the proof of [3, Theorem 5], one can show that

$$\alpha(T) = 1 - \theta_2(\theta_1 - 1) = 1 - \frac{\gamma_2^2(T)}{\alpha_\ell^2 \delta(T-\delta)} \qquad \text{for all } T \geq T^*, \tag{18}$$

where θ_1, θ_2 are defined as in Lemma 1.3 and T^* is chosen such that $\alpha(T^*) > 0$ holds. Now since $\gamma_2(T)$ is a bounded function and δ is fixed, we have

$$\lim_{T\to\infty} \alpha(T) = 1. \tag{19}$$

Therefore, asymptotically the RHC strategy is optimal.

The rest of paper is organized as follows: Sections 2, 3, and 4 deal, respectively, with the cases in which RHC enters as a distributed control, a Dirichlet boundary condition, and a Neumann boundary condition. In each of these sections, first well-posedness of the finite horizon optimal control problems (i.e., property P1) and the corresponding optimality conditions are investigated. Then, relying on observability conditions, properties P2 and P3 are analysed. Finally, in Section 5, we demonstrate numerical experiments in which Algorithm 1 is implemented for each type of the control actions. In addition, for each example the performance of RHC is evaluated and compared for different choices of the prediction horizon T and a fixed sampling time δ.

2. **Distributed Control.** In this section we are concerned with the controlled system (2). Here, the control operator B is the extension-by-zero operator defined by

$$(Bu)(x) := \begin{cases} u(x) & x \in \omega, \\ 0 & x \in \Omega \backslash \omega, \end{cases}$$

where the control domain ω is a nonempty open subset of $\Omega \in \mathbb{R}^n$ with smooth boundary $\partial\Omega$. We define $\mathcal{H}_1 := H_0^1(\Omega) \times L^2(\Omega)$, $\mathcal{U} := L^2(\omega)$, and the energy $\mathcal{E}(\cdot, y)$ along a trajectory y by

$$\mathcal{E}(t, y) := \|y(t)\|^2_{H_0^1(\Omega)} + \|\dot{y}(t)\|^2_{L^2(\Omega)}. \tag{20}$$

The incremental function $\ell : \mathcal{H}_1 \times L^2(\omega) \to \mathbb{R}_+$ is given by

$$\ell((y,z),u) := \frac{1}{2}\|(y,z)\|^2_{\mathcal{H}_1} + \frac{\beta}{2}\|u\|^2_{L^2(\omega)}. \tag{21}$$

First we recall different notions of solution to the following linear wave equation

$$\begin{cases} \ddot{y} - \Delta y = f & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ (y(0), \dot{y}(0)) = (y_0^1, y_0^2) & \text{on } \Omega. \end{cases} \tag{22}$$

Definition 2.1 (Weak solution)**.** Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_1$, and $f \in L^2(0,T; L^2(\Omega))$. Then $(y, \dot{y}) \in C^0([0,T]; \mathcal{H}_1)$ with $\ddot{y} \in L^2(0,T; H^{-1}(\Omega))$ is referred to as the weak solution of (22), if $(y(0), \dot{y}(0)) = (y_0^1, y_0^2)$ in $\mathcal{H}_1$, and for almost every $t \in (0,T)$ we have

$$\langle \ddot{y}(t), v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\nabla y(t), \nabla v)_{L^2(\Omega)} = (f(t), v)_{L^2(\Omega)} \qquad \text{for all } v \in H_0^1(\Omega). \tag{23}$$

Definition 2.2 (Very weak solution)**.** Let $T > 0$, $(y_0^1, y_0^2) \in L^2(\Omega) \times H^{-1}(\Omega)$, and $f \in L^2(0, T; H^{-1}(\Omega))$ be given. A function $y \in L^2(0, T; L^2(\Omega))$ is referred to as the very weak solution of (22), if the following inequality holds

$$\int_0^T (g(t), y(t))_{L^2(\Omega)}\, dt = \\ - (y_0^1, \dot{\vartheta}(0))_{L^2(\Omega)} + \langle y_0^2, \vartheta(0)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^T \langle f(t), \vartheta(t)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, dt,$$

for all $g \in L^2(0, T; L^2(\Omega))$, with ϑ the weak solution of the following backward in time problem

$$\begin{cases} \ddot{\vartheta} - \Delta\vartheta = g & \text{in } (0, T) \times \Omega, \\ \vartheta = 0 & \text{on } (0, T) \times \partial\Omega, \\ (\vartheta(T), \dot{\vartheta}(T)) = (0, 0) & \text{on } \Omega. \end{cases}$$

The very weak solution is also called solution by transposition.

We recall the following results for (22), see, e.g., [40, 42].

Proposition 2.3. *We have the following existence and regularity results for* (22)*:*

1. *Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_1$, and $f \in L^2(0, T; L^2(\Omega))$ be given. Then there exists a unique weak solution y with $(y, \dot{y}) \in C^0([0, T]; \mathcal{H}_1)$ to* (22) *which satisfies*

$$\|y\|_{C^0([0,T];H_0^1(\Omega))} + \|\dot{y}\|_{C^0([0,T];L^2(\Omega))} + \|\ddot{y}\|_{L^2(0,T;H^{-1}(\Omega))} \\ \leq c_1 \left(\|y_0^1\|_{H_0^1(\Omega)} + \|y_0^2\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))} \right), \tag{24}$$

where the constant c_1 is independent of y_0^1, y_0^2, and f. Moreover, we have the following hidden regularity

$$\partial_\nu y \in L^2(0, T; L^2(\partial\Omega)),$$

and the corresponding estimate

$$\|\partial_\nu y\|_{L^2(0,T;L^2(\partial\Omega))} \leq c_N \left(\|y_0^1\|_{H_0^1(\Omega)} + \|y_0^2\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))} \right),$$

where the constant c_N is independent of y_0^1, y_0^2, and f.

2. *For every $T > 0$, $f \in L^2(0, T; H^{-1}(\Omega))$, and every pair $(y_0^1, y_0^2) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique very weak solution to* (22) *in $C^1([0, T]; H^{-1}(\Omega)) \cap C^0([0, T]; L^2(\Omega))$ satisfying*

$$\|y\|_{C^0([0,T];L^2(\Omega))} + \|\dot{y}\|_{C^0([0,T];H^{-1}(\Omega))} \\ \leq \bar{c}_1 \left(\|y_0^1\|_{L^2(\Omega)} + \|y_0^2\|_{H^{-1}(\Omega)} + \|f\|_{L^2(0,T;H^{-1}(\Omega))} \right), \tag{25}$$

where the constant $\bar{c}_1$ is independent of y_0^1, y_0^2, and f.

Remark 2.4. From (24) it follows that (Est) holds for (2) with respect to the $\mathcal{H}_1$-norm.

2.1. **On the finite horizon optimal control problems.** For our subsequent work we need to gather some facts on the finite horizon optimal controls of the form (OP_T) given by

$$\min J_T(u; (y_0^1, y_0^2)) := \min \int_0^T \ell((y(t), \dot{y}(t)), u(t)) dt \tag{P_{dis}}$$

over all $u \in L^2(0,T;L^2(\omega))$, subject to

$$\begin{cases} \ddot{y} - \Delta y = Bu & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y(0),\dot{y}(0)) = (y_0^1, y_0^2) & \text{on } \Omega, \end{cases} \tag{26}$$

where $(y_0^1, y_0^2) \in \mathcal{H}_1$ and the incremental function ℓ is defined by (21). Property P1 is verified by means of the following proposition.

Proposition 2.5. *For every $T > 0$ and $(y_0^1, y_0^2) \in \mathcal{H}_1$, the optimal control problem (P_{dis}) admits a unique solution.*

Proof. For the proof we refer to [40]. □

In following we derive the first-order optimality conditions for (P_{dis}). Due to the presence of the tracking term for the velocity $\dot{y}$ in the performance index function of (P_{dis}), we will see that the solution of adjoint equation exists in the very weak sense.

Proposition 2.6. *Let $(\bar{y},\bar{u})$ be the optimal solution to (P_{dis}). It satisfies the following optimality conditions*

$$\begin{cases} \ddot{\bar{y}} - \Delta\bar{y} = B\bar{u} & \text{in } (0,T)\times\Omega, \\ \bar{y} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\bar{y}(0),\dot{\bar{y}}(0)) = (y_0^1, y_0^2) & \text{on } \Omega, \end{cases} \qquad \begin{cases} \ddot{\bar{p}} - \Delta\bar{p} = -\ddot{\bar{y}} - \Delta\bar{y} & \text{in } (0,T)\times\Omega, \\ \bar{p} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\bar{p}(T),\dot{\bar{p}}(T)) = (0, -\dot{\bar{y}}(T)) & \text{on } \Omega, \end{cases}$$

and

$$\beta\bar{u} = -B^*\bar{p} \quad \text{in } (0,T)\times\Omega,$$

where $\bar{p} \in C^1([0,T];H^{-1}(\Omega)) \cap C^0([0,T];L^2(\Omega))$ is the solution of the adjoint equation.

Proof. The proof is given in Appendix A.1. □

2.2. **Verification of P2 and P3.** In this subsection we deal with the verification of properties P2 and P3. Concerning this matter, we recall some aspects on the stabilizability of the wave equation with a distributed feedback law. Specifically, we consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = u(y) & \text{in } (0,\infty)\times\Omega, \\ y = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ (y(0),\dot{y}(0)) = (y_0^1, y_0^2) & \text{on } \Omega, \end{cases} \tag{27}$$

with the feedback control u given by $u(y) := -a(x)\dot{y}$, where the function $a \in L^\infty(\Omega)$ satisfies

$$a_1 \geq a(x) \geq a_0 > 0 \text{ for almost every } x \in \omega, \text{ and } a(x) = 0 \text{ in } \Omega\backslash\omega. \tag{28}$$

The following observability conditions will be used later.

To specify the required *observability conditions*, for any $(\phi_0^1, \phi_0^2) \in \mathcal{H}_1$ we denote by ϕ the weak solution of the following system

$$\begin{cases} \ddot{\phi} - \Delta\phi = 0 & \text{in } (0,T)\times\Omega, \\ \phi = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\phi(0),\dot{\phi}(0)) = (\phi_0^1, \phi_0^2) & \text{on } \Omega. \end{cases} \tag{29}$$

Then we can formulate the following observability conditions:

OB1: There exists $T_{ob1} > 0$ such that for every $T \geq T_{ob1}$, the weak solution ϕ to (29) with $(\phi, \dot{\phi}) \in C([0,T]; \mathcal{H}_1)$ satisfies the inequality

$$c_{ob1} \|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_1}^2 \leq \int_0^T \int_\omega |\dot{\phi}|^2 dx dt \quad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_1,$$

where the positive constant c_{ob1} depends only on T and $\omega \subseteq \Omega$.

OB2: There exists $T_{ob2} > 0$ such that for every $T \geq T_{ob2}$, the weak solution ϕ to (29) with $(\phi, \dot{\phi}) \in C([0,T]; \mathcal{H}_1)$ satisfies the inequality

$$c_{ob2} \|(\phi_0^1, \phi_0^2)\|_{\mathcal{H}_1}^2 \leq \int_0^T \int_{\Gamma_c} |\partial_\nu \phi|^2 dS dt \quad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_1,$$

where the positive constant c_{ob2} depends only on T and $\Gamma_c \subseteq \partial\Omega$.

The observability conditions OB1-OB2 are satisfied if and only if the Geometric Control Conditions (GCC) hold (see, e.g., [8, 11]). Roughly speaking, GCC for $(\Omega, \omega, T_{ob1})$ (resp. $(\Omega, \Gamma_c, T_{ob2})$) states that all rays of the geometric optics should enter in the domain ω (resp. meet the boundary Γ_c) in a time smaller than T_{ob1} (resp. T_{ob2}). For a comprehensive study, we refer to Reference [8].

The following equivalence is frequently mentioned in the literature. Since it is not straight forward to find a proof, we provide the arguments here.

Proposition 2.7. *Let $(y_0^1, y_0^2) \in \mathcal{H}_1$ and $a \in L^\infty(\Omega)$ satisfying (28) be given. Then for the controlled system (27) with the feedback law $u(y) := -a\dot{y}$ we have*

$$\mathcal{E}(t, y) \leq M e^{-\alpha t} \mathcal{E}(0, y) = M e^{-\alpha t} \|(y_0^1, y_0^2)\|_{\mathcal{H}_1}^2 \tag{30}$$

for positive constants M, and α independent of (y_0^1, y_0^2), if and only if the observability condition OB1 holds.

Proof. First we show that OB1 implies exponential stabilizability. We set $u(y) := -a\dot{y}$ in (27). In this case the resulting closed-loop system is well-posed (see, e.g., [12]) and for its unique weak solution we have

$$y \in C^0([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)).$$

Now, for an arbitrary $T > 0$ consider the following controlled system

$$\begin{cases} \ddot{y} - \Delta y = -a\dot{y} & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ (y(0), \dot{y}(0)) = (y_0^1, y_0^2) & \text{on } \Omega. \end{cases} \tag{31}$$

By taking the L^2-inner product of (31) with $\dot{y}$, and integrating over $[0,T]$, we obtain the following estimate

$$\mathcal{E}(T, y) - \mathcal{E}(0, y) \leq -2a_0 \int_0^T \|\dot{y}(t)\|_{L^2(\omega)}^2 dt. \tag{32}$$

Now by using a density argument and passing to the limit, it can be shown that the inequality (32) is also true for the weak solution of (31) with the initial data $(y_0^1, y_0^2) \in \mathcal{H}_1$. Moreover the solution y of (31) can be expressed as $y := \psi + \phi$ where $\phi \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega))$ is the weak solution to (29) with $(\phi_0^1, \phi_0^2) = (y_0^1, y_0^2)$, and $\psi \in C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H_0^1(\Omega))$ is the weak solution of

$$\begin{cases} \ddot{\psi} - \Delta \psi = -a\dot{y} & \text{in } (0,T) \times \Omega, \\ \psi = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\psi(0), \dot{\psi}(0)) = (0, 0) & \text{on } \Omega. \end{cases} \tag{33}$$

By the observability condition OB1, and estimate (24) for (33) we have

$$\begin{aligned}\mathcal{E}(0,y) = \|(y_0^1,y_0^2)\|^2_{\mathcal{H}_1} &\le \frac{1}{c_{ob1}}\int_0^{T_{ob1}} \|\dot{\phi}(t)\|^2_{L^2(\omega)}dt \\ &\le \frac{1}{c_{ob1}}\int_0^{T_{ob1}} \left(\|\dot{y}(t)\|^2_{L^2(\omega)} + \|\dot{\psi}(t)\|^2_{L^2(\omega)}\right) dt \le c_1'\int_0^{T_{ob1}} \|\dot{y}(t)\|^2_{L^2(\omega)}dt,\end{aligned} \tag{34}$$

for a constant $c_1' > 0$ which is independent of (y_0^1, y_0^2). By (32), (34) we obtain

$$\mathcal{E}(T_{ob1},y) - \mathcal{E}(0,y) \le -2a_0\int_0^{T_{ob1}} \|\dot{y}(t)\|^2_{L^2(\omega)}dt \le -\frac{2a_0}{c_1'}\mathcal{E}(0,y) \le -\frac{2a_0}{c_1'}\mathcal{E}(T_{ob1},y).$$

Setting $\alpha := \frac{\ln(1+\frac{2a_0}{c_1'})}{T_{ob1}}$, we have for every $k = 1, 2, \dots$

$$\mathcal{E}(kT_{ob1},y) \le e^{-\alpha T_{ob1}}\mathcal{E}((k-1)T_{ob1},y),$$

and, as a consequence, for every $t \in [kT_{ob1}, (k+1)T_{ob1}]$ we infer that

$$\begin{aligned}\mathcal{E}(t,y) \le \mathcal{E}(kT_{ob1},y) &\le e^{-\alpha k T_{ob1}}\mathcal{E}(0,y) \\ &= (1+\frac{2a_0}{c_1'})e^{-\alpha(k+1)T_{ob1}}\mathcal{E}(0,y) \le (1+\frac{2a_0}{c_1'})e^{-\alpha t}\mathcal{E}(0,y).\end{aligned}$$

Thus we conclude (30).

Next we show that the stabilizability property (30) implies the observability condition OB1 for (29) with an arbitrary initial pair $(y_0^1, y_0^2) \in \mathcal{H}_1$. Setting $u(y) := -a\dot{y}$ in (27) with $a \in L^\infty(\Omega)$ satisfying (28), taking the L^2-inner product of (27) with $\dot{y}$, and integrating over $[0,t]$ for $t > 0$, we obtain

$$\mathcal{E}(t,y) - \mathcal{E}(0,y) \ge -2a_1\int_0^t \|\dot{y}(t)\|^2_{L^2(\omega)}dt, \tag{35}$$

where a_1 is specified in (28). Further by (30), for a large enough $T' > 0$ we have

$$2a_1\int_0^{T'} \|\dot{y}(t)\|^2_{L^2(\omega)}dt \ge \frac{1}{2}\mathcal{E}(0,y). \tag{36}$$

Moreover, the solution ϕ to (29) with the initial pair (y_0^1, y_0^2) can be rewritten as $\phi := y - \psi$, where y is the weak solution to (31) and ψ is the weak solution to (33) for T' instead of T. Now assume that the solution of (33) is smooth enough. Taking the L^2-inner product of (33) with $\dot{\psi}$ and integrating over $[0, T']$ we have

$$\begin{aligned}0 \le \frac{1}{2}\big(\|\dot{\psi}(T')\|^2_{L^2(\Omega)} + \|\nabla\psi(T')\|^2_{L^2(\Omega)}\big) &= \int_0^{T'}\int_\Omega -a\dot{y}\dot{\psi}dxdt \\ &= \int_0^{T'}\int_\omega -a(\dot{\psi}+\dot{\phi})\dot{\psi}dxdt.\end{aligned} \tag{37}$$

By using a density argument and passing to the limit, it can be shown that the inequality (37) is also true for the weak solution of (33) with $-a\dot{y}$ as a forcing function. Moreover, (37) implies

$$\int_0^{T'} \|\dot{\psi}(t)\|^2_{L^2(\omega)}dt \le \frac{a_1^2}{a_0^2}\int_0^{T'} \|\dot{\phi}(t)\|^2_{L^2(\omega)}dt. \tag{38}$$

Note also that

$$\int_0^{T'} \|\dot{\phi}(t)\|^2_{L^2(\omega)}dt + \int_0^{T'} \|\dot{\psi}(t)\|^2_{L^2(\omega)}dt \ge \int_0^{T'} \|\dot{y}(t)\|^2_{L^2(\omega)}dt. \tag{39}$$

Combining (36), (38), and (39), we complete the proof with

$$\frac{1}{4a_1}\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_1} = \frac{1}{4a_1}\mathcal{E}(0,y) \le (1+\frac{a_1^2}{a_0^2})\int_0^{T'} \|\dot\phi(t)\|^2_{L^2(\omega)}dt.$$

□

Now we are in the position that we can investigate properties P2 and P3.

Proposition 2.8. *Suppose that the observability condition OB1 holds. Then for every $T>0$, there exists a control $\hat{u}\in L^2(0,T;L^2(\omega))$ for (26) such that*

$$V_T(\mathcal{Y}_0) \le J_T(\hat{u};\mathcal{Y}_0) \le \gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \tag{40}$$

for every initial pair $(y_0^1,y_0^2)=:\mathcal{Y}_0\in\mathcal{H}_1$, where $\gamma_2(\cdot)$ is a nondecreasing, continuous, and bounded function. Moreover, there exists a constant $\gamma_1(T)>0$ depending on T such that

$$V_T(\mathcal{Y}_0) \ge \gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_1} \tag{41}$$

for all $(y_0^1,y_0^2)=\mathcal{Y}_0\in\mathcal{H}_1$. Thus P2 and P3 hold.

Proof. Setting $u(t):=-\dot{y}(t)|_\omega$ in (26), and using Proposition 2.7 for the choice

$$a(x) := \begin{cases} 1 & x\in\omega,\\ 0 & \textit{otherwise},\end{cases}$$

we obtain

$$\mathcal{E}(t,y) \le Me^{-\alpha t}\mathcal{E}(0,y) \qquad \text{for all } t\in[0,T].$$

Here the constants M and α were defined in Proposition 2.7. Integrating from 0 to T implies that

$$\int_0^T \mathcal{E}(t,y)dt \le \frac{M}{\alpha}(1-e^{-\alpha T})\mathcal{E}(0,y).$$

By the definition of the value function V_T we have

$$\begin{aligned} V_T((y_0^1,y_0^2)) \le \int_0^T \left(\frac{1}{2}\mathcal{E}(t,y)+\frac{\beta}{2}\|\dot{y}(t)\|^2_{L^2(\omega)}\right)dt \le& \frac{(1+\beta)M}{2\alpha}(1-e^{-\alpha T})\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_1}\\ =&\gamma_2(T)\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_1},\end{aligned}$$

and thus (40) holds.

To verify (41), we use the superposition argument for equation (26) with an arbitrary control $u\in L^2(0,T;L^2(\omega))$. We rewrite the solution of (26) as $y=\phi+\varphi$ where ϕ is the solution to (29) with the initial pair (y_0^1,y_0^2) instead of (ϕ_0^1,ϕ_0^2), and φ is the solution to the following equation

$$\begin{cases} \ddot\varphi-\Delta\varphi = Bu & \text{in } (0,T)\times\Omega,\\ \varphi=0 & \text{on } (0,T)\times\partial\Omega,\\ (\varphi(0),\dot\varphi(0))=(0,0) & \text{on } \Omega.\end{cases} \tag{42}$$

By OB1 for (29) with the initial pair (y_0^1,y_0^2) and ω replaced by Ω, and (24) for (42), we obtain

$$\begin{aligned} \|(y_0^1,y_0^2)\|^2_{\mathcal{H}_1} &\le \frac{1}{c_{ob1}(T)}\int_0^T\|\dot\phi(t)\|^2_{L^2(\Omega)}dt \le \frac{1}{c_{ob1}(T)}\int_0^T\left(\|\dot{y}(t)\|^2_{L^2(\Omega)}+\|\dot\varphi(t)\|^2_{L^2(\Omega)}\right)dt\\ &\le \frac{1}{c_{ob1}(T)}\int_0^T\left(\|\dot{y}(t)\|^2_{L^2(\Omega)}+Tc_1^2\|u(t)\|^2_{L^2(\omega)}\right)dt\\ &\le c_1''(T)\int_0^T\left(\frac{1}{2}\|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_1}+\frac{\beta}{2}\|u(t)\|^2_{L^2(\omega)}\right)dt \le c_1''(T)\int_0^T \ell(\mathcal{Y}(t),u(t))dt.\end{aligned}$$

Since $u \in L^2(0,T;L^2(\omega))$ is arbitrary, we obtain (41) for a constant $c_1''(T)$ independent of u and (y_0^1, y_0^2). $\square$

Remark 2.9. Thus from Propositions 2.5 and 2.8, we conclude that Theorem 1.5 is applicable for the wave equation with distributed control and guarantees the exponential stability of RHC obtained by Algorithm 1.

3. **Dirichlet Boundary Control.** We consider the controlled system of the form (3), where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega := \overline{\Gamma_c} \cup \overline{\Gamma_0}$. The two disjoint components Γ_c, Γ_0 are relatively open in $\partial\Omega$ and $int(\Gamma_c) \neq \emptyset$. By setting $\mathcal{U} := L^2(\Gamma_c)$ and $\mathcal{H}_2 := L^2(\Omega) \times H^{-1}(\Omega)$, we are searching over all control functions $u \in L^2(0,\infty;\mathcal{U})$ for a given initial pair $(y_0^1, y_0^2) \in \mathcal{H}_2$. Let $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the Laplace operator with homogeneous Dirichlet boundary conditions, and define the operator $\mathcal{G} : H^{-1}(\Omega) \to H_0^1(\Omega)$ by $\mathcal{G} := (-\Delta)^{-1}$. Further, we denote the unique linear extension of $\mathcal{G}$ by $\overline{\mathcal{G}} : (H^2(\Omega) \cap H_0^1(\Omega))^* \to L^2(\Omega)$, where $(H^2(\Omega) \cap H_0^1(\Omega))^*$ stands for the dual space of $H^2(\Omega) \cap H_0^1(\Omega)$. The incremental function $\ell : \mathcal{H}_2 \times L^2(\Gamma_c) \to \mathbb{R}_+$ is given by

$$\ell((y,z),u) := \frac{1}{2}\|(y,z)\|^2_{\mathcal{H}_2} + \frac{\beta}{2}\|u\|^2_{L^2(\Gamma_c)}. \tag{43}$$

Moreover, we will use the space $H^1_{\Gamma_0}(\Omega) := \{q \in H^1(\Omega) : q|_{\Gamma_0} = 0\}$ and the control operator B_{bd} which is defined by

$$(B_{bd}u)(x) := \begin{cases} u(x) & x \in \Gamma_c, \\ 0 & x \in \Gamma_0. \end{cases}$$

Consider first the following linear wave equation with the inhomogeneous Dirichlet boundary condition imposed on the whole of the boundary

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T) \times \Omega, \\ y = h & \text{on } (0,T) \times \partial\Omega, \\ (y(0), \dot{y}(0)) = (y_0^1, y_0^2) & \text{on } \Omega. \end{cases} \tag{44}$$

Definition 3.1 (Very weak solution)**.** Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_2$, and $h \in L^2(0,T;L^2(\partial\Omega))$ be given. Then $y \in L^\infty(0,T;L^2(\Omega))$ is referred to as the very weak solution of (44), if the following equality holds

$$\begin{aligned} &\int_0^T (f(t), y(t))_{L^2(\Omega)} dt \\ &= -(y_0^1, \dot{\vartheta}(0))_{L^2(\Omega)} + \langle y_0^2, \vartheta(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_0^T (\partial_\nu \vartheta(t), h(t))_{L^2(\partial\Omega)}\, dt, \end{aligned} \tag{45}$$

for all $f \in L^1(0,T;L^2(\Omega))$, with ϑ the weak solution of the following backward in time problem

$$\begin{cases} \ddot{\vartheta} - \Delta \vartheta = f & \text{in } (0,T) \times \Omega, \\ \vartheta = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\vartheta(T), \dot{\vartheta}(T)) = (0,0) & \text{on } \Omega. \end{cases} \tag{46}$$

We have the following result for the very weak solution of (44), see, e.g., [40, 42].

Proposition 3.2. *For every $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_2$, and $h \in L^2(0,T;L^2(\partial\Omega))$, there exists a unique very weak solution to (44) belonging to the space $C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$, and satisfying*

$$\begin{aligned} \|y\|_{C^0([0,T];L^2(\Omega))} + &\|\dot{y}\|_{C^0([0,T];H^{-1}(\Omega))} + \|\ddot{y}\|_{L^2(0,T;(H_0^1(\Omega)\cap H^2(\Omega))^*)} \\ &\leq c_2 \left(\|y_0^1\|_{L^2(\Omega)} + \|y_0^2\|_{H^{-1}(\Omega)} + \|h\|_{L^2(0,T;L^2(\partial\Omega))}\right) \end{aligned} \tag{47}$$

for a constant c_2 depending on T and the domain Ω.

Remark 3.3. Thus (Est) for the controlled system (3) with respect to $\mathcal{H}_2$-norm follows from (47).

3.1. **On the finite horizon optimal control problems.** Here for $(y_0^1, y_0^2) \in \mathcal{H}_2$, we consider the finite horizon optimal control problems of the form (OP_T) given by

$$\min J_T(u;(y_0^1,y_0^2)) := \min \int_0^T \ell((y(t),\dot{y}(t)),u(t))dt \tag{OP_{dir}}$$

over all $u \in L^2(0,T;L^2(\Gamma_c))$, subject to

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T)\times\Omega, \\ y = u & \text{on } (0,T)\times\Gamma_c, \\ y = 0 & \text{on } (0,T)\times\Gamma_0, \\ (y(0),\dot{y}(0)) = (y_0^1,y_0^2) & \text{on } \Omega. \end{cases} \tag{48}$$

where the incremental function ℓ is given by (43). In the following proposition, we investigate property P1 for the controlled system (3).

Proposition 3.4. *For every $T > 0$ and $(y_0^1, y_0^2) \in \mathcal{H}_2$, the optimal control problem (OP_{dir}) admits a unique solution.*

Proof. The proof is similar to the one of Theorem 2.1 in [46]. □

Next we specify the first-order optimality conditions for (OP_{dir}). Since the objective function in (OP_{dir}) involves the tracking term of the velocity $\dot{y}$ in the space $L^2(0,T;H^{-1}(\Omega))$, the solution to the adjoint equation gains more regularity than the one to (48) and this solution exists in the weak sense.

Proposition 3.5. *Let $(\bar{y},\bar{u})$ be the optimal solution to (OP_{dir}). It satisfies the following optimality conditions*

$$\begin{cases} \ddot{\bar{y}} - \Delta\bar{y} = 0 & \text{in } (0,T)\times\Omega, \\ \bar{y} = \bar{u} & \text{on } (0,T)\times\Gamma_c, \\ \bar{y} = 0 & \text{on } (0,T)\times\Gamma_0, \\ (\bar{y}(0),\dot{\bar{y}}(0)) = (y_0^1,y_0^2) & \text{on } \Omega, \end{cases} \qquad \begin{cases} \ddot{\bar{p}} - \Delta\bar{p} = \bar{y} - \overline{\mathcal{G}}\ddot{\bar{y}} & \text{in } (0,T)\times\Omega, \\ \bar{p} = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\bar{p}(T),\dot{\bar{p}}(T)) = (0,-\overline{\mathcal{G}}\dot{\bar{y}}(T)) & \text{on } \Omega, \end{cases}$$

and

$$\beta\bar{u} = \partial_\nu\bar{p} \quad \text{on } (0,T)\times\Gamma_c,$$

where $\bar{p} \in C^1([0,T];L^2(\Omega)) \cap C^0([0,T];H_0^1(\Omega))$ is the solution of the adjoint equation.

Proof. The proof is given in Appendix A.2. □

3.2. **Verification of P2 and P3.** Similarly to the previous section, we show first that there exists a feedback law $u(y)$ that stabilizes the controlled system (3) with respect to the energy

$$\mathcal{E}(t,y) := \|y(t)\|^2_{L^2(\Omega)} + \|\dot{y}(t)\|^2_{H^{-1}(\Omega)} \tag{49}$$

which is defined along a trajectory y.

Lemma 3.6. *The observability condition OB1 is equivalent to the following observability inequality:*

OB3: *For every* $T \geq T_{ob1}$*, the very weak solution* ϕ *to* (29) *with* $(\phi, \dot{\phi}) \in C^0([0,T]; \mathcal{H}_2)$ *satisfies the inequality*

$$c_{ob1}\|(\phi_0^1, \phi_0^2)\|^2_{\mathcal{H}_2} \leq \int_0^T \int_\omega |\phi|^2 dxdt \quad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_2,$$

where the constants c_{ob1}*,* T_{ob1} *have been defined in the observability condition OB1.*

Similarly, the observability condition OB2 is equivalent to the following observability condition:

OB4: *For every* $T \geq T_{ob2}$*, the very weak solution* ϕ *to* (29) *with* $(\phi, \dot{\phi}) \in C^0([0,T]; \mathcal{H}_2)$ *satisfies the inequality*

$$c_{ob2}\|(\phi_0^1, \phi_0^2)\|^2_{\mathcal{H}_2} \leq \int_0^T \int_{\Gamma_c} |\partial_\nu \mathcal{G}\dot{\phi}|^2 dSdt \quad \text{for every } (\phi_0^1, \phi_0^2) \in \mathcal{H}_2,$$

where the constants c_{ob2}*,* T_{ob2} *have been defined in the observability condition OB2.*

Proof. The proof can be found in the literature, e.g., [1, 41]. □

Lemma 3.7. *Suppose that* $T > 0$ *and* $u \in L^2(0,T; L^2(\Gamma_c))$*. Then the linear wave equation*

$$\begin{cases} \ddot{\psi} - \Delta\psi = 0 & \text{in } (0,T) \times \Omega, \\ \psi = u & \text{on } (0,T) \times \Gamma_c, \\ \psi = 0 & \text{on } (0,T) \times \Gamma_0, \\ (\psi(0), \dot{\psi}(0)) = (0,0) & \text{on } \Omega, \end{cases} \tag{50}$$

admits a unique very weak solution $\psi \in C^0([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$*. Moreover,* $\partial_\nu(\mathcal{G}\psi) \in H^1(0,T; L^2(\Gamma_c))$ *and we have the following estimate*

$$\|\partial_\nu(\mathcal{G}\psi)\|_{H^1(0,T;L^2(\Gamma_c))} \leq c_2' \|u\|_{L^2(0,T;L^2(\Gamma_c))}, \tag{51}$$

where the constant c_2' *depends only on* T*.*

Proof. The proof can be found in, e.g., [1]. □

Proposition 3.8. *Suppose that* $(y_0^1, y_0^2) \in \mathcal{H}_2$ *is given. Then the solution of the controlled system* (3) *with the feedback law* $u(y) := \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$ *converges exponentially to zero with respect to* $\mathcal{H}_2$*, i.e.*

$$\mathcal{E}(t,y) \leq Me^{-\alpha t}\mathcal{E}(0,y) = Me^{-\alpha t}\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_2} \tag{52}$$

for positive constants M*,* α *independent of* (y_0^1, y_0^2)*, if and only if the observability condition OB2 holds.*

Proof. The proof of the first direction in the equivalence can be found in, e.g., [1]. Nevertheless, we provide here a proof for completeness. First assume that condition OB2 holds. We show the exponential decay inequality (52).

Setting $u(y) := \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c}$ in (3), the resulting closed-loop system is well-posed (see, e.g., [1, 35]), and for its unique solution we have

$$y \in C([0,\infty); L^2(\Omega)) \cap C^1([0,\infty); H^{-1}(\Omega)),$$

and $\partial_\nu(\mathcal{G}\dot y)|_{\Gamma_c} \in L^2(0,\infty; L^2(\Gamma_c))$. Now, for an arbitrary $T > 0$ consider the following controlled system

$$\begin{cases} \ddot y - \Delta y = 0 & \text{in } (0,T)\times\Omega,\\ y = \partial_\nu(\mathcal{G}\dot y) & \text{on } (0,T)\times\Gamma_c,\\ y = 0 & \text{on } (0,T)\times\Gamma_0,\\ (y(0),\dot y(0)) = (y_0^1, y_0^2) & \text{on } \Omega. \end{cases} \tag{53}$$

Suppose that the solution y of (53) is smooth enough. Taking the L^2-inner product of (53) with $\mathcal{G}\dot y$, and integrating over $[0,T]$, we obtain the following estimate

$$\|(y(T),\dot y(T))\|^2_{\mathcal{H}_2} - \|(y(0),\dot y(0))\|^2_{\mathcal{H}_2} = -2\int_0^T \|\partial_\nu(\mathcal{G}\dot y(t))\|^2_{L^2(\Gamma_c)}dt. \tag{54}$$

By using a density argument and passing to the limit, it can be shown that equality (54) is also true for the initial pair $(y_0^1,y_0^2)\in\mathcal{H}_2$ and its corresponding solution y. Further, the solution y of (53) can be rewritten as $y = \phi+\psi$, where ϕ is the solution of (29) with the initial pair (y_0^1,y_0^2) in the place of (ϕ_0^1,ϕ_0^2), and ψ is the solution to (50) with $u = \partial_\nu(\mathcal{G}\dot y)|_{\Gamma_c}$. By Lemma 3.6 we may use OB4 for (29) with the initial pair (y_0^1,y_0^2). Together with (51) for ψ with $u = \partial_\nu(\mathcal{G}\dot y)|_{\Gamma_c}$, we obtain

$$\begin{aligned}\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_2} &\le \frac{1}{c_{ob2}}\int_0^{T_{ob2}}\int_{\Gamma_c}|\partial_\nu(\mathcal{G}\dot\phi)|^2 dSdt\\ &\le \frac{1}{c_{ob2}}\int_0^{T_{ob2}}\int_{\Gamma_c}\left(|\partial_\nu(\mathcal{G}\dot y)|^2 + |\partial_\nu(\mathcal{G}\dot\psi)|^2\right)dSdt \le \frac{1+{c_2'}^2}{c_{ob2}}\int_0^{T_{ob2}}\int_{\Gamma_c}|\partial_\nu(\mathcal{G}\dot y)|^2 dSdt,\end{aligned} \tag{55}$$

for $T_{ob2} > 0$ defined in the observability condition OB2. Combining (54) and (55), we have

$$\begin{aligned}\|(y(T_{ob2}),\dot y(T_{ob2}))\|^2_{\mathcal{H}_2} - \|(y(0),\dot y(0))\|^2_{\mathcal{H}_2} &= -2\int_0^{T_{ob2}}\|\partial_\nu(\mathcal{G}\dot y(t))\|^2_{L^2(\Gamma_c)}dt\\ &\le \frac{-2c_{ob2}}{1+{c_2'}^2}\|(y(0),\dot y(0))\|^2_{\mathcal{H}_2} \le \frac{-2c_{ob2}}{1+{c_2'}^2}\|(y(T_{ob2}),\dot y(T_{ob2}))\|^2_{\mathcal{H}_2}.\end{aligned}$$

As a result, we have

$$\mathcal{E}(t,y) \le Me^{-\alpha t}\mathcal{E}(0,y) \qquad \text{for every } t>0,$$

where $\alpha := \frac{\ln(1+\frac{2c_{ob2}}{1+{c_2'}^2})}{T_{ob2}}$ and $M := (1+\frac{2c_{ob2}}{1+{c_2'}^2})$.

Next we show that the stabilizability property (52) implies the observability condition OB2 or equivalently OB4. Due to (52) and (54), there exists a $T' > 0$ such that

$$\int_0^{T'}\|\partial_\nu(\mathcal{G}\dot y(t))\|^2_{L^2(\Gamma_c)}dt \ge \frac14\mathcal{E}(0,y). \tag{56}$$

Moreover, the very weak solution ϕ to (29) with the initial pair $(y_0^1,y_0^2)\in\mathcal{H}_2$ can be rewritten as $\phi := y-\psi$, where y is the solution to (53) and ψ is the solution to (50) with $u = \partial_\nu(\mathcal{G}\dot y)|_{\Gamma_c}$ for T' instead of T. By taking the L^2-inner product of (50) with $\mathcal{G}\dot\psi$ for $u = \partial_\nu(\mathcal{G}\dot y)\in L^2(0,T';L^2(\Gamma_c))$, and integrating over $[0,T']$, we obtain

$$\begin{aligned}0 \le \frac12\left(\|\dot\psi(T')\|^2_{H^{-1}(\Omega)} + \|\psi(T')\|^2_{L^2(\Omega)}\right) &= \int_0^{T'}\int_{\Gamma_c} -\partial_\nu(\mathcal{G}\dot y)\partial_\nu(\mathcal{G}\dot\psi)dSdt\\ &= \int_0^{T'}\int_{\Gamma_c} -\partial_\nu(\mathcal{G}\dot\psi+\mathcal{G}\dot\phi)\partial_\nu(\mathcal{G}\dot\psi)dSdt,\end{aligned} \tag{57}$$

By a density argument, it can be shown that the inequality (57) is also true for the very weak solution of (50) with $u := \partial_\nu(\mathcal{G}\dot{y})|_{\Gamma_c} \in L^2(0,T';L^2(\Gamma_c))$. Next, (57) implies that

$$\int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\psi}(t))\|^2_{L^2(\Gamma_c)}dt \le \int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\phi}(t))\|^2_{L^2(\Gamma_c)}dt. \tag{58}$$

Using (56), (58), and the following inequality

$$\int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\phi}(t))\|^2_{L^2(\Gamma_c)}dt \ge \int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{y}(t))\|^2_{L^2(\Gamma_c)}dt - \int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\psi}(t))\|^2_{L^2(\Gamma_c)}dt,$$

we complete the proof with

$$\int_0^{T'} \|\partial_\nu(\mathcal{G}\dot{\phi}(t))\|^2_{L^2(\Gamma_c)}dt \ge \frac{1}{8}\mathcal{E}(0,y).$$

□

Now we are in the position to investigate properties P2 and P3 for (3).

Proposition 3.9. *Suppose that the observability conditions OB1-OB2 hold. Then for every $T > 0$, there exists a control $\hat{u} \in L^2(0,T;L^2(\Gamma_c))$ for (48) such that*

$$V_T(\mathcal{Y}_0) \le J_T(\hat{u};\mathcal{Y}_0) \le \gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \tag{59}$$

for every initial pair $(y_0^1, y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_2$, where $\gamma_2(\cdot)$ is a nondecreasing, continuous, and bounded function. Moreover, there exists a constant $\gamma_1(T) > 0$ depending on T such that

$$V_T(\mathcal{Y}_0) \ge \gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_2} \tag{60}$$

for all $(y_0^1, y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_2$. Thus P2 and P3 hold.

Proof. Setting $u(t) := \partial_\nu(\mathcal{G}\dot{y}(t))|_{\Gamma_c}$ in the controlled system (48), and using Proposition 3.8 and OB2, we obtain

$$\|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_2} \le Me^{-\alpha t}\|(y(0),\dot{y}(0))\|^2_{\mathcal{H}_2} \quad \text{for all } t \in [0,T],$$

where the constants M and α were defined in Proposition 3.8. By integrating from 0 to T we have

$$\int_0^T \|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_2}dt \le \frac{M}{\alpha}(1-e^{-\alpha T})\|(y(0),\dot{y}(0))\|^2_{\mathcal{H}_2}.$$

Moreover, by (52) and (54) we have

$$\begin{aligned}\int_0^T \|u(t)\|^2_{L^2(\Gamma_c)}dt &= \int_0^T \|\partial_\nu(\mathcal{G}\dot{y}(t))\|^2_{L^2(\Gamma_c)}dt \\ &\le \frac{1}{2}\left(\mathcal{E}(0,y) + \mathcal{E}(T,y)\right) \le \frac{(1+M)}{2}\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_2}.\end{aligned} \tag{61}$$

Using (43), (61), and the definition of the value function V_T, we have

$$\begin{aligned}V_T(y_0^1,y_0^2) &\le \int_0^T \left(\frac{1}{2}\|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_2} + \frac{\beta}{2}\|u(t)\|^2_{L^2(\Gamma_c)}\right)dt \\ &\le \left(\frac{M}{2\alpha}(1-e^{-\alpha T}) + \frac{\beta(1+M)}{4}\right)\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_2} = \gamma_2(T)\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_2},\end{aligned}$$

which gives (59).

To verify (60), we use the superposition argument for (48) with an arbitrary control $u \in L^2(0,T;L^2(\Gamma_c))$. We rewrite the solution of (48) as $y = \phi + \psi$ where ϕ is the solution to (29) with the initial pair (y_0^1, y_0^2) and ψ is the solution to (50).

By Lemma 3.6 we can use the observability condition OB3 for (29) with the initial pair (y_0^1, y_0^2) and Ω instead of ω. Together with (47) for (50), we obtain

$$\|(y_0^1, y_0^2)\|^2_{\mathcal{H}_2} \le \frac{1}{c_{ob1}} \int_0^T \|\phi(t)\|^2_{L^2(\Omega)} dt \le \frac{1}{c_{ob1}} \int_0^T \left(\|y(t)\|^2_{L^2(\Omega)} + \|\varphi(t)\|^2_{L^2(\Omega)} \right) dt$$
$$\le \frac{1}{c_{ob1}} \int_0^T \left(\|y(t)\|^2_{L^2(\Omega)} + T c_2^2 \|u(t)\|^2_{L^2(\Gamma_c)} \right) dt$$
$$\le c_2''(T) \int_0^T \left(\frac{1}{2} \|(y(t), \dot{y}(t))\|^2_{\mathcal{H}_2} + \frac{\beta}{2} \|u(t)\|^2_{L^2(\Gamma_c)} \right) dt = c_2''(T) \int_0^T \ell(\mathcal{Y}(t), u(t)) dt.$$

Since $u \in L^2(0, T; L^2(\Gamma_c))$ is arbitrary, we obtain (60) for a constant $c_2''(T)$ independent of u and (y_0^1, y_0^2). $\square$

Remark 3.10. From Propositions 3.4 and 3.9, it follows that Theorem 1.5 is applicable for (3).

4. Neumann Boundary Control.

In this section, we consider (4) where $L > 0$, $u \in L^2(0, \infty)$, and $(y_0^1, y_0^2) \in V \times L^2(0, L)$ with $V := \{q \in H^1(0, L) : q(0) = 0\}$. The function space V is equipped with the following scalar product

$$(\phi, \psi) := \int_0^L \phi_x \psi_x dx.$$

Moreover, V^* stands for the dual space of V. Similarly to the previous sections we define $\mathcal{H}_3 := V \times L^2(0, L)$ with its corresponding energy

$$\mathcal{E}(t, y) := \|y(t)\|^2_V + \|\dot{y}(t)\|^2_{L^2(0,L)},$$

along a trajectory y. The incremental function $\ell : V \times L^2(0, L) \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\ell((y, z), u) := \frac{1}{2} \|(y, z)\|^2_{\mathcal{H}_3} + \frac{\beta}{2} u^2. \tag{62}$$

Moreover, later we will use the space $V^2 := \{q \in H^2(0, L) \cap V : q_x(L) = 0\}$.

Remark 4.1. Note that, for the case $dim(\Omega) \ge 2$, the generalization of the controlled system (4) has the form

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0, \infty) \times \Omega, \\ \partial_\nu y = u & \text{on } (0, \infty) \times \Gamma_c, \\ y = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ (y(0), \dot{y}(0)) = (y_0^1, y_0^2) & \text{on } \Omega, \end{cases} \tag{63}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega := \overline{\Gamma_c} \cup \overline{\Gamma_0}$. The two components Γ_c, Γ_0 are relatively open in $\partial\Omega$, disjoint, and $int(\Gamma_c) \ne \emptyset$. Moreover $u \in L^2(0, \infty; L^2(\Gamma_c))$, $(y_0^1, y_0^2) \in \mathcal{H}_3$ with $\mathcal{H}_3 := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, and the functional space $H^1_{\Gamma_0}(\Omega)$ is defined by $H^1_{\Gamma_0}(\Omega) := \{f \in H^1(\Omega) : f|_{\Gamma_0} = 0\}$. For our framework we would require that the solution operator is continuous from $L^2(0, T; L^2(\Gamma_c))$ to $C^0([0, T]; \mathcal{H}_3)$. However, this property does not hold as was shown in [36]. In fact, the solution $(y(\cdot), \dot{y}(\cdot))$ belongs to a strictly larger space than $\mathcal{H}_3$. However, in the larger space we have no stabilizability results.

Consider now the following one dimensional wave equation with an inhomogeneous Neumann boundary condition

$$\begin{cases} \ddot{y} - y_{xx} = 0 & \text{in } (0,T)\times(0,L), \\ y(\cdot,0) = 0 & \text{in } (0,T), \\ y_x(\cdot,L) = u & \text{in } (0,T), \\ (y(0,\cdot),\dot{y}(0,\cdot)) = (y_0^1,y_0^2) & \text{in } (0,L). \end{cases} \tag{64}$$

Definition 4.2 (Weak solution)**.** Let $T > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_3$, and $u \in L^2(0,T)$ be given. Then y is referred to as the weak solution to (64) if $(y, \dot{y}) \in C^0([0,T]; \mathcal{H}_3)$, $(y(0), \dot{y}(0)) = (y_0^1, y_0^2)$, and for every $\vartheta \in C^1([0,T] \times [0,L])$ with $\vartheta(0,\tau) = 0, \forall \tau \in [0,T]$, it satisfies

$$\begin{aligned} \int_0^L \dot{y}(t,x)\vartheta(t,x)dx - \int_0^L \ y_0^2(x)\vartheta(0,x)dx \\ + \int_0^t \int_0^L \left(-\dot{\vartheta}\dot{y} + \vartheta_x y_x\right) dx d\tau - \int_0^t u(\tau)\vartheta(\tau,L)d\tau = 0 \end{aligned} \tag{65}$$

for almost every $t \in (0,T)$.

Proposition 4.3. *Let $T > 0$, $L > 0$, $(y_0^1, y_0^2) \in \mathcal{H}_3$, and $u \in L^2(0,T)$ be given. Then there exists a unique weak solution $y \in C^0([0,T]; V) \cap C^1([0,T]; L^2(0,L))$ satisfying*

$$\begin{aligned} \|y\|_{C^0([0,T];V)} + \|\dot{y}\|_{C^0([0,T];L^2(0,L))} + \|\ddot{y}\|_{L^2(0,T;V^*)} \\ \le c_3 \left(\|y_0^1\|_V + \|y_0^2\|_{L^2(0,L)} + \|u\|_{L^2(0,T)}\right), \end{aligned} \tag{66}$$

where the constant c_3 depends only on T and L. Furthermore, $y(\cdot,L) \in H^1(0,T)$ and we have

$$\|\dot{y}(\cdot,L)\|_{L^2(0,T)} \le c_4 \left(\|y_0^1\|_V + \|y_0^2\|_{L^2(0,L)} + \|u\|_{L^2(0,T)}\right), \tag{67}$$

for a constant c_4 depending only on L and T.

Proof. The proof is given in, e.g., [15, page 68]. □

Remark 4.4. Thus, (Est) follows from (66) for (4) with respect to $\mathcal{H}_3$-norm.

We will later use the following auxiliary problem.

$$\begin{cases} \ddot{y} - y_{xx} = f & \text{in } (0,T)\times(0,L), \\ y(\cdot,0) = 0 & \text{in } (0,T), \\ y_x(\cdot,L) = 0 & \text{in } (0,T), \\ (y(0,\cdot),\dot{y}(0,\cdot)) = (y_0^1,y_0^2) & \text{in } (0,L). \end{cases} \tag{68}$$

Definition 4.5 (Very weak solution)**.** Let $L > 0$, $T > 0$, $(y_0^1, y_0^2) \in L^2(0,L) \times V^*$, and $f \in L^2(0,T;V^*)$ be given. A function $y \in L^2(0,T;L^2(0,L))$ is referred to as the very weak solution of (68), if

$$\int_0^T (g(t),y(t))_{L^2(0,L)}\, dt = -(y_0^1, \dot{\vartheta}(0))_{L^2(0,L)} + \langle y_0^2, \vartheta(0)\rangle_{V^*,V} + \int_0^T \langle f(t), \vartheta(t)\rangle_{V^*,V}\, dt,$$

for all $g \in L^2(0,T;L^2(0,L))$, where $\vartheta \in C^1([0,T];L^2(0,L)) \cap C^0([0,T];V)$ is the weak solution of the following backward in time problem

$$\begin{cases} \ddot{\vartheta} - \vartheta_{xx} = g & \text{in } (0,T)\times(0,L), \\ \vartheta(\cdot,0) = 0 & \text{in } (0,T), \\ \vartheta_x(\cdot,L) = 0 & \text{in } (0,T), \\ (\vartheta(T,\cdot),\dot{\vartheta}(T,\cdot)) = (0,0) & \text{in } (0,L). \end{cases}$$

We have the following existence and regularity results for the very weak solution of (68), see, e.g., [40, 42].

Proposition 4.6. *For every $L > 0$, $T > 0$, $f \in L^2(0,T;V^*)$, and $(y_0^1, y_0^2) \in L^2(\Omega) \times V^*$, there exists a unique very weak solution to (68) in $C^1([0,T];V^*) \cap C^0([0,T];L^2(0,L))$, and we have the following estimate*

$$\|y\|_{C^0([0,T];L^2(0,L))} + \|\dot{y}\|_{C^0([0,T];V^*)} \le \bar{c}_4 \left(\|y_0^1\|_{L^2(0,L)} + \|y_0^2\|_{V^*} + \|f\|_{L^2(0,T;V^*)} \right), \tag{69}$$

where the constant $\bar{c}_4$ is independent of y_0^1, y_0^2, and f.

4.1. **On the finite horizon optimal control problems.** Analogously to the previous chapters, in this subsection we investigate well-posedness and the first-order optimality conditions for the following optimal control problem

$$\min \left\{ J_T(u;(y_0^1,y_0^2)) \mid (y,u) \text{ satisfies } (64), u \in L^2(0,T) \right\}, \tag{OP_{neu}}$$

where the performance index function J_T is given by

$$J_T(u;(y_0^1,y_0^2)) := \int_0^T \ell((y(t),\dot{y}(t)),u(t))dt$$

with the incremental function ℓ defined by (62) and $(y_0^1, y_0^2) \in \mathcal{H}_3$.

Proposition 4.7. *For every $T > 0$ and $(y_0^1, y_0^2) \in \mathcal{H}_3$, the optimal control problem (OP_{neu}) admits a unique solution.*

Proof. We use the standard argument of calculus of variation. Since the objective function $J_T(u;(y_0^1,y_0^2))$ is coercive and bounded from below, there is a bounded minimizing sequence $\{u^n\}_n \subset L^2(0,T)$ such that

$$J_T(u^n;(y_0^1,y_0^2)) \to \inf_{u \in L^2(0,T)} J_T(u;(y_0^1,y_0^2)) = \sigma < \infty.$$

Moreover, this sequence has a weakly convergent subsequence $u^n \rightharpoonup u^*$ with the limit $u^* \in L^2(0,T)$. Due to Proposition 4.3 and (66), there exists a bounded sequence of very weak solutions $\{y^n\}_n \subset L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(0,L))$ to (64) corresponding to the control sequence $\{u^n\}_n$. Hence, there are weakly-star convergent subsequences of $\{y^n\}_n$ and $\{\dot{y}^n\}_n$ such that

$$y^n \rightharpoonup^* y^* \text{ in } L^\infty(0,T;V), \quad \dot{y}^n \rightharpoonup^* \dot{y}^* \text{ in } L^\infty(0,T;L^2(0,L)), \quad \ddot{y}^n \rightharpoonup \ddot{y}^* \text{ in } L^2(0,T;V^*).$$

It remains to show that y^* is the weak solution to (64) corresponding to the control u^*. To see this, we only need to pass to the limit in the weak formulation (65) for the pair of sequences (y^n, u^n). For every $\vartheta \in C^1([0,T]\times[0,L])$ such that $\vartheta(0,\tau) = 0$ for all $\tau \in [0,T]$, we have for every $t \in [0,T]$

$$\int_0^t \int_0^L (\dot{y}^n(\tau,x) - \dot{y}^*(\tau,x))\vartheta(\tau,x)dxd\tau \to 0,$$
$$\int_0^t \int_0^L (\dot{y}^n(\tau,x) - \dot{y}^*(\tau,x))\dot{\vartheta}(\tau,x)dxd\tau \to 0,$$
$$\int_0^t \int_0^L ((y^n)_x(\tau,x) - \dot{y}_x^*(\tau,x))\vartheta_x(\tau,x)dxd\tau \to 0,$$
$$\int_0^t (u^n(\tau) - u^*(\tau))\vartheta(\tau,L)d\tau \to 0.$$

Moreover, due to (66), for every $t \in [0,T]$ the sequence $\{\dot{y}^n(t)\}_n$ is bounded in $L^2(0,L)$. Hence, it has a weakly convergent subsequence $\dot{y}^n(t) \rightharpoonup \bar{y}_t$ with limit

$\bar{y}_t \in L^2(0,L)$. For any $t \in [0,T]$, we define $\mathcal{I}_t : H^1(0,T;V^*) \to V^*$ by $p \mapsto p(t)$. This operator is continuous, moreover, for every $q \in V$ we have

$$\begin{aligned}(\bar{y}_t, q)_{L^2(0,L)} &= \lim_{n\to\infty} \langle \mathcal{I}_t \dot{y}^n, q\rangle_{V^*,V} = \lim_{n\to\infty} \langle \dot{y}^n, \mathcal{I}_t^* q\rangle_{H^1(0,T;V^*),(H^1(0,T;V^*))^*} \\ &= \langle \dot{y}^*, \mathcal{I}_t^* q\rangle_{H^1(0,T;V^*),(H^1(0,T;V^*))^*} = \langle \mathcal{I}_t \dot{y}^*, q\rangle_{V^*,V},\end{aligned} \tag{70}$$

where $\mathcal{I}_t^* : V \to (H^1(0,T;V^*))^*$ is the adjoint operator to $\mathcal{I}_t$. Therefore, for every $\vartheta \in C^1([0,T]\times[0,L])$ such that $\vartheta(0,\tau) = 0$ for all $\tau \in [0,T]$, we have for almost every $t \in [0,T]$

$$\int_0^L \dot{y}^n(t,x)\vartheta(t,x)dx \to \int_0^L \dot{y}^*(t,x)\vartheta(t,x)dx, \tag{71}$$

and, as a consequence, we can pass to the limit in (65) with y replaced by y^* and y^* is the weak solution to (64) corresponding to the control u^*. Now since the solution operator $S : L^2(0,T) \to L^\infty(0,T;\mathcal{H}_3)$ defined by $u \mapsto (y,\dot{y})$ is affine and continuous, the objective function $J_T(\cdot; y_0^1, y_0^2)$ is weakly lower semi-continuous and we have

$$0 \le J_T(u^*; (y_0^1, y_0^2)) \le \liminf_{n\to\infty} J_T(u^n; (y_0^1, y_0^2)) = \sigma.$$

As a result the pair (y^*, u^*) is optimal. Uniqueness follows from the strictly convexity of $J_T(\cdot; (y_0^1, y_0^2))$. □

We turn to the first-order optimality conditions for (OP_{neu}). Due to the presence of the tracking term for the velocity $\dot{y} \in L^2(0,T;L^2(0,L))$ in the objective function of (OP_{neu}), the solution to the ajdoint equation has less regularity than the one to (64) and exists in the very weak sense only.

Proposition 4.8. *Let $(\bar{y}, \bar{u})$ be the optimal solution to (OP_{neu}). It satisfies the following optimality conditions*

$$\begin{cases} \ddot{\bar{y}} - \bar{y}_{xx} = 0 & \text{in } (0,T)\times(0,L), \\ \bar{y}(\cdot,0) = 0 & \text{in } (0,T), \\ \bar{y}_x(\cdot,L) = \bar{u} & \text{in } (0,T), \\ (\bar{y}(0,\cdot), \dot{\bar{y}}(0,\cdot)) = (y_0^1, y_0^2) & \text{in } (0,L), \end{cases} \quad \begin{cases} \ddot{\bar{p}} - \bar{p}_{xx} = \ddot{\bar{y}} + \bar{y}_{xx} & \text{in } (0,T)\times(0,L), \\ \bar{p}(\cdot,0) = 0 & \text{in } (0,T), \\ \bar{p}_x(\cdot,L) = 0 & \text{in } (0,T), \\ (\bar{p}(T,\cdot), \dot{\bar{p}}(T,\cdot)) = (0, \dot{\bar{y}}(T,\cdot)) & \text{in } (0,L), \end{cases}$$

and

$$\beta\bar{u}(\cdot,L) = \bar{p}(\cdot,L) \quad \text{in } (0,T),$$

where $\bar{p}$ the solution of the adjoint equation belongs to the space $C^0([0,T];L^2(0,L)) \cap C^1([0,T];V^*)$.

Proof. The proof is given in Appendix A.3. □

4.2. **Verification of P2 and P3.** To specify the required *observability conditions*, for any given $(\phi_0^1, \phi_0^2) \in \mathcal{H}_3$, we denote by ϕ the weak solution of the following system

$$\begin{cases} \ddot{\phi} - \phi_{xx} = 0 & \text{in } (0,T)\times(0,L), \\ \phi(\cdot,0) = 0 & \text{in } (0,T), \\ \phi_x(\cdot,L) = 0 & \text{in } (0,T), \\ (\phi(0,\cdot), \dot{\phi}(0,\cdot)) = (\phi_0^1, \phi_0^2) & \text{in } (0,L). \end{cases} \tag{72}$$

Then we formulate the following observability inequalities:

OB5: There exists $T_{ob3} > 0$ such that for every $T \geq T_{ob3}$, the weak solution ϕ to (72) with $(\phi, \dot{\phi}) \in C^0([0,T];\mathcal{H}_3)$ satisfies the inequality

$$c_{ob3}\|(\phi_0^1,\phi_0^2)\|^2_{\mathcal{H}_3} \leq \int_0^T |\dot{\phi}(t,L)|^2 dt \quad \text{for every } (\phi_0^1,\phi_0^2) \in \mathcal{H}_3,$$

where the positive constant c_{ob3} depends only on T and L.

OB6: There exists $T_{ob4} > 0$ such that for every $T \geq T_{ob4}$, the weak solution ϕ to (72) with $(\phi, \dot{\phi}) \in C^0([0,T];\mathcal{H}_3)$ satisfies the inequality

$$c_{ob4}\|(\phi_0^1,\phi_0^2)\|^2_{\mathcal{H}_3} \leq \int_0^T \int_\omega |\dot{\phi}|^2 dx dt \quad \text{for every } (\phi_0^1,\phi_0^2) \in \mathcal{H}_3,$$

where the positive constant c_{ob4} depends only on T and $\omega \subseteq (0,L)$.

Proposition 4.9. *Suppose that $(y_0^1,y_0^2) \in \mathcal{H}_3$ is given. Then the solution of the controlled system (4) with the feedback law $u(t) := -\dot{y}(t,L)$ converges exponentially to zero with respect to $\mathcal{H}_3$, i.e.*

$$\mathcal{E}(t,y) \leq Me^{-\alpha t}\mathcal{E}(0,y) = Me^{-\alpha t}\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_3} \tag{73}$$

for positive constants M and α independent of (y_0^1,y_0^2), if and only if the observability condition OB5 holds.

Proof. The proof can be found in e.g., [50]. □

Proposition 4.10. *Suppose that the observability conditions OB5-OB6 hold. For every $T > 0$, there exists a control $\hat{u} \in L^2(0,T)$ for (64) such that*

$$V_T(\mathcal{Y}_0) \leq J_T(\hat{u};\mathcal{Y}_0) \leq \gamma_2(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_3} \tag{74}$$

for every initial pair $(y_0^1,y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_3$, where $\gamma_2(\cdot)$ is a nondecreasing, continuous, and bounded function. Moreover, there exists a constant $\gamma_1(T) > 0$ depending on T such that

$$V_T(\mathcal{Y}_0) \geq \gamma_1(T)\|\mathcal{Y}_0\|^2_{\mathcal{H}_3} \tag{75}$$

for all $(y_0^1,y_0^2) = \mathcal{Y}_0 \in \mathcal{H}_3$. Thus P2 and P3 hold.

Proof. Setting $u(t) := -\dot{y}(t,L)$ in the controlled system (64), we obtain the following closed-loop system

$$\begin{cases} \ddot{y} - y_{xx} = 0 & \text{in } (0,T)\times(0,L), \\ y(\cdot,0) = 0 & \text{in } (0,T), \\ y_x(\cdot,L) = -\dot{y}(\cdot,L) & \text{in } (0,T), \\ (y(0,\cdot),\dot{y}(0,\cdot)) = (y_0^1,y_0^2) & \text{in } (0,L). \end{cases} \tag{76}$$

Suppose that the solution y of (76) is smooth enough. By taking the L^2-inner product of (76) with $\dot{y}$, and integrating over $[0,T]$, we obtain

$$\|(y(T),\dot{y}(T))\|^2_{\mathcal{H}_3} - \|(y(0),\dot{y}(0))\|^2_{\mathcal{H}_3} = -2\int_0^T |\dot{y}(t,L)|^2 dt. \tag{77}$$

By using a density argument, it can be shown that (77) is also true for the weak solution of (76). Further, by Proposition 4.9 we obtain

$$\|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_3} \leq Me^{-\alpha t}\|(y(0),\dot{y}(0))\|^2_{\mathcal{H}_3} \quad \text{for all } t \in [0,T],$$

where the constants M and α were defined in Proposition 4.9. Integrating from 0 to T we have

$$\int_0^T \|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_3} dt \leq \frac{M}{\alpha}(1-e^{-\alpha T})\|(y(0),\dot{y}(0))\|^2_{\mathcal{H}_3}.$$

From (73) and (77) we conclude that

$$\int_0^T |u(t)|^2 dt = \int_0^T |\dot{y}(t,L)|^2 dt \leq \frac{1}{2}\left(\mathcal{E}(0,y)+\mathcal{E}(T,y)\right) \leq \frac{(1+M)}{2}\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_3}. \tag{78}$$

By (62), (78), and the definition of the value function V_T, we have

$$\begin{aligned} V_T(y_0^1,y_0^2) &\leq \int_0^T \left(\frac{1}{2}\|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_3} + \frac{\beta}{2}|u(t)|^2\right) dt \\ &\leq \left(\frac{M}{2\alpha}(1-e^{-\alpha T}) + \frac{\beta(1+M)}{4}\right)\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_3} = \gamma_2(T)\|(y_0^1,y_0^2)\|^2_{\mathcal{H}_3}. \end{aligned}$$

To verify (75), we use the superposition argument for (64) with an arbitrary control $u \in L^2(0,T)$. We express the solution of (64) as $y = \phi + \psi$ where ϕ is the solution to (72) with the initial pair (y_0^1,y_0^2), and ψ is the solution to the following problem

$$\begin{cases} \ddot{\psi} - \psi_{xx} = 0 & \text{in } (0,T)\times(0,L), \\ \psi(\cdot,0) = 0 & \text{in } (0,T), \\ \psi_x(\cdot,L) = u & \text{in } (0,T), \\ (\psi(0,\cdot),\dot{\psi}(0,\cdot)) = (0,0) & \text{in } (0,L). \end{cases} \tag{79}$$

By using the observability condition OB6 for (72) with the initial pair (y_0^1,y_0^2) and $(0,L)$ instead of ω, and using estimate (66) for (79) we obtain

$$\begin{aligned} \|(y_0^1,y_0^2)\|^2_{\mathcal{H}_3} &\leq \frac{1}{c_{ob4}}\int_0^T \|\dot{\phi}(t)\|^2_{L^2(0,L)} dt \leq \frac{1}{c_{ob4}}\int_0^T \left(\|\dot{y}(t)\|^2_{L^2(0,L)} + \|\dot{\varphi}(t)\|^2_{L^2(0,L)}\right) dt \\ &\leq \frac{1}{c_{ob4}}\int_0^T \left(\|\dot{y}(t)\|^2_{L^2(0,L)} + Tc_3^2|u(t)|^2\right) dt \\ &\leq c'(T)\int_0^T \left(\frac{1}{2}\|(y(t),\dot{y}(t))\|^2_{\mathcal{H}_3} + \frac{\beta}{2}|u(t)|^2\right) dt = c'(T)\int_0^T \ell(\mathcal{Y}(t),u(t))dt. \end{aligned}$$

Since $u \in L^2(0,T)$ is arbitrary, we obtain (75) for a constant $c'(T)$ independent of u and (y_0^1,y_0^2). □

Remark 4.11. From Propositions 4.7 and 4.10, it follows that Theorem 1.5 is applicable for (4).

5. Numerical Experiments.

This section is devoted to numerical simulations. In order to justify our theoretical results for the receding horizon Algorithm 1, we give numerical results for all the cases: Distributed control, Dirichlet boundary control, and Neumann boundary control. We give also a short description about the discretization of the control and the state, the optimization algorithm, and the implementation of Algorithm 1.

5.1. Discretization.

Among the many discretization approaches to the wave equation based on finite elements, we can mention the works [4, 5, 6, 7, 30, 31]. Here we follow the framework which was investigated in [7] and applied for optimal control problems in [32]. In this framework, the open-loop problems are discretized, temporally and spatially, by appropriate finite elements, for which the approaches optimize-discretize and discretize-optimize commute; see, e.g., [10]. In all cases, for the discretization of the state we write the equation as a system of first order equations in time. The spatial discretization was done by a conforming linear finite element scheme using continuous piecewise linear basis functions over a uniform mesh. This uniform mesh was generated by triangulation. For the temporal discretization of the state equation, a Petrov-Galerkin scheme based on continuous piecewise linear basis functions for the trial space and piecewise constant test functions was employed. By doing so, the resulting discretized system is equivalent to the system first discretized in space followed by the Crank-Nicolson time stepping method. Since the temporal test functions have been chosen to be piecewise constant, it is natural to also discretize the adjoint equation and also control by these functions. This implies that the approximated gradient is consistent with both continuous functional and the discrete functional. In the case of the Dirichlet boundary control, the inhomogeneous Dirichlet condition $y|_{\Gamma_c} = u$ was treated by interpreting u as the trace of a sufficiently smooth function $\hat{y}$ and solving the equation for $v = y - \hat{y}$ instead of y with homogeneous Dirichlet boundary conditions, see, e.g., [18, page 376] for more detail.

5.2. **Optimization.** Every discretized open-loop problem was first formulated as reduced problem. The resulting unconstrained optimization problem consists of minimizing a reduced objective function which depends only on the control variable u. Then these reduced problems were solved by applying the Barzilai-Borwein (BB) method [9] equipped with a nonmonotone line search [17]. The optimization algorithm was terminated as soon as the $L^2(0,T;\mathcal{U})$-norm of the gradient for the reduced objective function was less than the tolerance 10^{-6}.

5.3. **Implementation of RHC.** Turning to the numerical experiments, we considered three examples corresponding to the cases: distributed control (2), Dirichlet boundary control (3), and Neumann boundary control (4). We applied Algorithm 2 which is based on Algorithm 1. For a given initial pair $(y_0^1, y_0^2) =: \mathcal{Y}_0 \in \mathcal{H}$ and a

Algorithm 2 RHC$(\mathcal{Y}_0, T_\infty)$

Input: Let a final computational time horizon T_∞, and an initial state $\mathcal{Y}_0 := (y_0^1, y_0^2) \in \mathcal{H}$ be given.

1: Choose a prediction horizon $T < T_\infty$ and a sampling time $\delta \in (0, T]$.

2: Consider a grid $0 = t_0 < t_1 < \cdots < t_r = T_\infty$ on the interval $[0, T_\infty]$ where $t_i = i\delta$ for $i = 0, \dots, r$.

3: **for** $i = 0, \dots, r-1$ **do**

Solve the open-loop subproblem on $[t_i, t_i + T]$

$$\min \frac{1}{2}\int_{t_i}^{t_i+T} \|\mathcal{Y}(t)\|_{\mathcal{H}}^2 dt + \frac{\beta}{2}\int_{t_i}^{t_i+T} \|u(t)\|_{\mathcal{U}}^2 dt,$$

$$\text{subject to } \begin{cases} \dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{B}u \quad t \in (t_k, t_k + T), \\ \mathcal{Y}(t_i) = \mathcal{Y}_T^*(t_i) \text{ if } i \geq 1 \text{ or } \mathcal{Y}(t_i) = (y_0^1, y_0^2) \text{ if } i = 0, \end{cases}$$

where $\mathcal{Y}_T^*(\cdot)$ is the solution to the previous subproblem on $[t_{i-1}, t_{i-1} + T]$.

4: The model predictive pair $(\mathcal{Y}_{rh}^*(\cdot), u_{rh}^*(\cdot))$ is the concatenation of the optimal pairs $(\mathcal{Y}_T^*(\cdot), u_T^*(\cdot))$ on the finite horizon intervals $[t_i, t_{i+1}]$ with $i = 0, \dots, r-1$.

constant T_∞ defined as the final computation time, we ran Algorithm 2 for all the above mentioned cases. For every example, the receding horizon control u_{rh} was computed for the fixed sampling time $\delta = 0.25$ and different values of the prediction horizon T. In each example, the performance of the computed receding horizon controls for different prediction horizons are compared with each other. Moreover, in order to get more intuition about the stabilization problem, the results related to the uncontrolled problem are also reported. As performance criteria for our comparison, we considered the following quantities:

1. $J_{T_\infty}(u_{rh}, \mathcal{Y}_0) := \frac{1}{2}\int_0^{T_\infty} \|\mathcal{Y}_{rh}(t)\|^2_{\mathcal{H}} dt + \frac{\beta}{2}\int_0^{T_\infty} \|u_{rh}(t)\|^2_{\mathcal{U}} dt$,
2. $\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H})}$,
3. $\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}}$,
4. iter : the *total* number of iterations (BB-gradient steps) that the optimizer needs for all open-loop problems on the intervals (t_i, t_i+T) for $i = 0, \dots, r-1$.

5.4. **Numerical examples.** For the cases distributed control (2) and Dirichlet control (3), we considered the unit square $(0,1)^2 \subset \mathbb{R}^2$ as the spatial domain Ω. This spatial domain was discretized by using $N = 4225$ cells. Moreover for the case of Neumann control (4), the string equation is defined on the interval $(0,1)$. In this case, the spatial discretization was also done by the standard Galerkin method based on continuous piecewise linear basis functions with mesh-size $h = 0.0125$ and the time discretization described in Subsection 5.1 was used. For all examples, the step-size $\Delta t = 0.0025$ was chosen for time discretization. The numerical simulations were carried out on the MATLAB platform.

Example 5.1 (Distributed control)**.** In this example we applied Algorithm 2 to the infinite horizon problem (1)-(2) with ℓ defined by (21). We set $\mathcal{U} := L^2(\omega)$ with ω depicted in Figure 2(a), $\beta = 0.1$, $T_\infty = 15$, and

$$y_0^1(x) := 5e^{-20((x_1-0.5)^2+(x_2-0.5)^2)}, \qquad y_0^2(x) = 0,$$

where $x := (x_1, x_2) \in \Omega$. Before applying Algorithm 2, we investigate the uncontrolled system. For this case we obtained the following quantities:

$$\|\mathcal{Y}\|_{L^2(0,T_\infty;\mathcal{H}_1)} = 1.17 \times 10^3, \qquad \|\mathcal{Y}(T_\infty)\|_{\mathcal{H}_1} = 78.57.$$

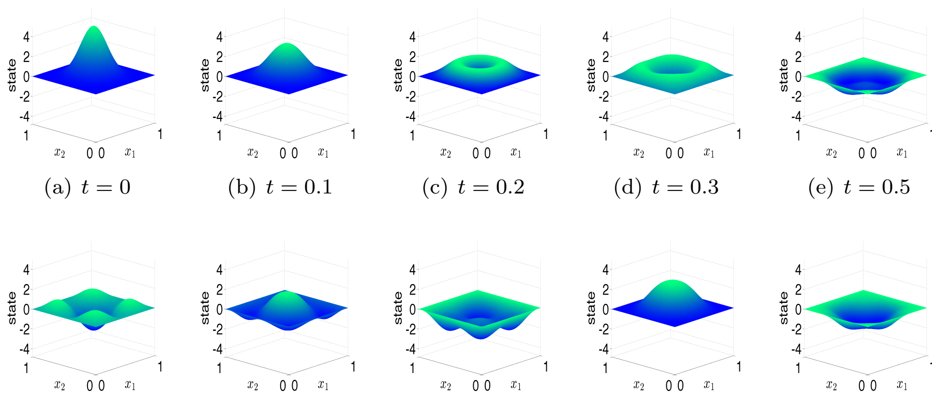
In fact, for this system the $\mathcal{H}_1$-energy is conserved in time, i.e.,

$$\|\mathcal{Y}(t)\|_{\mathcal{H}_1} = \|(y_0^1, y_0^2)\|_{\mathcal{H}_1} = 78.57 \qquad \text{for all } t \in [0, T_\infty],$$

where $\mathcal{H}_1 = H_0^1(\Omega) \times L^2(\Omega)$. As it is depicted by Figure 1, a single wave propagates and moves from the center of the domain to the boundaries. While moving to the boundaries, it decomposes into several small waves. After hitting the boundaries, the resulting small waves propagate and join together to form a single wave at the center of the domain. This process repeats constantly, as time progresses. We employed RHC computed by Algorithm 2 for different choices of the prediction horizon T and the fixed sampling time $\delta = 0.25$. The corresponding results are gathered in Table 1. Figure 4(a) demonstrates the evolution of the $\mathcal{H}_1$-energy of the receding horizon states for the different choices of T and fixed $\delta = 0.25$. The evolution of the $L^2(\omega)$-norm of the corresponding RHCs are plotted in Figure 3. Figure 5 shows the receding horizon state at different time points for the choice of $T = 1.5$. As expected longer T provides better stabilization performance but requires more iterations.

Prediction Horizon	J_{T_∞}	$\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H}_1)}$	$\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}_1}$	iter
$T = 1.5$	8.20×10^2	40.19	2.62×10^{-8}	1515
$T = 1$	1.13×10^3	47.40	3.03×10^{-6}	847
$T = 0.5$	3.13×10^3	79.10	2.00×10^{-3}	550
$T = 0.25$	1.94×10^4	197.43	3.79×10^{-1}	373

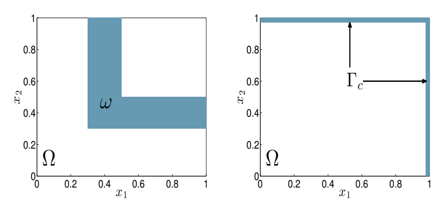
Table 1. Numerical results for Example 5.1



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.2$ (d) $t = 0.3$ (e) $t = 0.5$

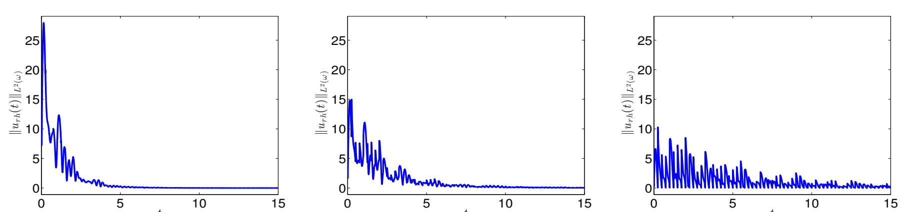
(f) $t = 1$ (g) $t = 2.5$ (h) $t = 5$ (i) $t = 10$ (j) $t = 15$

FIGURE 1. Snapshots of the uncontrolled state corresponding to Example 5.1



(a) Example 5.1 (b) Example 5.2

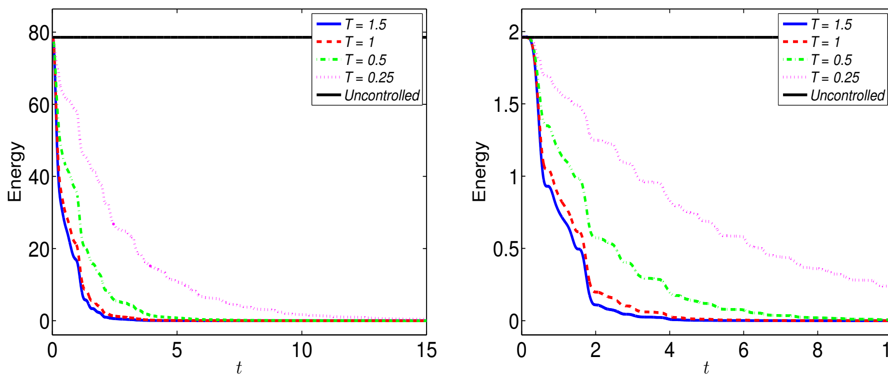
FIGURE 2. Control domains



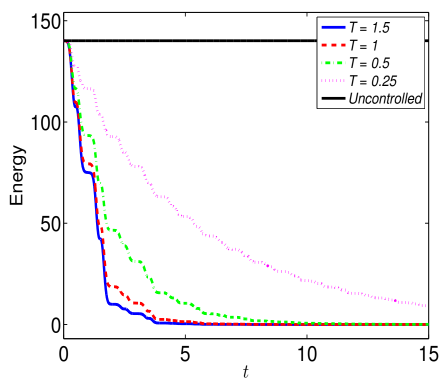
(a) $T = 1.5$ (b) $T = 0.5$ (c) $T = 0.25$

FIGURE 3. Evolution of $L^2(\omega)$-norm for RHC corresponding to Example 5.1 with different prediction horizons T

Example 5.2 (Dirichlet control)**.** Here we considered the stabilization of the wave equation (3) by Dirichlet boundary control. We set $\mathcal{U} := L^2(\Gamma_c)$, $T_\infty = 10$, $\beta = 1$, and chose the same initial pair (y_0^1, y_0^2) as in the previous example. The stabilization task was done with respect to the energy $\mathcal{H}_2 = L^2(\Omega) \times H^{-1}(\Omega)$ which is different from the one in the previous example. For the uncontrolled state, the $\mathcal{H}_2$-energy is conserved over the time. More precisely, we have $\|\mathcal{Y}(t)\|_{\mathcal{H}_2} = 1.96$ for all $t \in [0, T_\infty]$, and also $\|\mathcal{Y}\|_{L^2(0,T_\infty;\mathcal{H}_2)} = 19.60$. The receding horizon Dirichlet control is active on a subset $\Gamma_c \subset \partial\Omega$ as illustrated in Figure 2(b). Similarly to the previous example,

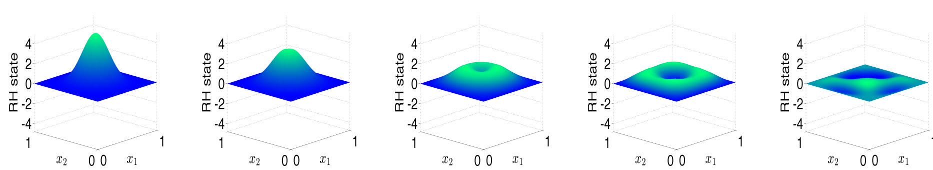


(a) Example 5.1 for energy $\mathcal{H}_1$ (b) Example 5.2 for energy $\mathcal{H}_2$

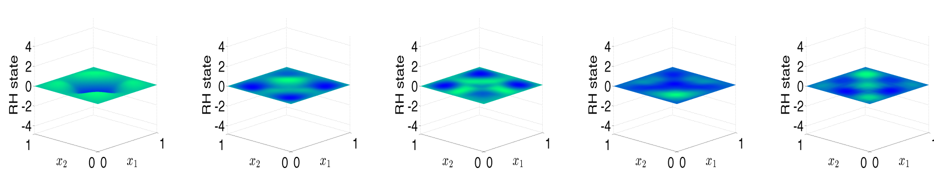


(c) Example 5.3 for energy $\mathcal{H}_3$

FIGURE 4. Evolution of $\|\mathcal{Y}_{rh}(t)\|_{\mathcal{H}}$ for different choices of T



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.2$ (d) $t = 0.3$ (e) $t = 0.5$



(f) $t = 1$ (g) $t = 2.5$ (h) $t = 5$ (i) $t = 10$ (j) $t = 15$

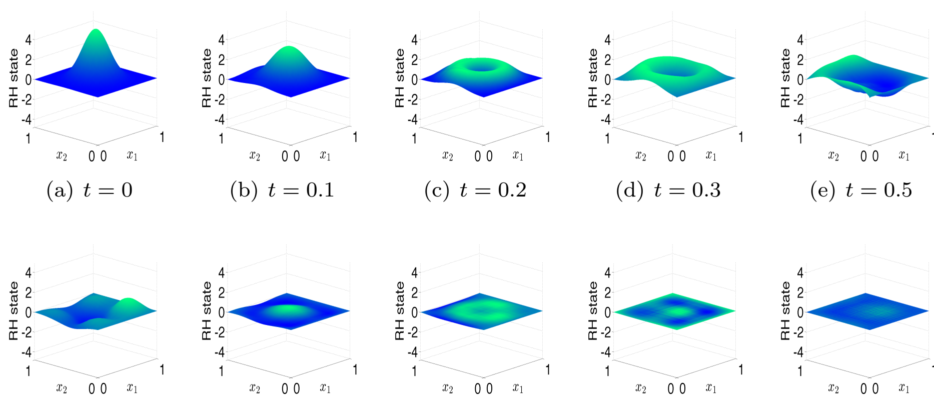
FIGURE 5. Snapshots of receding horizon state for the choice of $T = 1.5$ corresponding to Example 5.1

we implemented Algorithm 2 for different values of the prediction horizon T and fixed sampling time $\delta = 0.25$. The corresponding results are summarized in Table 2, Figure 4(b). Figure 6 shows the receding horizon state at different time points

Prediction Horizon	J_{T_∞}	$\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H}_2)}$	$\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}_2}$	iter
$T = 1.5$	2.20	1.93	2.11×10^{-6}	715
$T = 1$	2.75	2.23	3.42×10^{-5}	599
$T = 0.5$	6.77	3.64	6.00×10^{-3}	445
$T = 0.25$	33.75	8.20	2.36×10^{-1}	359

TABLE 2. Numerical results for Example 5.2

for the choice of $T = 1.5$ and $\delta = 0.25$.



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.2$ (d) $t = 0.3$ (e) $t = 0.5$

(f) $t = 1$ (g) $t = 2.5$ (h) $t = 4$ (i) $t = 5$ (j) $t = 10$

FIGURE 6. Snapshots of receding horizon state for the choice of $T = 1.5$ corresponding to Example 5.2

Example 5.3 (Neumann control)**.** Here, we consider the unilaterally controlled Neumann problem (4) and choose $L = 1$, $\mathcal{U} := \mathbb{R}$, $\beta = 15$, $T_\infty = 15$, and

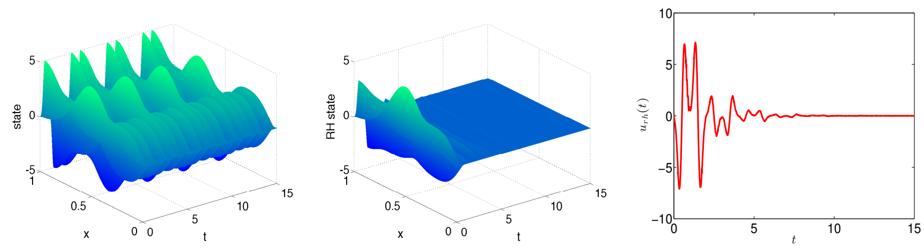
$$y_0^1(x) := 5e^{-20(x-0.5)^2}, \qquad y_0^2(x) = 0,$$

as the initial data. In the uncontrolled case, we have a vibrating string which is fixed at one end of the boundary, but whose other end keeps moving up and down in a periodic fashion. Similarly to the previous examples for the uncontrolled system, the $\mathcal{H}_3$-energy with $\mathcal{H}_3 = V \times L^2(0,1)$ is conserved for all times. Further we have $\|\mathcal{Y}\|_{L^2(0,T_\infty;\mathcal{H}_3)} = 2.10 \times 10^3$ and $\|\mathcal{Y}(T_\infty)\|_{\mathcal{H}_3} = 140.13$. The uncontrolled solution can be seen from Figure 7(a). The numerical results of RHC computed by Algorithm 2 for the different choices of the prediction horizon T and the fixed sampling time $\delta = 0.25$, are revealed by Table 3, and Figures 4(c). Figures 7(b) and 7(c) show, respectively, the receding horizon state and control for the choice of $T = 1.5$.

From Tables 1-3 and Figures 4(a), 4(b), and 4(c), we can assert that the results corresponding to the performance criteria are reasonable. Except for the case that

Prediction Horizon	J_{T_∞}	$\|\mathcal{Y}_{rh}\|_{L^2(0,T_\infty;\mathcal{H}_3)}$	$\|\mathcal{Y}_{rh}(T_\infty)\|_{\mathcal{H}_3}$	iter
$T = 1.5$	1.30×10^4	161.47	3.85×10^{-6}	5348
$T = 1$	1.67×10^4	182.97	7.08×10^{-5}	3303
$T = 0.5$	3.92×10^4	280.22	4.91×10^{-2}	1507
$T = 0.25$	2.41×10^5	694.40	9.26	823

TABLE 3. Numerical results for Example 5.3



(a) Uncontrolled solution (b) RH state for $T = 1.5$ (c) RHC for $T = 1.5$

FIGURE 7. Numerical results corresponding to Example 5.3

$\delta = T$, for all prediction horizons $T > \delta$ the underlying system was successfully stabilized as the theory in the previous sections suggests. Moreover, apparently the prediction horizon T plays an important role. As expected, increasing the prediction horizon T leads to a decrease of the stabilization indicators and more importantly the value of objective function J_{T_∞}. Moreover as can been seen from Figures 3, the corresponding RHCs are more regular, if the ratio of prediction horizon T to sampling time δ is large. On the other hand, the shorter prediction horizon T (i.e. the closer to the sampling time δ) is chosen, the fewer overall iterations and computational efforts are required.

Conclusion. Receding horizon control for the stabilization of linear wave equation with different boundary conditions was analysed and its numerical efficiency was investigated. The results encourage further investigations which may include the convergence analysis of the controls obtained by the receding horizon framework as $T \to \infty$, as well as nonlinear problems, and cost functionals different from quadratic ones, as for instance, sparsity promoting functionals.

A. **Appendix.**

A.1. **Proof of Proposition 2.6.**

Before establishing the first-order optimality conditions, we prove the following useful lemma.

Lemma A.1. *Consider the following linear wave equations*

$$\begin{cases} \ddot{y} - \Delta y = f & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y(0),\dot{y}(0)) = (0,0) & \text{on } \Omega, \end{cases} \quad (80) \qquad \begin{cases} \ddot{p} - \Delta p = g & \text{in } (0,T)\times\Omega, \\ p = 0 & \text{on } (0,T)\times\partial\Omega, \\ (p(T),\dot{p}(T)) = (p^1_T, p^2_T) & \text{on } \Omega, \end{cases} \quad (81)$$

where $f \in L^2(0,T;L^2(\Omega))$, $g \in L^2(0,T;H^{-1}(\Omega))$, and $(p^1_T,p^2_T) \in L^2(\Omega)\times H^{-1}(\Omega)$. Then the weak solution y to (80) and the very weak solution p to (81) satisfy the *following equality*

$$\begin{aligned}&\int_0^T (f(t),p(t))_{L^2(\Omega)}\,dt\\&=\int_0^T \langle g(t),y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt+(p_T^1,\dot y(T))_{L^2(\Omega)}-\langle p_T^2,y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned}\tag{82}$$

Proof. First, due to Proposition 2.3 and the time reversibility of the linear wave equation, the solution p to (81) belongs to the space $C^1([0,T];H^{-1}(\Omega))\cap C^0([0,T];L^2(\Omega))$. Moreover, equality (82) can be first established for a smooth solution of (81) by integration by parts and the Green formula. Moreover, for $(g,p_T^1,p_T^2)\in L^2(0,T;L^2(\Omega))\times \mathcal{H}_1$ the solution to (81) belongs to the space $C^0([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ (see, e.g., [40, 42]), and the spaces $L^2(0,T;L^2(\Omega))$ and $\mathcal{H}_1$ are dense in the spaces $L^2(0,T;H^{-1}(\Omega))$ and $L^2(\Omega)\times H^{-1}(\Omega)$, respectively. Next, (82) is derived by using a density argument and passing to the limit which is justified due to estimate (25). ☐

Now we are in a position to establish the optimality conditions. For sake of simplicity in notation, we remove the overbar in the notation of $(\bar y,\bar u)$. Let $(y_0^1,y_0^2)\in\mathcal{H}_1$ be given. Computing the directional derivative of $J_T(\cdot,(y_0^1,y_0^2))$ at u in the direction of an arbitrary $\delta u\in L^2(0,T;L^2(\omega))$ we obtain

$$\begin{aligned}&J_T'(u,(y_0^1,y_0^2))\delta u\\&=\int_0^T (y(t),\delta y(t))_{H_0^1(\Omega)}dt+\int_0^T(\dot y(t),\dot{\delta y}(t))_{L^2(\Omega)}dt+\beta\int_0^T(u(t),\delta u(t))_{L^2(\omega)}dt,\\&=\int_0^T\langle-\Delta y(t),\delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}dt+\int_0^T(\dot y(t),\dot{\delta y}(t))_{L^2(\Omega)}dt+\beta\int_0^T(u(t),\delta u(t))_{L^2(\omega)}dt,\end{aligned}\tag{83}$$

where $-\Delta:H_0^1(\Omega)\to H^{-1}(\Omega)$ is the Laplace operator with homogeneous Dirichlet boundary conditions which is an isomorphism. Moreover $\delta y\in C^0([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ is the weak solution of

$$\begin{cases}\ddot{\delta y}-\Delta\delta y=B\delta u & \text{in } (0,T)\times\Omega,\\ \delta y=0 & \text{on } (0,T)\times\partial\Omega,\\ (\delta y(0),\dot{\delta y}(0))=(0,0) & \text{on } \Omega.\end{cases}\tag{84}$$

Since the spaces $H^2(\Omega)\cap H_0^1(\Omega)$, $H_0^1(\Omega)$, and $L^2(0,T;H_0^1(\Omega))$ are dense in $H_0^1(\Omega)$, $L^2(\Omega)$, and $L^2(0,T;L^2(\Omega))$, respectively, by a density argument it can be shown that

$$\int_0^T(\dot y(t),\dot{\delta y}(t))_{L^2(\Omega)}dt=\langle\dot y(T),\delta y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}-\int_0^T\langle\ddot y(t),\delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}dt.\tag{85}$$

Due to (83) and (85), the first-order optimality condition is equivalent to the following equality

$$\begin{aligned}&\int_0^T\langle-\ddot y(t)-\Delta y(t),\delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}dt\\&\qquad+\langle\dot y(T),\delta y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\beta\int_0^T(u(t),\delta u(t))_{L^2(\omega)}dt=0,\end{aligned}\tag{86}$$

for all $\delta u \in L^2(0,T;L^2(\omega))$. Due to Lemma A.1 and using the equality (82) for equation (84), we have

$$\int_0^T \langle g(t), \delta y(t)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + (p_T^1, \dot{\delta y}(T))_{L^2(\Omega)} - \langle p_T^2, \delta y(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_0^T (p(t), B\delta u)_{L^2(\Omega)}\, dt = 0, \tag{87}$$

for any given $(g, p_T^1, p_T^2) \in L^2(0,T;H^{-1}(\Omega))\times L^2(\Omega)\times H^{-1}(\Omega)$ and its corresponding very weak solution $p \in C^1([0,T];H^{-1}(\Omega))\cap C^0([0,T];L^2(\Omega))$ to (81). By comparing (86) with (87) and since $\delta u \in L^2(0,T;L^2(\omega))$ is arbitrary, we infer that $\beta u = -B^*p$ in $L^2(0,T;L^2(\omega))$, $p_T^1 = 0$ in $L^2(\Omega)$, $p_T^2 = -\dot{y}(T)$ in $H^{-1}(\Omega)$, and $g = -\ddot{y} - \Delta y$ in $L^2(0,T;H^{-1}(\Omega))$.

A.2. **Proof of Proposition 3.5.** In order to show the optimality conditions, we need first to prove the following useful lemma.

Lemma A.2. *Consider the following linear wave equations*

$$\begin{cases} \ddot{y} - \Delta y = 0 & \text{in } (0,T)\times\Omega, \\ y = u & \text{on } (0,T)\times\Gamma_c, \\ y = 0 & \text{on } (0,T)\times\Gamma_0, \\ (y(0),\dot{y}(0)) = (0,0) & \text{on } \Omega, \end{cases} \tag{88}$$

$$\begin{cases} \ddot{p} - \Delta p = g & \text{in } (0,T)\times\Omega, \\ p = 0 & \text{on } (0,T)\times\partial\Omega, \\ (p(T),\dot{p}(T)) = (p_T^1, p_T^2) & \text{on } \Omega, \end{cases} \tag{89}$$

where $u \in L^2(0,T;L^2(\Gamma_c))$, $g \in L^2(0,T;L^2(\Omega))$, and $(p_T^1,p_T^2) \in H_0^1(\Omega)\times L^2(\Omega)$. Then the very weak solution y to (88) and the weak solution p to (89) satisfy the following equality

$$\int_0^T (g(t), y(t))_{L^2(\Omega)}\, dt + \langle p_T^1, \dot{y}(T))\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - (p_T^2, y(T))_{L^2(\Omega)} = -\int_0^T (u(t), \partial_\nu p(t))_{L^2(\Gamma_c)}\, dt. \tag{90}$$

Proof. Using that $y \in C^1([0,T];H^{-1}(\Omega))\cap C^0([0,T];L^2(\Omega))$, $p \in C^1([0,T];L^2(\Omega))\cap C^0([0,T];H_0^1(\Omega))$, and that $\partial_\nu p \in L^2(0,T;L^2(\partial\Omega))$, (90) can be verified by standard techniques analogous to Lemma A.1. □

Now we are in a position to prove Proposition 3.5. Again, we remove the overbar in the notation of $(\bar{y},\bar{u})$. Let $(y_0^1,y_0^2) \in \mathcal{H}_2$ be given. Computing the directional derivative of $J_T(\cdot,(y_0^1,y_0^2))$ at $\bar{u}$ in the direction of an arbitrary $\delta u \in L^2(0,T;L^2(\Gamma_c))$, we obtain

$$J_T'(u,(y_0^1,y_0^2))\delta u = \int_0^T (y(t), \delta y(t))_{L^2(\Omega)} dt + \int_0^T (\dot{y}(t), \dot{\delta y}(t))_{H^{-1}(\Omega)} dt + \beta\int_0^T (u(t), \delta u(t))_{L^2(\Gamma_c)} dt, \tag{91}$$

where $\delta y \in C^1([0,T];H^{-1}(\Omega))\cap C^0([0,T];L^2(\Omega))$ is the very weak solution of

$$\begin{cases} \ddot{\delta y} - \Delta \delta y = 0 & \text{in } (0,T)\times\Omega, \\ \delta y = \delta u & \text{on } (0,T)\times\Gamma_c, \\ \delta y = 0 & \text{on } (0,T)\times\Gamma_0, \\ (\delta y(0),\dot{\delta y}(0)) = (0,0) & \text{on } \Omega. \end{cases} \tag{92}$$

As defined $\overline{\mathcal{G}} : (H^2(\Omega)\cap H_0^1(\Omega))^* \to L^2(\Omega)$ denotes the unique linear extension of $\mathcal{G} : H^{-1}(\Omega) \to H_0^1(\Omega)$. Well-posedness of $\overline{\mathcal{G}}$ is justified since $H^{-1}(\Omega)$ is dense in

$(H^2(\Omega)\cap H^1_0(\Omega))^*$. Moreover one can show that $\overline{\mathcal{G}}$ is the inverse of the operator $(-\widetilde{\Delta})^* : L^2(\Omega) \to (H^2(\Omega)\cap H^1_0(\Omega))^*$, where $-\widetilde{\Delta} : (H^2(\Omega)\cap H^1_0(\Omega)) \to L^2(\Omega)$ is the Laplace operator with homogeneous Dirichlet boundary conditions. Next we show that

$$\int_0^T (\dot{y}(t), \dot{\delta y}(t))_{H^{-1}(\Omega)} dt = (\overline{\mathcal{G}}\dot{y}(T), \delta y(T))_{L^2(\Omega)} - \int_0^T (\overline{\mathcal{G}}\ddot{y}(t), \delta y(t))_{L^2(\Omega)} dt. \tag{93}$$

We proceed with the help of an approximation argument. The spaces $H^1_0(\Omega)$, $L^2(\Omega)$, and $H^2_0(0,T;H^{\frac{3}{2}}(\partial\Omega)) := \{q \in H^2(0,T;H^{\frac{3}{2}}(\partial\Omega)) : q(0) = \dot{q}(0) = 0\}$ are dense in the spaces $L^2(\Omega)$, $H^{-1}(\Omega)$, and $L^2(0,T;L^2(\partial\Omega))$, respectively, and the solutions of (48) (resp. (92)) is equal to the solution of (44) provided we choose $B_{bd}u \in L^2(0,T;L^2(\partial\Omega))$ (resp. $B_{bd}\delta u$) as the inhomogeneous Dirichlet part h and the pair (y^1_0, y^2_0) (resp. $(0,0)$) as the initial pair. Therefore, there exist sequences $\{y^{1n}_0\}_n \subset H^1_0(\Omega)$, $\{y^{2n}_0\}_n \subset L^2(\Omega)$, $\{h^n\}_n \subset H^2_0(0,T;H^{\frac{3}{2}}(\partial\Omega))$, and $\{\delta h^n\}_n \subset H^2_0(0,T;H^{\frac{3}{2}}(\partial\Omega))$ such that

$$\begin{aligned} &y^{1n}_0 \to y^1_0 \quad \text{in } L^2(\Omega) && h^n \to B_{bd}u \quad \text{in } L^2(0,T;L^2(\partial\Omega)),\\ &y^{2n}_0 \to y^2_0 \quad \text{in } H^{-1}(\Omega) && \delta h^n \to B_{bd}\delta u \quad \text{in } L^2(0,T;L^2(\partial\Omega)). \end{aligned}$$

For any triple $(y^{1n}_0, y^{2n}_0, h^n) \in H^1_0(\Omega)\times L^2(\Omega)\times H^2_0(0,T;H^{\frac{3}{2}}(\partial\Omega))$, the solution of y^n of (44) belongs to the space $C^1([0,T];L^2(\Omega))\cap C^0([0,T];H^1(\Omega))$ with $\ddot{y}^n \in L^2(0,T;H^{-1}(\Omega))$ (see, e.g., [43]), and similarly, for any triple $(0,0,\delta h^n) \in H^1_0(\Omega)\times L^2(\Omega)\times H^2_0(0,T;H^{\frac{3}{2}}(\partial\Omega))$, the solution of (44) belongs to the space $C^1([0,T];L^2(\Omega))\cap C^0([0,T];H^1(\Omega))$. By using estimate (47) we have

$$\begin{aligned} &\|y^n - y\|_{C^0([0,T];L^2(\Omega))} + \|\dot{y}^n - \dot{y}\|_{C^0([0,T];H^{-1}(\Omega))} + \|\ddot{y}^n - \ddot{y}\|_{L^2(0,T;(H^1_0(\Omega))\cap H^2(\Omega))^*)}\\ &\le c_2\left(\|y^{1n}_0 - y^1_0\|_{L^2(\Omega)} + \|y^{2n}_0 - y^2_0\|_{H^{-1}(\Omega)} + \|h^n - B_{bd}u\|_{L^2(0,T;L^2(\partial\Omega))}\right), \end{aligned}$$

and

$$\|\delta y^n - \delta y\|_{C^0([0,T];L^2(\Omega))} + \|\dot{\delta y}^n - \dot{\delta y}\|_{C^0([0,T];H^{-1}(\Omega))} \le c_2\|\delta h^n - B_{bd}\delta u\|_{L^2(0,T;L^2(\partial\Omega))}.$$

For a solution y^n of (44) with $(y^{1n}_0, y^{2n}_0, h^n) \in H^1_0(\Omega)\times L^2(\Omega)\times H^2_0(0,T;H^{\frac{3}{2}}(\partial\Omega))$ and a solution δy^n of (44) with $(0,0,\delta h^n) \in H^1_0(\Omega)\times L^2(\Omega)\times H^2_0(0,T;H^{\frac{3}{2}}(\partial\Omega))$, we have

$$\begin{aligned} \int_0^T (\dot{y}^n(t), \dot{\delta y}^n(t))_{H^{-1}(\Omega)} dt &= \int_0^T \langle \mathcal{G}\dot{y}^n(t), \dot{\delta y}^n(t)\rangle_{H^1_0(\Omega),H^{-1}(\Omega)} dt =\\ &(\mathcal{G}\dot{y}^n(T), \delta y^n(T))_{L^2(\Omega)} - \int_0^T (\mathcal{G}\ddot{y}^n(t), \delta y^n(t))_{L^2(\Omega)} dt. \end{aligned}$$

Passing to the limit we obtain

$$\begin{aligned} \int_0^T (\dot{y}^n(t), \dot{\delta y}^n(t))_{H^{-1}(\Omega)} dt &\to \int_0^T (\dot{y}(t), \dot{\delta y}(t))_{H^{-1}(\Omega)} dt,\\ (\mathcal{G}\dot{y}^n(T), \delta y^n(T))_{L^2(\Omega)} &\to (\mathcal{G}\dot{y}(T), \delta y(T))_{L^2(\Omega)} = (\overline{\mathcal{G}}\dot{y}(T), \delta y(T))_{L^2(\Omega)},\\ \int_0^T (\mathcal{G}\ddot{y}^n(t), \delta y^n(t))_{L^2(\Omega)} dt &\to \int_0^T (\overline{\mathcal{G}}\ddot{y}(t), \delta y(t))_{L^2(\Omega)} dt, \end{aligned}$$

which implies (93). Due to (91) and (93), the first-order optimality condition is equivalent to the following equality

$$\int_0^T (y(t) - \overline{\mathcal{G}}\ddot{y}(t), \delta y(t))_{L^2(\Omega)} dt + (\overline{\mathcal{G}}\dot{y}(T), \delta y(T))_{L^2(\Omega)} + \beta\int_0^T (u(t), \delta u(t))_{L^2(\Gamma_c)} dt = 0. \tag{94}$$

Due to Lemma A.2 and using equality (90) for equation (92), we have

$$\int_0^T (g(t), \delta y(t))_{L^2(\Omega)}\, dt + \langle p_T^1, \dot{\delta y}(T))\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - (p_T^2, \delta y(T))_{L^2(\Omega)} + \int_0^T (\partial_\nu p(t), \delta u(t))_{L^2(\Gamma_c)}\, dt = 0, \tag{95}$$

for an arbitrary triple $(g, p_T^1, p_T^2) \in L^2(0,T; L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ and its corresponding weak solution $p \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ to (89). Comparing (94) with (95) and since $\delta u \in L^2(0,T; L^2(\Gamma_c))$ is arbitrary, we infer that $\beta u = \partial_\nu p$ in $L^2(0,T; L^2(\Gamma_c))$, $p_T^1 = 0$ in $H_0^1(\Omega)$, $p_T^2 = -\overline{\mathcal{G}}\dot{y}(T)$ in $L^2(\Omega)$, and $g = y - \overline{\mathcal{G}}\ddot{y}$ in $L^2(0,T; L^2(\Omega))$.

A.3. **Proof of Proposition 4.8.** Prior to investigating the optimality conditions, we need first to prove the following useful lemma.

Lemma A.3. *Consider the following linear wave equations*

$$\begin{cases} \ddot{y} - y_{xx} = 0 & \text{in } (0,T) \times (0,L), \\ y(t,\cdot) = 0 & \text{in } (0,T), \\ y_x(\cdot, L) = u & \text{in } (0,T), \\ (y(0,\cdot), \dot{y}(0,\cdot)) = (0,0) & \text{in } (0,L), \end{cases} \tag{96}$$

$$\begin{cases} \ddot{p} - p_{xx} = g & \text{in } (0,T) \times (0,L), \\ p(\cdot, 0) = 0 & \text{in } (0,T), \\ p_x(\cdot, L) = 0 & \text{in } (0,T), \\ (p(T,\cdot), \dot{p}(T,\cdot)) = (p_T^1, p_T^2) & \text{in } (0,L), \end{cases} \tag{97}$$

where $u \in L^2(0,T)$, $g \in L^2(0,T; V^*)$ and $(p_T^1, p_T^2) \in L^2(0,L) \times V^*$. Then the weak solution y to (96) and the very weak solution p to (97) satisfy the following equality

$$\int_0^T u(t) p(t,L)\, dt = \int_0^T \langle g(t), y(t)\rangle_{V^*,V}\, dt + (p_T^1, \dot{y}(T))_{L^2(0,L)} - \langle p_T^2, y(T)\rangle_{V^*,V}. \tag{98}$$

Proof. Due to Proposition 4.6 and the time reversibility of the linear wave equation, the very weak solution of (97) belongs to the space $C^1([0,T]; V^*) \cap C^0([0,T]; L^2(0,L))$. Moreover, due to Proposition 4.3, the weak solution y of (96) belongs to the space $C^1([0,T]; L^2(0,L)) \cap C^0([0,T]; V)$. Therefore, the right hand side of (98) is well-posed. We show that $p(\cdot, L) \in L^2(0,T)$ is well-defined and can be associated to the very weak solution p to (97). Consider the following linear functional

$$\ell_{g,p_T^1,p_T^2}(u) := \int_0^T \langle g(t), y(t)\rangle_{V^*,V}\, dt + (p_T^1, \dot{y}(T))_{L^2(0,L)} - \langle p_T^2, y(T)\rangle_{V^*,V}, \tag{99}$$

where $y(u)$ is the solution of (96) depends on $u \in L^2(0,T)$. Due to (99) and (66) we have

$$\begin{aligned} |\ell_{g,p_T^1,p_T^2}(u)| &\le \|g\|_{L^2(0,T;V^*)} \|y\|_{L^2(0,T;V)} + \|\dot{y}(T)\|_{L^2(0,L)} \|p_T^1\|_{L^2(0,L)} + \|y(T)\|_V \|p_T^2\|_{V^*}, \\ &\le \hat{c}_4 \left(\|g\|_{L^2(0,T;V^*)} + \|p_T^1\|_{L^2(0,L)} + \|p_T^2\|_{V^*} \right) \|u\|_{L^2(0,T)}, \end{aligned} \tag{100}$$

for a constant $\hat{c}_4$ depending on T and L. Therefore, $\ell_{g,p_T^1,p_T^2} : L^2(0,T) \to \mathbb{R}$ is a continuous linear functional. By the Riesz representation theorem and (100), there exists a unique object $p(\cdot, L) \in L^2(0,T)$ such that

$$\int_0^T u(t) p(t,L)\, dt = \ell_{g,p_T^1,p_T^2}(u), \tag{101}$$

and we have

$$\|p(\cdot, L)\|_{L^2(0,T)} \le \hat{c}_4 \left(\|g\|_{L^2(0,T;V^*)} + \|p_T^1\|_{L^2(0,L)} + \|p_T^2\|_{V^*} \right). \tag{102}$$

Next, we show that $p(\cdot, L)$ is the trace of the solution p to (97). Since the spaces V, $L^2(0,L)$, and $L^2(0,T; L^2(0,L))$ are dense in the spaces $L^2(0,L)$, V^*, and $L^2(0,T; V^*)$, respectively, there exist sequences $\{p_T^{1n}\}_n \subset V$, $\{p_T^{2n}\}_n \subset L^2(0,L)$, and $\{g^n\}_n \subset L^2(0,T;L^2(0,L))$ such that

$$p_T^{1n} \to p_T^1 \text{ in } L^2(0,L), \quad p_T^{2n} \to p_T^2 \text{ in } V^*, \quad g^n \to g \text{ in } L^2(0,T;V^*).$$

Moreover, for any triple $(p_T^{1n}, p_T^{2n}, g^n) \in V \times L^2(0,L) \times L^2(0,T;L^2(0,T))$, the solution p^n of (97) belongs to the space $C^1([0,T];L^2(0,L)) \cap C^0([0,T];V)$ (see, e.g., [42]). By (69) we have

$$\begin{aligned}&\|p^n - p\|_{C^0([0,T];L^2(0,L))} + \|\dot{p}^n - \dot{p}\|_{C^0([0,T];V^*)}\\ &\leq \bar{c}_4 \left(\|p_T^{1n} - p_T^1\|_{L^2(0,L)} + \|p_T^{2n} - p_T^2\|_{V^*} + \|g^n - g\|_{L^2(0,T;V^*)}\right).\end{aligned} \tag{103}$$

For the solution p^n of (97) with $(p_T^{1n}, p_T^{2n}, g^n) \in V \times L^2(0,L) \times L^2(0,T;L^2(0,L))$ and the solution y of (96) we have

$$\int_0^T u(t)p^n(t,L)\,dt = \int_0^T \langle g^n(t), y(t)\rangle_{V^*,V}\,dt + (p_T^{1n}, \dot{y}(T))_{L^2(0,L)} - \langle p_T^{2n}, y(T)\rangle_{V^*,V}. \tag{104}$$

From (102), we deduce that

$$\|p^n(\cdot,L)\|_{L^2(0,T)} \leq \hat{c}_4 \left(\|g^n\|_{L^2(0,T;V^*)} + \|p_T^{1n}\|_{L^2(0,L)} + \|p_T^{2n}\|_{V^*}\right).$$

Therefore the sequence $p^n(\cdot,L)$ is bounded in $L^2(0,T)$ and thus there is a weakly convergent subsequence $\{p^n(\cdot,L)\}_n$ such that $p^n(\cdot,L) \rightharpoonup p^*(\cdot,L)$ with a function $p^*(\cdot,L) \in L^2(0,T)$. Passing to the limit we have

$$\begin{aligned}\int_0^T \langle g^n(t), y(t)\rangle_{V^*,V}\,dt &\to \int_0^T \langle g(t), y(t)\rangle_{V^*,V}\,dt,\\ (p_T^{1n}, \dot{y}(T))_{L^2(0,L)} &\to (p_T^1, \dot{y}(T))_{L^2(0,L)},\\ \langle p_T^{2n}, y(T)\rangle_{V^*,V} &\to \langle p_T^2, y(T)\rangle_{V^*,V},\\ \int_0^T u(t)p^n(t,L)\,dt &\to \int_0^T u(t)p^*(t,L)\,dt.\end{aligned}$$

and, as a consequence, by using (104) we obtain

$$\int_0^T u(t)p^*(t,L)\,dt = \int_0^T \langle g(t), y(t)\rangle_{V^*,V}\,dt + (p_T^1, \dot{y}(T))_{L^2(0,L)} - \langle p_T^2, y(T)\rangle_{V^*,V}. \tag{105}$$

Moreover, due to (103), we infer that $p^n \to p$ in $C^1([0,T];V^*) \cap C^0([0,T];L^2(0,L))$. Finally, (99), (101), and (105) imply

$$\int_0^T u(t)p^*(t,L)\,dt = \int_0^T u(t)p(t,L)\,dt = \ell_{g,p_T^1,p_T^2}(u). \tag{106}$$

for all $u \in L^2(0,T)$. We conclude $p^*(\cdot,L) = p(\cdot,L)$ in $L^2(0,T)$. ☐

Now the proof of Proposition 4.8 can be obtained with standard arguments similarly to those for Propositions 2.6 and 3.5.

REFERENCES

[1] K. Ammari, *Dirichlet boundary stabilization of the wave equation*, Asymptot. Anal., 30 (2002), pp. 117–130.

[2] K. Ammari and M. Tucsnak, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM Control Optim. Calc. Var., 6 (2001), pp. 361–386 (electronic).

[3] B. Azmi and K. Kunisch, *On the Stabilizability of the Burgers Equation by Receding Horizon Control*, SIAM J. Control Optim., 54 (2016), pp. 1378–1405.

[4] L. Bales and I. Lasiecka, *Continuous finite elements in space and time for the nonhomogeneous wave equation*, Comput. Math. Appl., 27 (1994), pp. 91–102.

[5] ———, *Negative norm estimates for fully discrete finite element approximations to the wave equation with nonhomogeneous L_2 Dirichlet boundary data*, Math. Comp., 64 (1995), pp. 89–115.
[6] W. Bangerth, M. Geiger, and R. Rannacher, *Adaptive Galerkin finite element methods for the wave equation*, Comput. Methods Appl. Math., 10 (2010), pp. 3–48.
[7] W. Bangerth and R. Rannacher, *Adaptive finite element techniques for the acoustic wave equation*, J. Comput. Acoust., 9 (2001), pp. 575–591.
[8] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024–1065.
[9] J. Barzilai and J. M. Borwein, *Two-point step size gradient methods*, IMA J. Numer. Anal., 8 (1988), pp. 141–148.
[10] R. Becker, D. Meidner, and B. Vexler, *Efficient numerical solution of parabolic optimization problems by finite element methods*, Optim. Methods Softw., 22 (2007), pp. 813–833.
[11] N. Burq and P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes*, C. R. Acad. Sci. Paris Sér. I Math., 325 (1997), pp. 749–752.
[12] G. Chen, *Control and stabilization for the wave equation in a bounded domain*, SIAM Journal on Control and Optimization, 17 (1979), pp. 66–81.
[13] H. Chen and F. Allgöwer, *A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability*, Automatica J. IFAC, 34 (1998), pp. 1205–1217.
[14] N. Cîndea and A. Münch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations*, Calcolo, 52 (2015), pp. 245–288.
[15] J.-M. Coron, *Control and nonlinearity*, vol. 136 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007.
[16] R. Curtain and K. Morris, *Transfer functions of distributed parameter systems: a tutorial*, Automatica J. IFAC, 45 (2009), pp. 1101–1116.
[17] Y.-H. Dai and H. Zhang, *Adaptive two-point stepsize gradient algorithm*, Numer. Algorithms, 27 (2001), pp. 377–385.
[18] A. Ern and J.-L. Guermond, *Theory and practice of finite elements*, vol. 159 of Applied Mathematical Sciences, Springer-Verlag, New York, 2004.
[19] F. Flandoli, *Invertibility of Riccati operators and controllability of related systems*, Systems Control Lett., 9 (1987), pp. 65–72.
[20] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, *Model predictive control: for want of a local control Lyapunov function, all is not lost*, IEEE Trans. Automat. Control, 50 (2005), pp. 546–558.
[21] L. Grüne, *Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems*, SIAM J. Control Optim., 48 (2009), pp. 1206–1228.
[22] L. Grüne and A. Rantzer, *On the infinite horizon performance of receding horizon controllers*, IEEE Trans. Automat. Control, 53 (2008), pp. 2100–2111.
[23] M. Gugat, *Exponential stabilization of the wave equation by Dirichlet integral feedback*, SIAM J. Control Optim., 53 (2015), pp. 526–546.
[24] M. Gugat, G. Leugering, and G. Sklyar, *L^p-optimal boundary control for the wave equation*, SIAM J. Control Optim., 44 (2005), pp. 49–74.
[25] M. Gugat, E. Trélat, and E. Zuazua, *Optimal Neumann control for the 1D wave equation: finite horizon, infinite horizon, boundary tracking terms and the turnpike property*, Systems Control Lett., 90 (2016), pp. 61–70.
[26] W. Guo and B.-Z. Guo, *Adaptive output feedback stabilization for one-dimensional wave equation with corrupted observation by harmonic disturbance*, SIAM J. Control Optim., 51 (2013), pp. 1679–1706.
[27] E. Hendrickson and I. Lasiecka, *Numerical approximations of solutions to Riccati equations arising in boundary control problems for the wave equation*, in Optimal control of differential equations (Athens, OH, 1993), vol. 160 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1994, pp. 111–132.
[28] K. Ito and K. Kunisch, *Receding horizon optimal control for infinite dimensional systems*, ESAIM Control Optim. Calc. Var., 8 (2002), pp. 741–760 (electronic). A tribute to J. L. Lions.
[29] A. Jadbabaie and J. Hauser, *On the stability of receding horizon control with a general terminal cost*, IEEE Trans. Automat. Control, 50 (2005), pp. 674–678.
[30] C. Johnson, *Discontinuous Galerkin finite element methods for second order hyperbolic problems*, Comput. Methods Appl. Mech. Engrg., 107 (1993), pp. 117–129.

[31] O. Karakashian and C. Makridakis, *Convergence of a continuous Galerkin method with mesh modification for nonlinear wave equations*, Math. Comp., 74 (2005), pp. 85–102.

[32] A. Kröner, K. Kunisch, and B. Vexler, *Semismooth Newton methods for optimal control of the wave equation with control constraints*, SIAM J. Control Optim., 49 (2011), pp. 830–858.

[33] K. Kunisch, P. Trautmann, and B. Vexler, *Optimal control of the undamped linear wave equation with measure valued controls*, SIAM J. Control Optim., 54 (2016), pp. 1212–1244.

[34] J. E. Lagnese, *Note on boundary stabilization of wave equations*, SIAM J. Control Optim., 26 (1988), pp. 1250–1256.

[35] I. Lasiecka and R. Triggiani, *Uniform exponential energy decay of wave equations in a bounded region with $L_2(0,\infty;L_2(\Gamma))$-feedback control in the Dirichlet boundary conditions*, J. Differential Equations, 66 (1987), pp. 340–390.

[36] ———, *Sharp regularity theory for second order hyperbolic equations of Neumann type. I. L_2 nonhomogeneous data*, Ann. Mat. Pura Appl. (4), 157 (1990), pp. 285–367.

[37] ———, *Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory*, vol. 164 of Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1991.

[38] ———, *Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions*, Appl. Math. Optim., 25 (1992), pp. 189–224.

[39] ———, *Algebraic Riccati equations arising from systems with unbounded input-solution operator: applications to boundary control problems for wave and plate equations*, Nonlinear Anal., 20 (1993), pp. 659–695.

[40] J.-L. Lions, *Optimal control of systems governed by partial differential equations.*, Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170, Springer-Verlag, New York-Berlin, 1971.

[41] ———, *Contrôle des systèmes distribués singuliers*, vol. 13 of Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science], Gauthier-Villars, Montrouge, 1983.

[42] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[43] ———, *Non-homogeneous boundary value problems and applications. Vol. II*, Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.

[44] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, *Constrained model predictive control: stability and optimality*, Automatica J. IFAC, 36 (2000), pp. 789–814.

[45] B. S. Mordukhovich and J.-P. Raymond, *Neumann boundary control of hyperbolic equations with pointwise state constraints*, SIAM J. Control Optim., 43 (2004/05), pp. 1354–1372 (electronic).

[46] B. S. Mordukhovich and J.-P. Raymond, *Dirichlet boundary control of hyperbolic equations in the presence of state constraints*, Appl. Math. Optim., 49 (2004), pp. 145–157.

[47] A. Münch and A. F. Pazoto, *Uniform stabilization of a viscous numerical approximation for a locally damped wave equation*, ESAIM Control Optim. Calc. Var., 13 (2007), pp. 265–293 (electronic).

[48] M. Reble and F. Allgöwer, *Unconstrained model predictive control and suboptimality estimates for nonlinear continuous-time systems*, Automatica J. IFAC, 48 (2012), pp. 1812–1817.

[49] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Rev., 20 (1978), pp. 639–739.

[50] L. T. Tebou and E. Zuazua, *Uniform boundary stabilization of the finite difference space discretization of the $1-d$ wave equation*, Adv. Comput. Math., 26 (2007), pp. 337–365.

[51] R. Triggiani, *Exact boundary controllability on $L_2(\Omega)\times H^{-1}(\Omega)$ of the wave equation with Dirichlet boundary control acting on a portion of the boundary $\partial\Omega$, and related problems*, Appl. Math. Optim., 18 (1988), pp. 241–277.

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