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Oblique projection local feedback stabilization of nonautonomous semilinear damped wave-like equations

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OBLIQUE PROJECTION LOCAL FEEDBACK STABILIZATION OF NONAUTONOMOUS SEMILINEAR DAMPED WAVE-LIKE EQUATIONS

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ABSTRACT. The stabilization of a class of nonlinear weakly damped wave equations by means of a finite-dimensional feedback control is investigated. The stabilizing control is constructed based on an appropriate oblique projection and it enters as a time-dependent linear combination of a finite numbers of suitable indicator functions supported in small regions. Firstly, it is shown that an oblique projection feedback is able to globally exponentially stabilize linear nonautonomous weakly damped wave equations. Then, relying on this result, the local stabilization for semilinear equations is proven for a suitable class of nonlinearities. Finally, numerical experiments are given which validate the theoretical results.

1. INTRODUCTION

We are concerned with the stabilization of controlled systems governed by weakly damped wave equations

$$\tfrac{\partial^2}{\partial t^2} y - \nu\Delta y + \varsigma\tfrac{\partial}{\partial t} y + ay + \mathcal{N}(y) = \sum_{i=1}^{M} u_i 1_{\omega_i^M}, \qquad \mathcal{G}y|_\Gamma = 0, \tag{1.1a}$$

$$y(0) = y_0, \qquad \tfrac{\partial}{\partial t} y(0) = y_1. \tag{1.1b}$$

with a time depending control vector $u = (u_1(t), \dots, u_M(t)) \in \mathbb{R}^M$ with a positive integer M. The equation evolves in a spatial bounded smooth domain $\Omega \subset \mathbb{R}^d$, which is a located locally on one side of its boundary $\Gamma = \partial\Omega$, for a positive integer d. The unknown $y = y(t, x) \in \mathbb{R}$, defined for $(t, x) \in (0, +\infty) \times \Omega$, stands for the state. The coefficient $\nu > 0$ and the damping coefficient $\varsigma > 0$ are positive constants and the function $a = a(t, x) \in \mathbb{R}$ is fixed. Further, our actuators are the indicator functions $1_{\omega_i^M} = 1_{\omega_i^M}(x)$, where $\omega_i^M \subset \Omega$ are small domains for $i = 1, 2, \dots, M$. The term $\mathcal{N}(\cdot)$ stands for the nonlinearity which will be specified later. Finally the relation $\mathcal{G}y|_\Gamma = 0$ sets the boundary conditions up. Our results will cover both Dirichlet and Neumann boundary conditions, respectively $\mathcal{G} = \mathbf{1}$ and $\mathcal{G} = \frac{\partial}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla$, where $\mathbf{n}$ stands for the unit outward normal vector to the boundary Γ.

Throughout this manuscript the symbol $\mathbf{1}$ stands for the identity operator on a given linear space, such space will be clear from the context.

Our objective is to construct an oblique projection based feedback control $u \in L^2((0, +\infty), \mathbb{R}^M)$ that is able to *locally* exponentially stabilize systems as (1.1).

First of all, to simplify the exposition, it will be convenient to guarantee the coercivity of the diffusion operator (in particular, for Neumann boundary conditions), in other words we would like to have $-\nu\Delta + \mathbf{1}$ instead of $-\nu\Delta$ (cf. [21, Eq. (6a)]). For this we will rewrite (1.1) in the equivalent

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form

$$\tfrac{\partial^2}{\partial t^2} y + (-\nu\Delta + \mathbf{1})y + \varsigma \tfrac{\partial}{\partial t} y + (a-1)y + \mathcal{N}(y) = \sum_{i=1}^{M} u_i 1_{\omega_i^M}, \quad \mathcal{G}y|_\Gamma = 0, \tag{1.2a}$$

$$y(0) = y_0, \quad \tfrac{\partial}{\partial t} y(0) = y_1. \tag{1.2b}$$

1.1. **Main results.** In order to introduce the oblique projection feedback we need some notation. The eigenvalues of $-\nu\Delta + \mathbf{1}$, repeated accordingly with their multiplicity, are denoted by $\alpha_i \in \mathbb{R}$ for $i \in \mathbb{N}_0 := \{1, 2, 3, \dots\}$, and satisfy

$$0 < \alpha_i \le \alpha_{i+1}, \qquad \lim_{i \to +\infty} \alpha_i = +\infty.$$

Then we fix a complete orthonormal system of eigenfunctions e_i,

$$-\nu\Delta e_i + \mathbf{1} e_i = \alpha_i e_i, \qquad i \ge 1.$$

Let $\mathbb{M} := \{1, 2, 3, \dots, M\}$ and let $E_\mathbb{M} = \operatorname{span}\{e_i \mid 1 \le i \le M\} \subset L^2(\Omega)$ be the linear space spanned by the first M eigenfunctions. $E_\mathbb{M}^\perp$ is the orthogonal complement to $E_\mathbb{M}$ in $L^2(\Omega)$, and $U_M = \operatorname{span}\{1_{\omega_i^M} \mid 1 \le i \le M\} \subset L^2(\Omega)$ is the linear space spanned by our M actuators.

As in the case of parabolic equations [20], stabilization will be guaranteed for a large enough number M of actuators. Therefore, we will allow ourselves to take such number of actuators. However, as in [20] we are interested in the case where the total volume covered by the actuators is uniformly bounded, that is,

$$\operatorname{vol}(\textstyle\bigcup_{i=1}^M \omega_i^M) \le r \operatorname{vol}(\Omega), \qquad 0 < r < 1, \tag{1.3}$$

where r is given independently of M.

The construction of the feedback operator here is based on a suitable oblique (non-orthogonal) projection. Let two closed subspaces $F \subseteq H$ and $G \subseteq H$, and a Hilbert space H be given. Further assume that F and G are complementary in H, that is, $H = F + G$ and $F \bigcap G = \{\mathbf{0}\}$. Then the oblique projection in H onto F along G is denoted by $P_F^G : H \to F$ and, for every $u \in H$ it is defined by $u \mapsto u_F$ where u_F is given by

$$u = u_F + u_G \quad \text{with} \quad (u_F, u_G) \in F \times G.$$

Note that the continuity of the oblique projection is well known (see. e.g., [10, Sect. 2.4]. Further, due to the definition, it follows immediately that $\mathbf{1} - P_F^G = P_G^F$. The projection P_F^G is orthogonal if, and only if, $G = F^\perp$. Hereafter, we shall denote orthogonal projections simply by

$$P_F := P_F^{F^\perp}$$

Now, let $P_{U_M}^{E_\mathbb{M}^\perp}$ be the oblique projection in $L^2(\Omega)$ onto U_M along $E_\mathbb{M}^\perp$.

Next we give sufficient conditions for stabilization.

Sufficient conditions for stabilization. *For each M, we can find the actuators $\{1_{\omega_i^M} \mid 1 \le i \le M\}$ so that:*

$$L^2(\Omega) = U_M + E_\mathbb{M}^\perp, \qquad U_M \bigcap E_\mathbb{M}^\perp = \{\mathbf{0}\}, \quad \textit{and} \tag{1.4a}$$

$$\left| P_{U_M}^{E_\mathbb{M}^\perp} \right|_{\mathcal{L}(L^2(\Omega))} \le C_P < +\infty, \quad \textit{with } C_P \ge 1 \textit{ independent of } M, \tag{1.4b}$$

Condition (1.4a) is necessary and sufficient for the projection $P_{U_M}^{E_\mathbb{M}^\perp}$ to be well defined. Condition (1.4b) is satisfied for a suitable placement of the actuators in 1D domains $\Omega \subset \mathbb{R}$. For instance, for bounded intervals $\Omega = (0, L)$ with $L > 0$, we refer to [33]. Then, an analogous condition follows also for rectangular domains $\Omega \subset \mathbb{R}^d$ with $d \ge 2$. This follows from the results in [20, Sect. 4.8.1] (with a slightly different subspace $E_\mathbb{M}$). For general (nonrectangular) domains the satisfiability of (1.4b) is still an interesting open question.

We are now ready to present one typical explicit oblique projection feedback control operator. For every given pair of constants $\lambda_1 > 0$ and $\lambda_2 > 0$, we consider the control function u defined by

$$\sum_{i=1}^{M} u_i 1_{\omega_i^M} = \mathbf{K}_{U_M}^{\lambda} y := P_{U_M}^{E_{\mathbb{M}}^{\perp}} \left((-\nu\Delta+\mathbf{1})y + \varsigma\tfrac{\partial}{\partial t}y + (a-1)y - \lambda_1 y - \lambda_2 \dot{y}\right). \tag{1.5}$$

To present our main results for both Dirichlet, $\mathcal{G} = \mathbf{1}$, and Neumann, $\mathcal{G} = \frac{\partial}{\partial \mathbf{n}}$, boundary conditions we define the Hilbert spaces

$$V = V_{\mathcal{G}} = V_{\mathbf{1}} := H_0^1(\Omega) := \{f \in H^1(\Omega) \mid f|_{\Gamma} = 0\},$$
$$V = V_{\mathcal{G}} = V_{\frac{\partial}{\partial \mathbf{n}}} := H^1(\Omega).$$

For both of the boundary conditions we also set $H := L^2(\Omega)$.

Theorem 1.1. *Let $d \in \{1, 2, 3\}$ and a real number $1 < r \le 3$. Under the conditions in (1.4), there exists $M \in \mathbb{N}$ large enough such that the system*

$$\tfrac{\partial^2}{\partial t^2}y + (-\nu\Delta + \mathbf{1})y + \varsigma\tfrac{\partial}{\partial t}y + (a-1)y - |y|_{\mathbb{R}}^{r-1}\, y = \mathbf{K}_{U_M}^{\lambda} y, \quad \mathcal{G}y|_{\Gamma} = 0, \tag{1.6a}$$
$$y(0) = y_0, \quad \tfrac{\partial}{\partial t}y(0) = y_1. \tag{1.6b}$$

is locally exponentially stable in $V \times H$, with rate $-\frac{\mu}{2}$. That is, there are positive constants $\epsilon > 0$, $\mu > 0$, and $C \ge 1$, such that

$$|(y_0, y_1)|_{V\times H} < \epsilon \quad \Longrightarrow \quad \left|(y(t), \tfrac{\partial}{\partial t}y(t))\right|_{V\times H}^2 \le C\mathrm{e}^{-\mu t}\, |(y_0, y_1)|_{V\times H}^2 . \tag{1.7}$$

Theorem 1.1 is a corollary of a more abstract result that we will present later on. However, the particular class of nonlinearities in (1.6) is interesting in the stabilization context, because the corresponding free dynamics (uncontrolled) solutions are unstable and may blow-up in finite time. The task of our feedback here is to drive the solution exponentially to zero. In particular, the feedback must be able to avoid the blow-up of the solutions, provided the initial condition is small enough.

The goal (1.7) will follow from the global stability of the corresponding linear closed-loop system (system (1.6) without the nonlinear term). Hence, we will start by proving the following.

Theorem 1.2. *Under conditions in 1.4, there exists $M \in \mathbb{N}$ large enough such that the system*

$$\tfrac{\partial^2}{\partial t^2}y + (-\nu\Delta + \mathbf{1})y + \varsigma\tfrac{\partial}{\partial t}y + (a-1)y = \mathbf{K}_{U_M}^{\lambda} y, \quad \mathcal{G}y|_{\Gamma} = 0, \tag{1.8a}$$
$$y(0) = y_0, \quad \tfrac{\partial}{\partial t}y(0) = y_1. \tag{1.8b}$$

is (globally) exponentially stable in $V \times H$, with rate $-\mu$. That is, there are positive constants $\mu > 0$ and $C \ge 1$, such that

$$\textit{for all } (y_0, y_1) \in V \times H, \quad \left|(y(t), \tfrac{\partial}{\partial t}y(t))\right|_{V\times H}^2 \le C\mathrm{e}^{-\mu t}\, |(y_0, y_1)|_{V\times H}^2 . \tag{1.9}$$

Then we will use a suitable fixed point argument to prove that (1.7) holds true for the solution of (1.6).

1.2. **State of the art concerning oblique projection feedback controls.** The oblique projection based feedback in (1.5) was introduced in [20] in the setting of linear parabolic-like equations. The assumptions in [20] are clearly motivated by the latter type of equations and it is not difficult to see that they are not fulfilled by linear wave-like equations. Namely, the results in [20] concern the stability of linear evolutionary system in the general (abstract) form

$$\dot{z} + \mathbf{A}z + \mathbf{A}_{\mathrm{rc}}z = \sum_{i=1}^{M} u_i \Phi_i, \tag{1.10}$$

where our M actuators are now the Φ_is. Attempting to write the "linearized" (i.e., with $\mathcal{N}=0$) nonautonomous damped wave equation (1.2) in the abstract form (1.10), we obtain by defining $z^w := \begin{bmatrix} y \\ \frac{\partial}{\partial t}y \end{bmatrix}$ that

$$\mathbf{A}^w = \begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ -\nu\Delta + \mathbf{1} & \varsigma\mathbf{1} \end{bmatrix}, \qquad \mathbf{A}^w_{\mathrm{rc}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ a-1 & \mathbf{0} \end{bmatrix}, \qquad \text{and} \qquad \Phi_i = \begin{bmatrix} \mathbf{0} \\ 1_{\omega_i^M} \end{bmatrix}. \tag{1.11}$$

Unfortunately, the operator $\mathbf{A}^w$ in (1.11) does not satisfy the symmetry assumption required for the "diffusion-like" operator in [20, 21]. Therefore, we cannot directly use the results in [20] for our damped wave equation with the operators $\mathbf{A}^w$ and $\mathbf{A}^w_{\mathrm{rc}}$ defined in (1.11), and we are not making any further attempt on this direction. Instead, we are dealing directly with the damped wave equation.

There exists however a particular property shared by both of the operator $\mathbf{A}^w$ and the operator $-\nu\Delta + \mathbf{1}$ (taking the role of $\mathbf{A}$ in the parabolic case), namely both dynamical systems

$$\dot{z}^w + \mathbf{A}^w z^w = 0 \qquad \text{and} \qquad \dot{z} + (-\nu\Delta + \mathbf{1})z = 0$$

are stable. It is this property that will allow us to extend the results in [20] for linear weakly-damped wave equations. Of course, for such extension we will also have to deal with the regularity differences between the two types of equations.

Finally, concerning the local stabilization of the nonlinear systems, we follow a standard, but nontrivial, fixed point argument. Here, we follow mainly the ideas from [27, 21].

1.3. **On the stabilization of linear nonautonomous equations.** Up to now, results on stabilization of nonautonomous systems with a finite number of actuators, concern mainly parabolic-like equations. The development of appropriate mathematical tools to tackle this problem was initiated in [6] for the Navier–Stokes equations. This fact is also mentioned in [12, Sect. 7.1]. Indeed, the standard spectral properties used to investigate the stability of autonomous systems are not appropriate for dealing with nonautonomous systems, as shown by the examples in [38]. In [6], appropriate truncated observability inequalities were proven and used to derive the internal stabilizability result. Truncated observability inequalities were also used in [30, 31] to derive the boundary stabilizability result. These inequalities were also used in [2] to derive the internal stabilizability result for weakly damped wave equations.

A different approach was proposed in [19] (see also [18]) for a one-dimensional domain $\Omega \subset \mathbb{R}$. This approach relies on an appropriate direct finite-dimensional approximation of an infinite-dimensional internal control which drives the solution to zero at a fixed finite time $T > 0$. One advantage of this approach is that it allows to easily derive estimates on the number of actuators needed to guarantee the stabilization result. Later, the same idea was used to derive the internal and boundary stabilization results for higher dimensions $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, see [9, 27].

The above stabilizability results also include a feedback control which is given through the solution of a differential Riccati operator. Since solving a differential Riccati equation for accurate approximations (fine discretizations, large number of degrees of freedom) and a relatively long time-horizon can be a difficult numerical task, a new tool has been proposed recently in [20] to tackle parabolic equations. Namely, the explicit oblique projection feedback as in (1.5), whose numerical computation amounts essentially to the computation of the inverse of an $M \times M$ matrix. Thus its numerical difficulty is essentially independent of the degrees of freedom of the discretized equations.

Here, we shall show that a similar explicit feedback is able to stabilize weakly damped linear wave equations.

1.4. **Further comments on stabilization of partial differential equations.** Stabilization of nonautonomous systems appear in applications when we want to stabilize our system to one of its trajectories. If such targeted trajectory is time-dependent then the problem can be rewritten as the stabilization (to zero) of a nonautonomous system, by considering the dynamics of the difference to the target (see, e.g., [6, Sect. 2.2], [27]). The most studied case is the case where the targeted trajectory is time-independent, that is, a steady state (an equilibrium). Since such steady states will not exist if our dynamical system is subject to time-dependent external forces, we conclude that stabilization to time-dependent trajectories is no less important than stabilization to steady states. Of course, in the particular case where steady states exist, such time-independent trajectories are "naturally" desirable targets. Stabilization to steady states reduces to the stabilization (to zero) of autonomous systems, where usually the spectral properties of the (linearized) dynamics operator are the main tool used to derive the stabilization results. For works dealing with such autonomous systems, and finite-dimensional controls we refer the reader to [7, 5, 24, 22, 4, 29] and references therein.

The above results concern the stabilization of parabolic-like equations. Concerning wave-like equations, for stabilization results by using a finite number of actuators we refer the reader to [2, 17, 23]. In these works the nonlinearity is such that the solutions of the uncontrolled systems do not blow-up in finite time. We are interested in finite-dimensional controls, because in applications we will have only a finite number of actuators at our disposal. Though infinite-dimensional controls are not practical for real world applications, an impressive amount of research has been devoted to the stabilization by using such controls. This is still an interesting mathematical problem and could be seen as a first step towards the construction of a practical finite-dimensional control. For results on the stabilization of wave-like equations, by means of infinite-dimensional controls, and on the stability of damped wave-like equations, we refer the reader to [3, 15, 1, 40, 39, 36, 14]. We refer also to [28] and references therein, for analogous stabilization results concerning the Navier–Stokes equations (which is parabolic like) and to [8], concerning the stabilization of a coupled parabolic-ODE system.

1.5. **Contents and general notation.** The rest of the paper is organized as follows. In Section 2 we consider damped wave-like equations in an abstract form and introduce the general properties required for the operators involved in the dynamics. Further, we investigate a relaxed form of the set of conditions (1.4) for the oblique projection feedback operator. In Section 3 we prove our main global stability result for the linear case. Then this result is strengthened to the local stability result for the nonlinear case in Section 4. Section 5 presents numerical experiments which validate the theoretical results in the previous sections. Short final comments are given in Section 6. Finally, the appendix gathers proofs of auxiliary results used in the main text.

Concerning the notation, we write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_r := (r, +\infty)$ and $\overline{\mathbb{R}_r} := [r, +\infty)$, for $r \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

For an open interval $I \subseteq \mathbb{R}$ and two Banach spaces X, Y, we write $W(I, X, Y) := \{y \in L^2(I, X) \mid \dot{y} \in L^2(I, Y)\}$, where $\dot{y} := \frac{\mathrm{d}}{\mathrm{d}t} y$ is taken in the sense of distributions. This space is endowed with the natural norm $|y|_{W(I, X, Y)} := \left(|y|^2_{L^2(I, X)} + |\dot{y}|^2_{L^2(I, Y)}\right)^{1/2}$.

If the inclusions $X \subseteq Z$ and $Y \subseteq Z$ are continuous for a Hausdorff topological space Z, then we can define the Banach spaces $X \times Y$, $X \cap Y$, and $X + Y$, endowed with the norms defined as $|(a, b)|_{X\times Y} := \left(|a|^2_X + |b|^2_Y\right)^{\frac{1}{2}}$, $|a|_{X\cap Y} := |(a, a)|_{X\times Y}$, and $|a|_{X+Y} := \inf_{(a^X, a^Y)\in X\times Y} \left\{|(a^X, a^Y)|_{X\times Y} \mid a = a^X + a^Y\right\}$, respectively. In case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead.

If the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \overset{\mathrm{d}}{\hookrightarrow} Y$, respectively $X \overset{\mathrm{c}}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact.

The space of continuous linear mappings from X into Y is denoted by $\mathcal{L}(X, Y)$. In case $X = Y$ we write $\mathcal{L}(X) := \mathcal{L}(X, X)$. The continuous dual of X is denoted $X' := \mathcal{L}(X, \mathbb{R})$.

The space of continuous functions from X into Y is denoted by $C(X, Y)$.

The orthogonal complement to a subset $S \subseteq H$ of a Hilbert space H, with scalar product $(\cdot, \cdot)_H$, is denoted by $S^\perp := \{h \in H \mid (h, s)_H = 0 \text{ for all } s \in S\}$.

By $\overline{C}_{[a_1,\dots,a_n]}$ we denote a nonnegative function that increases in each of its nonnegative arguments a_i, $1 \le i \le n$.

Finally, C, C_i, $i = 0, 1, \dots,$ stand for unessential positive constants.

2. Assumptions

As we will recall later on, with either Dirichlet or Neumann boundary conditions, system (1.2) can be rewritten as the evolutionary system

$$\ddot{y} + Ay + \varsigma\dot{y} + A_{\mathrm{r}}y + \mathcal{N}(y) = h, \tag{2.1a}$$

$$y(0) = y_0, \quad \dot{y}(0) = y_1, \tag{2.1b}$$

evolving in a suitable Hilbert space. Here we have taken a general external forcing h in the place of the control.

2.1. **Assumptions on the dynamical operators.** Let H and V be separable Hilbert spaces, with $V \subseteq H$. We will consider H as pivot space, that is, $H' = H$.

Assumption 2.1. *$A \in \mathcal{L}(V, V')$ is an isomorphism from V onto V', A is symmetric, and $(y, z) \mapsto \langle Ay, z\rangle_{V',V}$ is a complete scalar product on V.*

From now on we suppose that V is endowed with the scalar product $(y, z)_V \coloneqq \langle Ay, z\rangle_{V',V}$, which still makes V a Hilbert space. Therefore, $A\colon V \to V'$ is an isometry.

Assumption 2.2. *The inclusion $V \subseteq H$ is continuous, dense, and compact.*

Necessarily, we have that the operator A is densely defined in H, with domain $\mathrm{D}(A) \coloneqq \{u \in V \mid Au \in H\}$ endowed with the scalar product $(y, z)_{\mathrm{D}(A)} \coloneqq (Ay, Az)_H$, and the inclusions

$$\mathrm{D}(A) \xhookrightarrow{\mathrm{d,c}} V \xhookrightarrow{\mathrm{d,c}} H \xhookrightarrow{\mathrm{d,c}} V' \xhookrightarrow{\mathrm{d,c}} \mathrm{D}(A)'.$$

Further, A has compact inverse $A^{-1}\colon H \to \mathrm{D}(A)$, and we can find a nondecreasing system of (repeated) eigenvalues $(\alpha_i)_{i\in\mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_i)_{i\in\mathbb{N}_0}$:

$$0 < \alpha_1 \le \alpha_2 \le \dots \le \alpha_i \le \alpha_{i+1} \to +\infty \quad \text{and} \quad Ae_i = \alpha_i e_i.$$

For the time-dependent operators we assume the following:

Assumption 2.3. *For (almost) all $t > 0$ we have $A_{\mathrm{r}}(t) \in \mathcal{L}(H)$, and there is a nonnegative constant C_{r} such that, $|A_{\mathrm{r}}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H))} \le C_{\mathrm{r}}$.*

Assumption 2.4. *There exist constants $C_{\mathcal{N}} \ge 0$, $\zeta \ge 0$, and $\eta \ge 1$ such that for (almost) all $t \ge 0$,*

$$\zeta + \eta > 1 \quad \textit{and} \quad |\mathcal{N}(t, y) - \mathcal{N}(t, z)|_H \le C_{\mathcal{N}} \left(|y|_V^\zeta + |z|_V^\zeta\right) |y - z|_V^\eta . \tag{2.2}$$

Remark 2.5. We can also consider nonlinearities satisfying, for some integer $n > 1$, $|\mathcal{N}(t, y) - \mathcal{N}(t, z)|_H \le C_{\mathcal{N}} \sum\limits_{i=1}^{n} \left(|y|_V^{\zeta_i} + |z|_V^{\zeta_i}\right) |y - z|_V^{\eta_i}$, with $\eta_i \ge 1$ and $\zeta_i + \eta_i > 1$. For the sake of simplicity of the exposition, we restrict ourselves to the case $n = 1$.

2.2. **Assumptions on the feedback control construction.** Looking at conditions (1.4), we consider sequences of subspaces

$$E_{\{1\}},\ E_{\{1,2\}}, \dots,\ E_{\{1,2,3,\dots,M\}} \qquad \text{and} \qquad U_1,\ U_2, \dots,\ U_M, \tag{2.3}$$

where the Mth term of each sequence is an M-dimensional space, that is, $\dim E_{\mathbb{M}} = M = \dim U_M$.

Motivated by the results in [20, Sect. 4.8] (see also [20, Rem. 3.9]), in order to prove the boundedness of the norm $\left|P_{U_M}^{E_{\mathbb{M}}^\perp}\right|_{\mathcal{L}(H)} \le C_P$, uniformly on M, it may be convenient to consider different sequences.

To simplify the exposition, we denote by $\#Z \in \mathbb{N}$ the number of elements of a given finite set $Z \subseteq Y$, see [13, Sect. 13]. For $N \in \mathbb{N}_0$, $\#Z = N$ simply means that there exists a one-to-one correspondence from $\{1, 2, \dots, N\}$ onto Z. Of course $\#Z = 0$ means that $Z = \emptyset$, the empty set. We also denote the collection

$$\mathfrak{P}_N(Y) = \{Z \subseteq Y \mid \#Z = N\}.$$

Now instead of (2.3), for every $M \in \mathbb{N}_0$ we consider the following more general sequences

$$E_{\{\sigma_1^1,\sigma_2^1,\ldots,\sigma_{n(1)}^1\}},\ E_{\{\sigma_1^2,\sigma_2^2,\ldots,\sigma_{n(2)}^2\}},\ldots,\ E_{\{\sigma_1^M,\sigma_2^M,\ldots,\sigma_{n(M)}^M\}},$$

and

$$U_{n(1)},\ U_{n(2)},\ldots,\ U_{n(M)}.$$

For each $M \in \mathbb{N}_0$, we denote $\mathbb{M}_\sigma \coloneqq \{\sigma_1^M,\sigma_2^M,\ldots,\sigma_{n(M)}^M\}$ where the notation "$\mathbb{M}_\sigma$" underlines the association with M. The set $\mathbb{M}_\sigma$ is specified by an injective sequence $\sigma^M : \{1,2,\ldots,n(M)\} \to \mathbb{N}_0$ which, in particular, gives us the cardinality of $\mathbb{M}_\sigma$, $\#\mathbb{M}_\sigma = n(M)$. That is, we have the bijection $\sigma^M \colon \{1,2,\ldots,n(M)\} \to \mathbb{M}_\sigma \in \mathfrak{P}_{\#\mathbb{M}_\sigma}(\mathbb{N}_0)$ defined by $i \mapsto \sigma_i^M$. Further, we define

$$E_{\mathbb{M}_\sigma} \coloneqq \operatorname{span}\{e_i \mid i \in \mathbb{M}_\sigma\} \qquad \text{and} \qquad U_{\#\mathbb{M}_\sigma} \coloneqq U_{n(M)}, \tag{2.4}$$

with the property that $\dim E_{\mathbb{M}_\sigma} = \#\mathbb{M}_\sigma = \dim U_{\#\mathbb{M}_\sigma} = n(M)$.

With this notation, $\#\mathbb{M}_\sigma$ will correspond to the number of actuators.

For each $M \in \mathbb{N}_0$, we also consider the two particular eigenvalues defined as

$$\alpha_{\mathbb{M}_\sigma} \coloneqq \max\{\alpha_i \mid i \in \mathbb{M}_\sigma\}, \qquad \alpha_{\mathbb{M}_{\sigma+}} \coloneqq \min\{\alpha_i \mid i \notin \mathbb{M}_\sigma\}. \tag{2.5}$$

Notice that, the sequence (2.3) is the particular case of (2.4) where $\sigma_i^M = i$, $\#\mathbb{M}_\sigma = M$, $\mathbb{M}_\sigma = \mathbb{M} = \{1,\ldots,M\}$, $\alpha_{\mathbb{M}_\sigma} = \alpha_M$, and $\alpha_{\mathbb{M}_{\sigma+}} = \alpha_{M+1}$.

The results in [20] tell us that the linear closed-loop parabolic-like system

$$\dot z + Az + A_{\mathrm r}(t)z - \widehat{\mathbf{K}}_{U_{\#\mathbb{M}_\sigma}}^{\widehat{\mathcal F}_{\mathbb{M}_\sigma}}(t,z) = 0, \quad z(0) = z_0 \in H, \tag{2.6}$$

is *globally* exponentially stable, with the explicit feedback control

$$z \mapsto \widehat{\mathbf{K}}_{U_{\#\mathbb{M}_\sigma}}^{\widehat{\mathcal F}_{\mathbb{M}_\sigma}}(t,z) = P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp}\left(Az + A_{\mathrm r}(t)z - \widehat{\mathcal F}_{\mathbb{M}_\sigma} z)\right),$$

provided (a corollary of) (1.4) holds true, with $(\mathbb{M}_\sigma, \widehat{\mathcal F}_{\mathbb{M}_\sigma}) = (\mathbb{M}, \lambda\mathbf{1})$.

In [32], dealing with parabolic equations and suitable nonlinearities, it was observed that we (may) need to take $\widehat{\mathcal F}_{\mathbb{M}_\sigma} \neq \lambda\mathbf{1}$, namely $\widehat{\mathcal F}_{\mathbb{M}_\sigma} = A + \lambda\mathbf{1}$. This is the reason why we will consider in this section (see Assumption 2.6) a general linear bounded operator $\mathcal F_{\mathbb{M}_\sigma} \in \mathcal L(E_{\mathbb{M}_\sigma} \times E_{\mathbb{M}_\sigma}, E_{\mathbb{M}_\sigma})$.

We are going to show that also the linear wave like-system

$$\ddot y + Ay + \varsigma\dot y + A_{\mathrm r}y = \mathbf{K}_{U_{\#\mathbb{M}_\sigma}}^{\mathcal F_{\mathbb{M}_\sigma}} y, \tag{2.7a}$$

$$y(0) = y_0, \quad \dot y(0) = y_1, \tag{2.7b}$$

is *globally* exponentially stable, with the analogous explicit feedback control

$$y \mapsto \mathbf{K}_{U_{\#\mathbb{M}_\sigma}}^{\mathcal F_{\mathbb{M}_\sigma}}(t,y) \coloneqq P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp}\left(Ay + A_{\mathrm r}(t)y + \varsigma\dot y - \mathcal F_{\mathbb{M}_\sigma}(P_{E_{\mathbb{M}_\sigma}}y, P_{E_{\mathbb{M}_\sigma}}\dot y)\right), \tag{2.8}$$

provided conditions analogous to (1.4) are satisfied. To introduce such conditions, first we consider the following dynamical equation

$$\ddot q = -\mathcal F_{\mathbb{M}_\sigma}(q,\dot q), \qquad (q(s),\dot q(s)) = (q_{0,s}, q_{1,s}) \in E_{\mathbb{M}_\sigma} \times E_{\mathbb{M}_\sigma}, \qquad t > s, \tag{2.9}$$

evolving in the finite-dimensional space $E_{\mathbb{M}_\sigma} \times E_{\mathbb{M}_\sigma} \subset H \times H$ for an initial time $s \geq 0$. The dynamics of (2.9) will be required to be exponentially stable. This fact motivates the following requirements for the triple $(E_{\mathbb{M}_\sigma}, U_{\#\mathbb{M}_\sigma}, \mathcal F_{\mathbb{M}_\sigma})$ defining the feedback operator.

Assumption 2.6. *For the sequence of triples* $(E_{\mathbb{M}_\sigma}, U_{\#\mathbb{M}_\sigma}, \mathcal{F}_{\mathbb{M}_\sigma})_{M\in\mathbb{N}_0}$ *it holds:*

- $\mathcal{F}_{\mathbb{M}_\sigma} \in \mathcal{L}(E_{\mathbb{M}_\sigma} \times E_{\mathbb{M}_\sigma}, E_{\mathbb{M}_\sigma})$; (2.10a)
- *There are* $C_{\mathcal{F}} \ge 1$ *and* $\lambda > 0$, *independent of* $s \ge 0$, *such that the solution of* (2.9) *satisfies: for all* $s \ge 0$ *and all* $(q_{0,s}, q_{1,s}) \in E_{\mathbb{M}_\sigma} \times E_{\mathbb{M}_\sigma}$,
$$\left|(q(t), \dot q(t))\right|_{V\times H} \le C_{\mathcal{F}} \mathrm{e}^{-\lambda(t-s)} \left|(q_{0,s}, q_{1,s})\right|_{V\times H}; \tag{2.10b}$$
- $\lim\limits_{M\to+\infty} \alpha_{\mathbb{M}_{\sigma}+} \to +\infty$; (2.10c)
- $L^2(\Omega) = U_{\#\mathbb{M}_\sigma} \oplus E_{\mathbb{M}_\sigma}^\perp$; (2.10d)
- $\left|P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp}\right|_{\mathcal{L}(H)} \le C_P < +\infty$, *with* $C_P > 0$ *independent of* M. (2.10e)

As we will see, while for the parabolic system (2.6) the operator $\widehat{\mathcal{F}}_{\mathbb{M}_\sigma}$ imposes the dynamics of the component $q_z := P_{E_{\mathbb{M}_\sigma}} z$, for the wave system (2.7) the operator $\mathcal{F}_{\mathbb{M}_\sigma}$ imposes the dynamics of the component $(q_y, \dot q_y) := (P_{E_{\mathbb{M}_\sigma}} y, P_{E_{\mathbb{M}_\sigma}} \dot y)$. In particular, for the choice of $\widehat{\mathcal{F}}_{\mathbb{M}_\sigma} P_{E_{\mathbb{M}_\sigma}} z = \lambda \mathbf{1} P_{E_{\mathbb{M}_\sigma}} z$ for (2.6), we obtain the following stable dynamical system

$$\dot q_z(t) = -\lambda q_z(t) \quad t \ge s, \qquad q_z(s) = q_s. \tag{2.11}$$

For every solution of (2.11), we have $q_z(t) = \mathrm{e}^{-\lambda(t-s)} q_z(s)$ for $t \ge s \ge 0$. Analogously, for the linear wave-like system (2.7), by setting

$$\mathcal{F}_{\mathbb{M}_\sigma}(P_{E_{\mathbb{M}_\sigma}} y, P_{E_{\mathbb{M}_\sigma}} \dot y) = \lambda_1 \mathbf{1} P_{E_{\mathbb{M}_\sigma}} y + \lambda_2 \mathbf{1} P_{E_{\mathbb{M}_\sigma}} \dot y, \tag{2.12}$$

we come up with the following stable dynamical system

$$\ddot q_y = -\lambda_1 q_y - \lambda_2 \dot q_y \quad t \ge s, \quad (q_y(s), \dot q_y(s)) = (q_{0,s}, q_{1,s}). \tag{2.13}$$

Note that, with $\mathcal{A} := \lambda_1 \mathbf{1}$, $\mathcal{H} := E_{\mathbb{M}_\sigma} =: \mathcal{V}$, and $\varsigma := \lambda_2$, system (2.13) takes the form of a damped wave-like equation as

$$\ddot q + \mathcal{A} q + \varsigma \dot q = 0 \quad t \ge 0, \quad (q(0), \dot q(0)) = (q_0, q_1) \in \mathcal{V} \times \mathcal{H}, \tag{2.14}$$

which is stable provided that Assumptions 2.1–2.2 hold with $(\mathcal{A}, \mathcal{V}, \mathcal{H})$ in the place of (A, V, H). More precisely, we have the following.

Lemma 2.7. *There are constants* $C_{\mathrm{w}} \ge 1$ *and* $\mu_{\mathrm{w}} > 0$ *such that for all* $(y_0, y_1) \in \mathcal{V} \times \mathcal{H}$ *the solution of* (2.14) *satisfies*

$$\left|(q(t), \dot q(t))\right|_{\mathcal{V}\times\mathcal{H}} \le C_{\mathrm{w}} \mathrm{e}^{-\mu_{\mathrm{w}} t} \left|(q_0, q_1)\right|_{\mathcal{V}\times\mathcal{H}}. \tag{2.15}$$

Therefore, we have the exponential stability of system (2.13).

Proofs of Lemma 2.7 can be found in [37, 35]. The rate $-\mu_{\mathrm{w}}$ depends on ς, for example in [37, Ch. IV, Sect. 1.2] we find that $\mu_{\mathrm{w}} \le \min\{\frac{\varsigma}{4}, \frac{\alpha_1}{2\varsigma}\}$, while in [35, Ch. 3, Sect. 3.8.4, Cor. 38.8] we find $\mu_{\mathrm{w}} \le \min\{1, \frac{\varsigma}{2}, \frac{\alpha_1}{1+\varsigma}\}$, where $\alpha_1 > 0$ is the first (smallest) eigenvalue of $\mathcal{A}$. Those different estimates also suggest that the "best" possible $\overline{\mu}_{\mathrm{w}} := \sup\{\mu_{\mathrm{w}} \mid (2.15) \text{ holds true for some } C_{\mathrm{w}}\}$ is not trivial question. In particular, a large ς does not necessarily lead to a large $\overline{\mu}_{\mathrm{w}}$, see [11].

3. The linear system

In this section, we prove Theorem 1.2.

First of all, observe that by setting $Q := P_{E_{\mathbb{M}_\sigma}^\perp} y$, the dynamics of (2.7) can be rewritten as

$$\ddot q = -\mathcal{F}_{E_{\mathbb{M}_\sigma}}(q, \dot q), \tag{3.1a}$$

$$\ddot Q + P_{E_{\mathbb{M}_\sigma}^\perp}\Big(Ay + \varsigma \dot y + A_{\mathrm{r}} y\Big) = P_{E_{\mathbb{M}_\sigma}^\perp} P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp}\Big(Ay + \varsigma \dot y + A_{\mathrm{r}} y - \mathcal{F}_{\mathbb{M}_\sigma}(q, \dot q)\Big). \tag{3.1b}$$

Using the fact that $\mathbf{1} - P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} = P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}}$, system (3.1) can be expressed as

$$\ddot{q} = -\mathcal{F}_{E_{\mathbb{M}_\sigma}}(q, \dot{q}), \tag{3.2a}$$

$$\ddot{Q} + P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} \Big(Ay + \varsigma \dot{y} + A_{\mathrm{r}} y \Big) = -P_{E_{\mathbb{M}_\sigma}^\perp} P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} \mathcal{F}_{\mathbb{M}_\sigma}(q, \dot{q}). \tag{3.2b}$$

Now since $\xi \coloneqq \mathcal{F}_{\mathbb{M}_\sigma}(q, \dot{q}) \in E_{\mathbb{M}_\sigma}$, we have

$$\begin{aligned} P_{E_{\mathbb{M}_\sigma}^\perp} P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} \xi &= P_{E_{\mathbb{M}_\sigma}^\perp} P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} P_{E_{\mathbb{M}_\sigma}} \xi = (\mathbf{1} - P_{E_{\mathbb{M}_\sigma}}) P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} P_{E_{\mathbb{M}_\sigma}} \xi = P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} \xi - \xi \\ &= -P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} \xi \end{aligned}$$

and, as a consequence, due to (3.2), we can rewrite (2.7) as

$$\ddot{q} = -\mathcal{F}_{E_{\mathbb{M}_\sigma}}(q, \dot{q}), \tag{3.3a}$$

$$q(s) = q_0 \in E_{\mathbb{M}_\sigma}, \quad \dot{q}(s) = q_1 \in E_{\mathbb{M}_\sigma}, \tag{3.3b}$$

$$\ddot{Q} + P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} \Big(Ay + \varsigma \dot{y} + A_{\mathrm{r}} y \Big) = P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} \mathcal{F}_{\mathbb{M}_\sigma}(q, \dot{q}), \tag{3.3c}$$

$$Q(s) = Q_0 \in V \bigcap E_{\mathbb{M}_\sigma}^\perp, \quad \dot{Q}(s) = Q_1 \in E_{\mathbb{M}_\sigma}^\perp. \tag{3.3d}$$

Since the dynamics of (1.8) can be written as (2.7), and hence as (3.3), then Theorem 1.2 shall follow as a corollary of the following result.

Theorem 3.1. *Let Assumptions 2.1–2.3 and 2.6 hold true. Then system (2.7) is exponentially stable, for M large enough. There are constants $\overline{C} = \overline{C}_{[C_\mathcal{F}]} \ge 1$ and $0 < \overline{\mu} < 2\lambda$ such that for all $(y_0, y_1) \in V \times H$, we have $(y, \dot{y}) \in C(\overline{\mathbb{R}_0}; V \times H)$ and*

$$|y(t), \dot{y}(t)|_{V \times H} \le \overline{C} \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)} |y(s), \dot{y}(s)|_{V \times H}, \qquad t \ge s \ge 0. \tag{3.4}$$

The constants $\overline{C}$ and $\overline{\mu}$ may also depend on M.

For the proof of Theorem 3.1 we will need the following proposition, whose proof is given in the Appendix, Section A.1.

Proposition 3.2. *Let $\mu_1 > 0$ and $\mu_2 > 0$. For every $0 \le \mu_0 \le \min\{\mu_1, \mu_2\}$, with $\max\{\mu_1, \mu_2\} > \mu_0$, we have the inequality*

$$\int_s^t \mathrm{e}^{-\mu_1(t-\tau)} \mathrm{e}^{-\mu_2(\tau - s)} \, \mathrm{d}\tau \le \overline{C}_{\left[\frac{1}{\max\{\mu_1,\mu_2\} - \mu_0}\right]} \mathrm{e}^{-\mu_0(t-s)}.$$

Another auxiliary result is the following, whose proof can be found in [20].

Lemma 3.3. *Let $E_{\mathbb{M}_\sigma}^{\perp, V'} \subset V'$ be the orthogonal complement to $U_{\#\mathbb{M}_\sigma} \subset V'$ in V'. Then the oblique projection, in V' onto $U_{\#\mathbb{M}_\sigma}$ along $E_{\mathbb{M}_\sigma}^{\perp, V'}$, $P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^{\perp, V'}} \in \mathcal{L}(V')$ is an extension of the analogous oblique projection in H, $P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} = P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^{\perp, H}} \in \mathcal{L}(H)$. That is, $P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^{\perp, V'}} h = P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp} h$, for all $h \in H$.*

Hereafter, for simplicity, as in [20] we shall still denote $P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^{\perp, V'}}$ by $P_{U_{\#\mathbb{M}_\sigma}}^{E_{\mathbb{M}_\sigma}^\perp}$.

3.1. **Proof of Theorem 3.1.** The existence and uniqueness of the solution y to (2.7) follows from standard arguments for linear wave equations, see [37, Sect. 4] for the details. Here we restrict ourselves to the formal derivation of suitable a priori-like estimates leading us to the stability result.

Observe that we can write system (3.3c)–(3.3d) as a damped wave system

$$\ddot{Q} + AQ + \varsigma \dot{Q} + P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} A_{\mathrm{r}} Q = h(q, \dot{q}), \tag{3.5a}$$

$$Q(s) = Q_0 \in V \bigcap E_{\mathbb{M}_\sigma}^\perp, \quad \dot{Q}(s) = Q_1 \in E_{\mathbb{M}_\sigma}^\perp, \tag{3.5b}$$

for time $t \geq s \geq 0$, with the external forcing

$$h(q, \dot{q}) \coloneqq -P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} \left(Aq + \varsigma \dot{q} + A_{\rm r} q \right) + P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} \mathcal{F}_{\mathbb{M}_\sigma}(q, \dot{q}). \tag{3.5c}$$

Some of the following arguments are a slight variation of standards ones. We start by looking at the dynamics of $W \coloneqq \dot{Q} + \varepsilon Q$, with $\varepsilon \in (0, \varsigma)$,

$$\begin{aligned} \dot{W} = \ddot{Q} + \varepsilon \dot{Q} &= -AQ - (\varsigma - \varepsilon)\dot{Q} - P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} A_{\rm r} Q + h \\ &= -AQ - (\varsigma - \varepsilon) W + \varepsilon(\varsigma - \varepsilon) Q - P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} A_{\rm r} Q + h \end{aligned}$$

and, by taking the duality product $\langle \cdot, \cdot \rangle_{V',V}$ with W, we obtain

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} |W|_H^2 &= -2(Q, W)_V - 2(\varsigma - \varepsilon) |W|_H^2 \\ &\quad + 2\varepsilon(\varsigma - \varepsilon)(Q, W)_H - 2(P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} A_{\rm r} Q, W)_H + 2(h, W)_H. \end{aligned}$$

Recalling (2.5) and using the fact that $|Q|_H \leq \alpha_{\mathbb{M}_{\sigma+}}^{-\frac{1}{2}} |Q|_V$, we find

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \left(|W|_H^2 + |Q|_V^2 \right) &\leq -2\varepsilon |Q|_V^2 - 2(\varsigma - \varepsilon) |W|_H^2 + 2\varepsilon(\varsigma - \varepsilon)\alpha_{\mathbb{M}_{\sigma+}}^{-\frac{1}{2}} |Q|_V |W|_H \\ &\quad + 2 \left| P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} A_{\rm r} \right|_{\mathcal{L}(H)} \alpha_{\mathbb{M}_{\sigma+}}^{-\frac{1}{2}} |Q|_V |W|_H + 2 |h|_H |W|_H . \end{aligned}$$

Next, by using the appropriate Young inequalities, we find that, for an arbitrary $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (0, +\infty)^3$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(|W|_H^2 + |Q|_V^2 \right) \leq -\Psi_Q |Q|_V^2 - \Psi_W |W|_H^2 + \gamma_3^{-1} |h|_H^2 , \tag{3.6a}$$

with

$$\Psi_Q = \left(2\varepsilon - \gamma_1^{-1}\varepsilon^2(\varsigma - \varepsilon)^2 \alpha_{\mathbb{M}_{\sigma+}}^{-1} - \gamma_2^{-2} \left| P_{E_{\mathbb{M}_\sigma}^\perp}^{U_{\#\mathbb{M}_\sigma}} A_{\rm r} \right|_{\mathcal{L}(H)}^2 \alpha_{\mathbb{M}_{\sigma+}}^{-1} \right), \tag{3.6b}$$

$$\Psi_W = \quad (2(\varsigma - \varepsilon) - \gamma_1 - \gamma_2 - \gamma_3). \tag{3.6c}$$

Since $\varsigma > 0$, it is clear that we can choose a quadruple $(\varepsilon, \gamma_1, \gamma_2, \gamma_3) \in (0, +\infty)^4$, with small enough coordinates, such that

$$0 < \varepsilon < \varsigma \quad \text{and} \quad \Psi_W > 0.$$

Then for such a fixed quadruple and for M large enough it follows that $\Psi_Q > 0$, due to (2.10c) and (2.10e). Thus, we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \left(|W|_H^2 + |Q|_V^2 \right) \leq -\mu \left(|W|_H^2 + |Q|_V^2 \right) + \gamma_3^{-1} |h|_H^2 , \tag{3.7}$$

with $\mu \coloneqq \min\{\Psi_W, \Psi_Q\}$.

From (3.5c) and (2.10b), we have that

$$|h(q(t), \dot{q}(t))|_H \leq C_1 |(q(t), \dot{q}(t))|_{V \times H} \leq C_2 \mathrm{e}^{-\lambda(t-s)} |(q(s), \dot{q}(s))|_{V \times H} .$$

By the Gronwall inequality it follows that, with $Z(t) \coloneqq |W(t)|_H^2 + |Q(t)|_V^2$ and $z(t) \coloneqq |(q(t), \dot{q}(t))|_{V \times H}^2$,

$$\begin{aligned} Z(t) &\leq \mathrm{e}^{-\mu(t-s)} Z(s) + \int_s^t \mathrm{e}^{-\mu(t-\tau)} \gamma_3^{-1} |h(\tau)|_H^2 \, \mathrm{d}\tau \\ &\leq \mathrm{e}^{-\mu(t-s)} Z(s) + \gamma_3^{-1} C_2^2 z(s) \int_s^t \mathrm{e}^{-\mu(t-\tau)} \mathrm{e}^{-2\lambda(\tau - s)} \, \mathrm{d}\tau. \end{aligned}$$

Hence, by Proposition 3.2 it follows that, for any given $\overline{\mu} < \min\{\mu, 2\lambda\}$ and a suitable constant $D = D(\overline{\mu}, \mu, 2\lambda) > 0$,

$$Z(t) \leq \mathrm{e}^{-\mu(t-s)} Z(s) + D\gamma_3^{-1} C_2^2 z(s) \mathrm{e}^{-\overline{\mu}(t-s)}.$$

Now, from $W = \dot{Q} + \varepsilon Q$, we have that the norm $(|Q|_V^2 + |W|_H^2)^{\frac{1}{2}} + |(q,\dot{q})|_{V\times H}$ is equivalent to the norm $\left|(Q+q,\dot{q}+\dot{Q})\right|_{V\times H}$, that is, there are constants $C_3 > 0$ and $C_4 > 0$ such that

$$\begin{aligned} C_3 \left|(q+Q,\dot{q}+\dot{Q})\right|_{V\times H} &\le (|Q|_V^2 + |W|_H^2)^{\frac{1}{2}} + |(q,\dot{q})|_{V\times H} \\ &\le C_4 \left|(q+Q,\dot{q}+\dot{Q})\right|_{V\times H}. \end{aligned}$$

Thus, using (2.10b), we conclude that

$$\begin{aligned} |(y(t),\dot{y}(t))|_{V\times H} &= \left|(q(t)+Q(t),\dot{q}(t)+\dot{Q}(t))\right|_{V\times H} \le C_3^{-1}\left(Z(t)^{\frac{1}{2}} + z(t)^{\frac{1}{2}}\right) \\ &\le C_3^{-1}\max\left\{1, D^{\frac{1}{2}}\gamma_3^{-\frac{1}{2}}C_2, C_{\mathcal{F}}\right\} \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)}\left(Z(s)^{\frac{1}{2}} + z(s)^{\frac{1}{2}}\right) \\ &\le C_4 C_3^{-1}\max\left\{D^{\frac{1}{2}}\gamma_3^{-\frac{1}{2}}C_2, C_{\mathcal{F}}\right\} \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)}\left|y(s),\dot{y}(s)\right|_{V\times H}, \end{aligned} \tag{3.8}$$

since $C_{\mathcal{F}} \ge 1$. This finishes the verification of (3.4).

It remains to show that $(y,\dot{y}) \in C(\overline{\mathbb{R}_0}; V\times H)$, which is equivalent to showing that $(y,\dot{y}) \in C([0,T]; V\times H)$ for arbitrary $T>0$. Here, we follow the argument given in [37, Ch. II, Thm. 4.1]. For that purpose, we fix an arbitrary $T>0$ and start by showing that $\ddot{y} \in L^2([0,T];V')$ and $|(y,\dot{y})|_{V\times H}^2 \in C([0,T];\mathbb{R})$.

Observe that from $(y,\dot{y}) \in L^\infty(\mathbb{R}_0; V\times H)$, it follows that $Ay + A_{\mathrm{r}}y + \varsigma\dot{y} \in L^\infty(\mathbb{R}_0; V')$. By (2.10a) we also have that $\mathcal{F}_{\mathbb{M}_\sigma} \in \mathcal{L}(H\times H, H)$ which implies that $\mathcal{F}_{\mathbb{M}_\sigma} \in \mathcal{L}(V\times H, H)$, because $E_{\mathbb{M}_\sigma}$ is finite-dimensional.

Thus by (2.7), (2.8), $H \hookrightarrow V'$, and Lemma 3.3, we derive that $\ddot{y} \in L^\infty(\mathbb{R}_0; V')$.

Note that $\kappa \coloneqq \mathbf{K}_{U_{\#\mathbb{M}_\sigma}}^{\mathcal{F}_{\mathbb{M}_\sigma}} y \in L^\infty([0,T],H)$, since $\kappa \in L^\infty([0,T],V')$ and $U_{\#\mathbb{M}_\sigma} \subset H$ is finite-dimensional. Hence we find

$$(y,\dot{y},\ddot{y},\kappa) \in L^\infty((0,T); V\times H\times V'\times H), \tag{3.9a}$$

$$\ddot{y} + Ay \in L^\infty((0,T);H). \tag{3.9b}$$

By multiplying (2.7) with $\dot{y}$ (cf. [37, Ch. II, Lem. 4.1]), we arrive at

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left|(y,\dot{y})\right|_{V\times H}^2 = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(|y|_V^2 + |\dot{y}|_H^2\right) = -2\varsigma\left|\dot{y}\right|_H^2 - 2(A_{\mathrm{r}}y + \kappa, \dot{y})_H \tag{3.9c}$$

which gives us $\frac{\mathrm{d}}{\mathrm{d}t}\left|(y,\dot{y})\right|_{V\times H}^2 \in L^1((0,T),\mathbb{R})$ and $|(y,\dot{y})|_{V\times H}^2 \in C([0,T],\mathbb{R})$.

The continuity of the norm $|(y,\dot{y})|_{V\times H}$ together with the weak continuity of $(y,\dot{y})$ from $[0,T]$ to $V\times H$, which can be derived from [37, Ch. II, Lem. 3.3], lead us to $(y,\dot{y}) \in C([0,T], V\times H)$. □

3.2. **Proof of Theorem 1.2.** Observe that we can write (1.8) as (2.7) by setting

$$\begin{aligned} &A = -\nu\Delta + \mathbf{1}; \quad A_{\mathrm{r}} = (a-1)\mathbf{1}; \\ &H = L^2(\Omega); \quad D(A) = \{z \in H^2(\Omega) \mid \mathcal{G}z|_{\partial\Omega} = 0\}; \end{aligned}$$

and with

$$\begin{aligned} &V = \{z \in H^1(\Omega) \mid z|_{\partial\Omega} = 0\} \text{ when } \mathcal{G} = \mathbf{1} \text{ (Dirichlet boundary conditions)}; \\ &V = H^1(\Omega) \text{ when } \mathcal{G} = \mathbf{n}\cdot\nabla \text{ (Neumann boundary conditions)}. \end{aligned}$$

Therefore, Theorem 1.2 will follow from Theorem 3.1 with $X_{\mathcal{G}} = V\times H$, provided that Assumptions 2.1–2.3 and also conditions (2.10a)–(2.10c) in Assumption 2.6 are satisfied. It is straightforward to check that Assumptions 2.1–2.2 are indeed satisfied. It is also straightforward to check that Assumption 2.3 is satisfied for $a \in L^\infty(\mathbb{R}_0\times\Omega)$. Condition (2.10a) is clearly satisfied with $\mathbb{M}_\sigma = \mathbb{M} = \{1,2,\dots,M\}$ and $\mathcal{F}_{\mathbb{M}_\sigma}(q,\dot{q}) = \lambda_1 q + \lambda_2\dot{q}$. Then condition (2.10b) also follows, with a suitable $\lambda > 0$, because $\ddot{q} = -\lambda_1 q - \lambda_2\dot{q}$ is a damped wave-like equation (recall system (2.13) and take $\lambda = \mu_{\mathrm{w}}$ as in

Lemma 2.7). Condition (2.10c) now reads $\lim_{M\to+\infty} \alpha_{M+1} \to +\infty$, which is known to be true for the ordered eigenvalues of $-\nu\Delta + \mathbf{1}$.

Finally, notice that the assumption (1.4) in Theorem 1.2 corresponds to assumptions (2.10d)–(2.10e). □

3.3. **On the best achievable decreasing rate.** A natural question now could be what is the best exponential decreasing rate that we can achieve with such explicit oblique projection feedback.

Theorem 3.4. *Let an arbitrary $\hat\mu$ with $0 < \hat\mu < \varsigma$ be given. If $2\lambda > \varsigma$, then there exist M large enough and $\hat C \ge 1$ so that for all $(y_0, y_1) \in V \times H$, the solution of system* (3.3) *satisfies*

$$|y(t), \dot y(t)|_{V\times H} \le \hat C \mathrm{e}^{-\frac{\hat\mu}{2}(t-s)} |y(s), \dot y(s)|_{V\times H}, \qquad t \ge s.$$

Proof. Let $\varepsilon \in (0, \varsigma)$ be arbitrary given. Looking at (3.6), we can infer that for any given $\delta > 0$, we can choose the coordinates of $\gamma \in (0, +\infty)^3$ small enough, and then M large enough so that $\Phi_W \ge 2(\varsigma - \varepsilon) - \delta$ and $\Phi_Q \ge 2\varepsilon - \delta$. Hence, with $0 < \delta < \min\{2(\varsigma - \varepsilon), 2\varepsilon\}$ we can take $\mu \coloneqq \min\{2(\varsigma - \varepsilon) - \delta, 2\varepsilon - \delta\}$ in (3.7), and an arbitrary

$$\overline{\mu} < \min\{2(\varsigma - \varepsilon) - \delta, 2\varepsilon - \delta, 2\lambda\}$$

in (3.8). Since $2\lambda > \varsigma$, choosing $\varepsilon = \frac{\varsigma}{2}$ and $\delta = \frac{\varsigma - \hat\mu}{2} < \varsigma - \hat\mu$, we have that we may take $\overline{\mu} < \min\{\varsigma - \delta, 2\lambda\} = \varsigma - \delta = \frac{\varsigma + \hat\mu}{2}$. Thus, we may take $\overline{\mu} = \hat\mu$, because $\hat\mu < \frac{\varsigma+\hat\mu}{2}$. □

Theorem 3.5. *With the oblique projection feedback, we cannot obtain an exponential stabilization rate strictly better than $-\frac{\varsigma}{2}$. That is, we cannot take $\hat\mu > \varsigma$ in Theorems 3.1 and 3.4.*

Proof. Let us consider the particular case of a constant reaction term $A_{\mathrm{r}} = -\rho\mathbf{1}$, with $\rho > 0$. In this case with an initial condition $(y_0, y_1) \in (V \bigcap E^\perp_{\mathbb{M}_\sigma}) \times E^\perp_{\mathbb{M}_\sigma}$ we observe that system (3.3) reads

$$\begin{aligned}
&\ddot q = -\mathcal{F}_{E_{\mathbb{M}_\sigma}}(q, \dot q), \qquad q(s) = 0 = \dot q(s),\\
&\ddot Q + P^{U_{\#\mathbb{M}_\sigma}}_{E^\perp_{\mathbb{M}_\sigma}}\Big(A(Q+q) + \varsigma(\dot Q + \dot q) - \rho(Q+q)\Big) = P^{U_{\#\mathbb{M}_\sigma}}_{E^\perp_{\mathbb{M}_\sigma}} \mathcal{F}_{\mathbb{M}_\sigma}(q, \dot q),\\
&Q(s) = Q_0 = y_0 \in V \textstyle\bigcap E^\perp_{\mathbb{M}_\sigma}, \quad \dot Q(s) = Q_1 = y_1 \in E^\perp_{\mathbb{M}_\sigma}.
\end{aligned}$$

From (2.10b) it follows that $(q, \dot q) = (0, 0)$, thus $(y, \dot y, \ddot y) = (Q, \dot Q, \ddot Q)$ and

$$\ddot Q + AQ + \varsigma \dot Q - \rho Q = 0, \qquad (Q(s), \dot Q(s)) \in (V \textstyle\bigcap E^\perp_{\mathbb{M}_\sigma}) \times E^\perp_{\mathbb{M}_\sigma}, \tag{3.10}$$

whose dynamics can be written, by setting $Z = \dot Q$, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} Q \\ Z \end{bmatrix} = \mathbb{A}\begin{bmatrix} Q \\ Z \end{bmatrix}, \qquad \text{with} \qquad \mathbb{A} \coloneqq \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -A + \rho\mathbf{1} & -\varsigma\mathbf{1} \end{bmatrix}.$$

Following an argument as in [11], we observe that every eigenvalue ζ of $\mathbb{A}$, and one of its associated eigenvectors $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$, must satisfy

$$V_2 = \zeta V_1 \quad \text{and} \quad -AV_1 + \rho V_1 - \varsigma V_2 = \zeta V_2,$$

which implies that $-\zeta(\zeta + \varsigma) + \rho$ must be an eigenvalue of A, as

$$AV_1 = (-\zeta(\zeta+\varsigma) + \rho)V_1.$$

That is, there exists $i \in \mathbb{N}_0$ with $i \notin \mathbb{M}_\sigma$, such that $-\zeta(\zeta + \varsigma) + \rho = \alpha_i$. From

$$\zeta^2 + \varsigma\zeta - \rho + \alpha_i = 0 \quad \Longleftrightarrow \quad \zeta = \zeta^\pm = \tfrac{-\varsigma \pm \sqrt{\varsigma^2 - 4(\alpha_i - \rho)}}{2}, \tag{3.11}$$

we conclude that the real part $\mathrm{Re}(\zeta)$ of ζ satisfies

$$\begin{aligned}
&\mathrm{Re}(\zeta^+) > -\tfrac{\varsigma}{2}, \qquad \text{if} \quad 4(\alpha_i - \rho) < \varsigma^2,\\
&\mathrm{Re}(\zeta^\pm) = -\tfrac{\varsigma}{2}, \qquad \text{if} \quad 4(\alpha_i - \rho) \ge \varsigma^2.
\end{aligned}$$

Hence, independently of α_i, we have always eigenvalues ζ with $\mathrm{Re}(\zeta) \geq -\frac{\varsigma}{2}$.

Therefore the best exponential decreasing rate that we can expect is $-\frac{\varsigma}{2}$. □

3.4. **Further remarks on the case of a constant reaction.** For the constant reaction example in the proof of Theorem 3.5, from (3.11) we can see that $\mathrm{Re}(\zeta^+) > 0$ if $\alpha_i < \rho$ for $i \notin \mathbb{M}_\sigma$. In this case, system (3.10) is not stable. Also $\mathrm{Re}(\zeta) = 0 \Longleftrightarrow \zeta = \zeta^+ = 0 \Longleftrightarrow \alpha_i = \rho$ and in this case the eigenvectors associated to ζ are of the form $\begin{bmatrix} e_i \\ 0 \end{bmatrix}$ where $Ae_i = \alpha_i e_i$ with $i \notin \mathbb{M}_\sigma$. Such an eigenvector is a steady state for system (3.10), therefore the system is not exponentially stable (with a strictly negative rate $-\frac{\mu}{2} < 0$).

Exponential stability of system (3.10) will follow if $\alpha_i > \rho$ for all $i \notin \mathbb{M}_\sigma$, that is, if $\alpha_{\mathbb{M}_\sigma +} > \rho$. Because in this case for any $\alpha_i \geq \alpha_{\mathbb{M}_\sigma +} > \rho$, we will have that $\mathrm{Re}(\zeta^\pm) < 0$. As a consequence also system (3.3) is exponentially stable.

In other words, $\alpha_{\mathbb{M}_\sigma +} > \rho$ is a necessary and sufficient condition for the exponential stability of system (3.3) with a suitable rate $-\frac{\mu}{2} < 0$.

By taking a larger M, so that $\varsigma^2 - 4(\alpha_i - \rho) < 0$, system (3.10) is exponentially stable with rate $-\frac{\varsigma}{2}$. Then, if (2.10b) holds with $\lambda > -\frac{\varsigma}{2}$, it follows that also system (3.3) is exponentially stable with (the best possible) rate $-\frac{\varsigma}{2}$.

3.5. **Short comparison to parabolic equations.** Note that due to Theorem 3.5, the exponential rate $\hat{\mu}$ in Theorem 3.4 for weakly damped wave-like equations (2.7) is bounded from above by ς. This fact highlights a difference to parabolic-like equations (2.6), in which $\widehat{\mathcal{F}}_{\mathbb{M}_\sigma} \in \mathcal{L}(E_{\mathbb{M}_\sigma})$ is chosen such that the dynamics $\dot{q} = -\widehat{\mathcal{F}}_{\mathbb{M}_\sigma} q$ is stable,

$$|q(t)|_H \leq C_{\mathcal{F}} \mathrm{e}^{-\lambda(t-s)} |q(s)|_H \qquad \text{(cf. Assumption 2.6).}$$

Indeed, for system (2.6) under Assumptions 2.1–2.3 and (2.10c)–(2.10e), any a priori given exponential stability rate $\frac{\overline{\mu}}{2}$ can be achieved, provided that $\lambda > \frac{\overline{\mu}}{2}$ and M is large enough.

Moreover, for parabolic-like equations (2.6), setting $\widehat{\mathcal{F}}_{\mathbb{M}_\sigma} = -\lambda \mathbf{1}$ will clearly guarantee the exponential stability of the dynamics $\dot{q} = -\widehat{\mathcal{F}}_{\mathbb{M}_\sigma} q$. However, for damped wave-like equations (2.7) it is not enough to set $\mathcal{F}_{\mathbb{M}_\sigma} \in \mathcal{L}(E_{\mathbb{M}_\sigma} \times E_{\mathbb{M}_\sigma}, E_{\mathbb{M}_\sigma})$ as $\mathcal{F}_{\mathbb{M}_\sigma}(q, \dot{q}) = -\lambda q$ because this would imply that $\ddot{q} = -q$ and that $|\dot{q}(t)|^2_{E_{\mathbb{M}_\sigma}} + |q(t)|^2_{E_{\mathbb{M}_\sigma}} = |\dot{q}(0)|^2_{E_{\mathbb{M}_\sigma}} + |q(0)|^2_{E_{\mathbb{M}_\sigma}}$. Therefore condition (2.10b) will not be satisfied.

Analogously it is not enough to set $\mathcal{F}_{\mathbb{M}_\sigma}(q, \dot{q}) = -\lambda \dot{q}$ because this would imply $\dot{q}(t) = \mathrm{e}^{-\lambda t} \dot{q}(0)$, $q(t) = q(0) + \frac{(1-\mathrm{e}^{-\lambda t})}{\lambda} \dot{q}(0)$, and $\lim_{t \to +\infty} |q(t)|_{E_{\mathbb{M}_\sigma}} = \left| q(0) + \frac{\dot{q}(0)}{\lambda} \right|_{E_{\mathbb{M}_\sigma}}$. Hence, condition (2.10b) will not hold.

4. The nonlinear system

In this section we prove Theorem 1.1. We shall prove the result for a general class of weakly damped wave-like equations of the form

$$\ddot{y} + Ay + \varsigma \dot{y} + A_{\mathrm{r}} y + \mathcal{N}(y) = \mathbf{K}_{U_{\#\mathbb{M}_\sigma}}^{\mathcal{F}_{\mathbb{M}_\sigma}}(y, \dot{y}), \tag{4.1a}$$

$$y(s) = y_0, \quad \dot{y}(s) = y_1, \qquad s \geq 0. \tag{4.1b}$$

with a nonlinearity $\mathcal{N} = \mathcal{N}(t, y)$ satisfying Assumption 2.4. We will consider (4.1) as a perturbation of the linear system (2.7), where the nonlinear perturbation $\mathcal{N}$ satisfies (2.2).

Theorem 4.1. *Let Assumptions 2.1–2.4 and 2.6 hold true. Then system* (4.1) *is locally exponentially stable, for M large enough. There are constants $\epsilon > 0$, $\overline{C} = \overline{C}_{[C_{\mathcal{N}}]} \geq 1$, and $0 < \overline{\mu} < 2\lambda$ such that for all $(y_0, y_1) \in V \times H$, satisfying $|(y_0, y_1)|_{V \times H} < \epsilon$, for the solution $(y, \dot{y}) \in C(\overline{\mathbb{R}_s}; V \times H)$ to* (4.1) *it holds*

$$|(y(t), \dot{y}(t))|_{V \times H} \leq \overline{C} \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)} |y_0, y_1|_{V \times H} \qquad t \geq s \geq 0. \tag{4.2}$$

The constants $\overline{C}$ and $\overline{\mu}$ may also depend on M. Furthermore, the solution $(y,\dot y)$ is unique in the vector space $L^\infty(\mathbb{R}_s, V\times H)$.

Proof. We follow a standard fixed point argument. Let us consider the subspace of essentially bounded functions from $\mathbb{R}_s=(s,+\infty)$ into $V\times H$ defined by

$$\mathcal{Z}^{\overline{\mu}} := \Big\{\mathbf{z}\in L^\infty(\mathbb{R}_s,V\times H) \mid \mathrm{e}^{\frac{\overline{\mu}}{2}(\cdot-s)}\mathbf{z}(\cdot)\in L^\infty(\mathbb{R}_s,V\times H)\Big\}$$

and endowed with the norm

$$|\mathbf{z}|_{\mathcal{Z}^{\overline{\mu}}} := \left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\cdot-s)}\mathbf{z}(\cdot)\right|_{L^\infty(\mathbb{R}_s,V\times H)} = \operatorname*{ess\,sup}_{t>s}\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(t-s)}\mathbf{z}(t)\right|_{V\times H}.$$

Further, for any $\varrho>0$ we define the set

$$\mathcal{Z}^{\overline{\mu}}_{\varrho} := \left\{\mathbf{z}\in\mathcal{Z}^{\overline{\mu}} \mid |\mathbf{z}|_{\mathcal{Z}^{\overline{\mu}}}\le\varrho\,|(y_0,y_1)|_{V\times H}\right\},$$

and the mapping

$$\Psi\colon \mathcal{Z}^{\overline{\mu}}_{\varrho}\to\mathcal{Z}^{\overline{\mu}},\qquad (\mathbf{z}_1,\mathbf{z}_2)\mapsto\Psi(\mathbf{z}_1,\mathbf{z}_2),$$

for a suitable value of ϱ, where $\Psi(\mathbf{z}_1,\mathbf{z}_2)=\Psi(\mathbf{z}_1):=(y,\dot y)$ is the solution of

$$\ddot y + Ay + \varsigma\dot y + A_{\mathrm{r}}y - \mathbf{K}^{\mathcal{F}_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}(y,\dot y) = -\mathcal{N}(\mathbf{z}_1), \tag{4.3a}$$
$$y(s)=y_0,\quad \dot y(s)=y_1. \tag{4.3b}$$

Next, we will show that, for a suitable (large enough) $\varrho>0$, we can choose $\epsilon>0$ small enough so that the mapping Ψ is a contraction in the subset $\mathcal{Z}^{\overline{\mu}}_{\varrho}$ provided that $|(y_0,y_1)|_{V\times H}<\epsilon$. Note that a fixed point of Ψ solves system (4.1).

Let $\overline{C}$ and $\overline{\mu}$ be as in Theorem 3.1 and set $\varrho>\overline{C}$.

Ⓢ Step 1: *for small ϵ, Ψ maps $\mathcal{Z}^{\overline{\mu}}_{\varrho}$ into itself.* Let us fix an arbitrary $\mathbf{z}\in\mathcal{Z}^{\overline{\mu}}_{\varrho}$. We denote by $\mathbf{y}$ the solution of the linear wave system (2.7) for time $t\ge\tau\ge s$, and $\mathfrak{W}(t,s)$ stands for evolution operator which takes (y_0,y_1), at initial time s, to $(y_0(t),y_1(t))$, at time $t\ge s$. For simplicity, we write

$$\mathbf{y}_0 := (y_0,y_1)\quad\text{and}\quad \mathbf{y}(t):=(y(t),\dot y(t)) = \mathfrak{W}(t,s)\mathbf{y}_0 = \mathfrak{W}(t,\tau)\mathbf{y}(\tau).$$

Note that (4.3) is a perturbation of (2.7), and by Duhamel formula we have

$$\begin{aligned}|\mathbf{y}(t)|_{V\times H} &= \left|\mathfrak{W}(t,s)\mathbf{y}_0 + \int_s^t \mathfrak{W}(t,\tau)\big(0,-\mathcal{N}(\tau,\mathbf{z}_1(\tau))\big)\,\mathrm{d}\tau\right|_{V\times H},\\ &\le \overline{C}\mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)}\,|\mathbf{y}_0|_{V\times H} + \overline{C}\int_s^t \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-\tau)}\,|\mathcal{N}(\tau,\mathbf{z}_1(\tau))|_H\,\mathrm{d}\tau.\end{aligned} \tag{4.4}$$

Due to (2.2) from Assumption 2.4, we obtain that

$$|\mathcal{N}(\tau,\mathbf{z}_1(\tau))|_H \le C_{\mathcal{N}}\,|\mathbf{z}_1(\tau)|_V^{\zeta+\eta} = C_{\mathcal{N}}\mathrm{e}^{-(\zeta+\eta)\frac{\overline{\mu}}{2}(\tau-s)}\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{z}_1(\tau)\right|_V^{\zeta+\eta},$$

which, together with (4.4) and $\mathbf{z}\in\mathcal{Z}^{\overline{\mu}}_{\varrho}$, give us

$$\begin{aligned}|\mathbf{y}(t)|_{V\times H} &\le \overline{C}\mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)}\,|\mathbf{y}_0|_{V\times H} + \overline{C}C_{\mathcal{N}}\int_s^t \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-\tau)}\mathrm{e}^{-(\zeta+\eta)\frac{\overline{\mu}}{2}(\tau-s)}\varrho^{\zeta+\eta}\,|\mathbf{y}_0|_{V\times H}^{\zeta+\eta}\,\mathrm{d}\tau\\ &= \overline{C}\mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)}\,|\mathbf{y}_0|_{V\times H} + \overline{C}C_{\mathcal{N}}\varrho^{\zeta+\eta}\,|\mathbf{y}_0|_{V\times H}^{\zeta+\eta}\int_s^t \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-\tau)}\mathrm{e}^{-(\zeta+\eta)\frac{\overline{\mu}}{2}(\tau-s)}\,\mathrm{d}\tau.\end{aligned}$$

Hence, using Proposition 3.2 and (2.2), we infer that there exists a constant $D=\overline{C}_{\left[\frac{1}{\zeta+\eta-1}\right]}>0$ such that

$$|\mathbf{y}(t)|_{V\times H} \le \overline{C}\mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)}\,|\mathbf{y}_0|_{V\times H} + \overline{C}C_{\mathcal{N}}\varrho^{\zeta+\eta}\,|\mathbf{y}_0|_{V\times H}^{\zeta+\eta}\,D\mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)},$$

which gives us

$$\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(t-s)}\mathbf{y}(t)\right|_{V\times H} \le \overline{C}\left(1 + C_{\mathcal{N}}\varrho^{\zeta+\eta}\left|\mathbf{y}_0\right|_{V\times H}^{\zeta+\eta-1}D\right)\left|\mathbf{y}_0\right|_{V\times H}.$$

We can see now that if $\left|\mathbf{y}_0\right|_{V\times H} \le \epsilon$ where

$$\begin{aligned}\overline{C}\left(1 + C_{\mathcal{N}}\varrho^{\zeta+\eta}D\epsilon^{\zeta+\eta-1}\right) \le \varrho \quad &\iff \quad \epsilon^{\zeta+\eta-1} \le \tfrac{\varrho-\overline{C}}{C_{\mathcal{N}}\varrho^{\zeta+\eta}D\overline{C}}\\ &\iff \quad \epsilon \le \left(\tfrac{\varrho-\overline{C}}{C_{\mathcal{N}}D\overline{C}}\right)^{\frac{1}{\zeta+\eta-1}}\varrho^{-\frac{\zeta+\eta}{\zeta+\eta-1}},\end{aligned} \tag{4.5}$$

then

$$\left|\Psi(\mathbf{z})\right|_{\mathcal{Z}^{\overline{\mu}}} = \left|\mathbf{y}\right|_{\mathcal{Z}^{\overline{\mu}}} = \operatorname*{ess\,sup}_{t>s}\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(t-s)}\mathbf{y}(t)\right|_{V\times H} \le \varrho\left|\mathbf{y}_0\right|_{V\times H},$$

and we can conclude that $\Psi(\mathcal{Z}^{\overline{\mu}}_{\varrho}) \subseteq \mathcal{Z}^{\overline{\mu}}_{\varrho}$.

Ⓢ Step 2: *for smaller ϵ, Ψ is a contraction in $\mathcal{Z}^{\overline{\mu}}_{\varrho}$.* Note that, given $(\mathbf{z},\mathbf{w}) \in \mathcal{Z}^{\overline{\mu}}_{\varrho} \times \mathcal{Z}^{\overline{\mu}}_{\varrho}$, we find that the difference $D_\Psi \coloneqq \Psi(\mathbf{z}) - \Psi(\mathbf{w})$ solves (4.3) with $\mathcal{N}(\mathbf{w}_1) - \mathcal{N}(\mathbf{z}_1)$ in the place of $-\mathcal{N}(\mathbf{z}_1)$. Again, due to Duhamel formula and Assumption 2.4, we arrive at

$$\begin{aligned}\left|D_\Psi(t)\right|_{V\times H} &= \left|\int_s^t \mathfrak{W}(t,\tau)\Big(0, \mathcal{N}(\tau,\mathbf{w}_1(\tau)) - \mathcal{N}(\tau,\mathbf{z}_1(\tau))\Big)\,\mathrm{d}\tau\right|_{V\times H}\\ &\le \overline{C}C_{\mathcal{N}}\int_s^t \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-\tau)}\left(\left|\mathbf{w}_1(\tau)\right|_V^\zeta + \left|\mathbf{z}_1(\tau)\right|_V^\zeta\right)\left|\mathbf{w}_1(\tau) - \mathbf{z}_1(\tau)\right|_V^\eta\,\mathrm{d}\tau\\ &= \overline{C}C_{\mathcal{N}}\int_s^t \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-\tau)}\mathrm{e}^{-(\zeta+\eta)\frac{\overline{\mu}}{2}(\tau-s)}\Xi(\mathbf{w}_1,\mathbf{w}_1)\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{d}_1(\tau)\right|_V^\eta\,\mathrm{d}\tau,\end{aligned} \tag{4.6}$$

with

$$\begin{aligned}\Xi(\mathbf{w}_1,\mathbf{w}_1) &\coloneqq \left(\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{w}_1(\tau)\right|_V^\zeta + \left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{z}_1(\tau)\right|_V^\zeta\right),\\ \mathbf{d} &\coloneqq \mathbf{z} - \mathbf{w}, \quad \text{and} \quad \mathbf{d}_1 \coloneqq \mathbf{z}_1 - \mathbf{w}_1.\end{aligned}$$

From $(\mathbf{z},\mathbf{w}) \in \mathcal{Z}^{\overline{\mu}}_{\varrho} \times \mathcal{Z}^{\overline{\mu}}_{\varrho}$, we obtain that

$$\Xi(\mathbf{w}_1,\mathbf{w}_1) = \left(\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{w}_1(\tau)\right|_V^\zeta + \left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{z}_1(\tau)\right|_V^\zeta\right) \le 2\varrho^\zeta\left|\mathbf{y}_0\right|_{V\times H}^\zeta \tag{4.7}$$

and, since $\eta \ge 1$, using the triangle inequality and [26, Prop. 2.6] we obtain

$$\begin{aligned}\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}(\mathbf{w}_1(\tau) - \mathbf{z}_1(\tau))\right|_V^{\eta-1} &\le \left(\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{z}_1(\tau)\right|_V + \left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{w}_1(\tau)\right|_V\right)^{\eta-1}\\ &\le (1+2^{\eta-2})\left(\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{z}_1(\tau)\right|_V^{\eta-1} + \left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}\mathbf{w}_1(\tau)\right|_V^{\eta-1}\right)\\ &\le (2+2^{\eta-1})\varrho^{\eta-1}\left|\mathbf{y}_0\right|_{V\times H}^{\eta-1}.\end{aligned} \tag{4.8}$$

Note also that

$$\mathcal{M} \coloneqq \operatorname*{ess\,sup}_{\tau>s}\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(\tau-s)}(\mathbf{w}_1(\tau) - \mathbf{z}_1(\tau))\right|_V \le \left|\mathbf{z} - \mathbf{w}\right|_{\mathcal{Z}^{\overline{\mu}}}. \tag{4.9}$$

From (4.6), (4.7), (4.8), (4.9), and Proposition 3.2, it follows

$$\begin{aligned}\left|D_\Psi(t)\right|_{V\times H} &\le (4+2^\eta)\overline{C}C_{\mathcal{N}}\varrho^{\zeta+\eta-1}\left|\mathbf{y}_0\right|_{V\times H}^{\zeta+\eta-1}\mathcal{M}\int_s^t \mathrm{e}^{-\frac{\overline{\mu}}{2}(t-\tau)}\mathrm{e}^{-(\zeta+\eta)\frac{\overline{\mu}}{2}(\tau-s)}\,\mathrm{d}\tau\\ &\le (4+2^\eta)\overline{C}C_{\mathcal{N}}\varrho^{\zeta+\eta-1}\left|\mathbf{y}_0\right|_{V\times H}^{\zeta+\eta-1}D\mathrm{e}^{-\frac{\overline{\mu}}{2}(t-s)}\left|\mathbf{z} - \mathbf{w}\right|_{\mathcal{Z}^{\overline{\mu}}}.\end{aligned}$$

which implies

$$|\Psi(\mathbf{z})-\Psi(\mathbf{w})|_{\mathcal{Z}^{\overline{\mu}}}=\operatorname*{ess\,sup}_{t>s}\left|\mathrm{e}^{\frac{\overline{\mu}}{2}(t-s)}D_\Psi(t)\right|_{V\times H}\le\varkappa|\mathbf{z}-\mathbf{w}|_{\mathcal{Z}^{\overline{\mu}}},$$

with

$$\varkappa=(4+2^\eta)\overline{C}C_{\mathcal{N}}\varrho^{\zeta+\eta-1}\left|\mathbf{y}_0\right|_{V\times H}^{\zeta+\eta-1}D.$$

We see that if $|\mathbf{y}_0|_{V\times H}\le\epsilon$ where

$$(4+2^\eta)\overline{C}C_{\mathcal{N}}\varrho^{\zeta+\eta-1}\epsilon^{\zeta+\eta-1}D<1\quad\iff\quad\epsilon<\left(\tfrac{1}{(4+2^\eta)\overline{C}C_{\mathcal{N}}D}\right)^{\frac{1}{\zeta+\eta-1}}\varrho^{-1},\tag{4.10}$$

we find that $|\Psi(\mathbf{z})-\Psi(\mathbf{w})|_{\mathcal{Z}^{\overline{\mu}}}<\varkappa|\mathbf{z}-\mathbf{w}|_{\mathcal{Z}^{\overline{\mu}}}$ with $\varkappa<1$.

Hence, if ϵ satisfies both (4.5) and (4.10), then Ψ is a contraction in $\mathcal{Z}^{\overline{\mu}}_\varrho$.

Ⓢ Step 3: *verification of* $(y,\dot y)\in C(\overline{\mathbb{R}_s};V\times H)$. We can follow a similar argument as in the proof of Theorem 3.1. For that we just need to observe that, for arbitrary $T>s$,

$$|\mathcal{N}(y)|_{L^\infty([s,T],H)}\le C_{\mathcal{N}}\left|y\right|^{\zeta+\eta}_{L^\infty([s,T],V)},$$

which allows us to obtain (3.9) with $\widehat\kappa:=\kappa+\mathcal{N}(y)$, in the place of $\kappa=\mathbf{K}^{\mathcal{F}_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}y$.

Ⓢ Step 4: *the solution exists, and is unique, in* $L^\infty(\mathbb{R}_s,V\times H)$. By the contraction mapping principle there exists one fixed point $\widehat{\mathbf{z}}\in\mathcal{Z}^{\overline{\mu}}_\varrho$ for Ψ. This means that $\widehat{\mathbf{y}}:=\Psi(\widehat{\mathbf{z}})=\widehat{\mathbf{z}}$ solves (4.1).

It remains to prove that such fixed point $\widehat{\mathbf{y}}=:(\widehat y,\dot{\widehat y})$ is the unique solution for (4.1) in $L^\infty(\mathbb{R}_s,V\times H)$. Note that the contraction mapping principle also tells us that $\widehat{\mathbf{y}}$ is unique in the set $\mathcal{Z}^{\overline{\mu}}_\varrho$, but it does not guarantee uniqueness in the entire vector space $L^\infty(\mathbb{R}_s,V\times H)$.

Let $\widetilde{\mathbf{y}}\in L^\infty(\mathbb{R}_s,V\times H)=:(\widetilde y,\dot{\widetilde y})$ be another solution for (4.1). The difference $\mathfrak{d}:=\widehat{\mathbf{y}}-\widetilde{\mathbf{y}}=(\widehat y-\widetilde y,\dot{\widehat y}-\dot{\widetilde y})=:(\delta,\dot\delta)$ solves

$$\begin{aligned}&\ddot\delta+A\delta+\varsigma\dot\delta+A_{\mathrm{r}}\delta+\mathcal{N}(\widehat y)-\mathcal{N}(\widetilde y)=\mathbf{K}^{\mathcal{F}_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}(\delta,\dot\delta),\\&\delta(s)=0,\quad\dot\delta(s)=0,\end{aligned}$$

and setting $w:=\dot\delta+\varepsilon\delta$, with $\varepsilon\in(0,\varsigma)$, we find

$$\dot w=\ddot\delta+\varepsilon\dot\delta=-A\delta-(\varsigma-\varepsilon)\dot\delta+A_{\mathrm{r}}\delta+\mathbf{K}^{\mathcal{F}_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}(\delta,\dot\delta)-\mathcal{N}(\widehat y)+\mathcal{N}(\widetilde y),$$

from which we obtain

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\left|w\right|_H^2&=-2(\delta,w)_V-2(\varsigma-\varepsilon)\left|w\right|_H^2+(A_{\mathrm{r}}\delta,w)_H\\&\quad+2\varepsilon(\varsigma-\varepsilon)(\delta,w)_H+2(\mathbf{K}^{\mathcal{F}_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}(\delta,\dot\delta),w)_H+2(\mathcal{N}(\widetilde y)-\mathcal{N}(\widehat y),w)_H,\end{aligned}$$

and consequently,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\left|w\right|_H^2+\left|\delta\right|_V^2\right)&\le-2\varepsilon\left|\delta\right|_V^2-2(\varsigma-\varepsilon)\left|w\right|_H^2+2\varepsilon(\varsigma-\varepsilon)\alpha_1^{-\frac12}\left|\delta\right|_V\left|w\right|_H\\&\quad+2C_{\mathrm{r}}\left|\delta\right|_H\left|w\right|_H+2\left|P^{E^\perp_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}\right|_{\mathcal{L}(H)}\left|P_{E_{\mathbb{M}_\sigma}}\left(A\delta+\varsigma\dot\delta\right)\right|_H\left|w\right|_H\\&\quad+2C_{\mathrm{r}}\left|P^{E^\perp_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}\right|_{\mathcal{L}(H)}\alpha_1^{-\frac12}\left|\delta\right|_V\left|w\right|_H\\&\quad+2\left|P^{E^\perp_{\mathbb{M}_\sigma}}_{U_{\#\mathbb{M}_\sigma}}\right|_{\mathcal{L}(H)}\left|\mathcal{F}_{\mathbb{M}_\sigma}\right|_{\mathcal{L}(H\times H,H)}\left|(\delta,\dot\delta)\right|_{H\times H}\left|w\right|_H\\&\quad+2\left|\mathcal{N}(\widetilde y)-\mathcal{N}(\widehat y)\right|_H\left|w\right|_H.\end{aligned}\tag{4.11}$$

Next, due to the definition of w we have the estimates

$$\begin{aligned}\left|P_{E_{\mathbb{M}_\sigma}}\left(A\delta+\varsigma\dot{\delta}\right)\right|_H &= \left|\left(AP_{E_{\mathbb{M}_\sigma}}\delta+\varsigma P_{E_{\mathbb{M}_\sigma}}\dot{\delta}\right)\right|_H \le \alpha_{\mathbb{M}_\sigma}\left|\delta\right|_H+\varsigma\left|\dot{\delta}\right|_H\\ &\le \alpha_{\mathbb{M}_\sigma}\left|\delta\right|_H+\varsigma\left|w\right|_H+\varsigma\varepsilon\left|\delta\right|_H\\ &\le \alpha_1^{-\frac{1}{2}}\left(\alpha_{\mathbb{M}_\sigma}+\varsigma\varepsilon\right)\left|\delta\right|_V+\varsigma\left|w\right|_H.\end{aligned} \tag{4.12}$$

and

$$\left|(\delta,\dot{\delta})\right|_{H\times H}\le\left|\delta\right|_H+\left|\dot{\delta}\right|_H\le\left|\delta\right|_H+\left|w\right|_H+\epsilon\left|\delta\right|_H\le\Theta_1^{\frac{1}{2}}\left|\delta\right|_V+\left|w\right|_H, \tag{4.13}$$

where $\Theta_1 := \alpha_1^{-1}(1+\epsilon)^2$, with α_1 the first eigenvalue of A. Further, we observe that, due to Assumption 2.4, we have

$$\left|\mathcal{N}(\widetilde{y})-\mathcal{N}(\widehat{y})\right|_H^2\le C_{\mathcal{N}}^2\left(\left|\widetilde{y}\right|_V^{\zeta}+\left|\widehat{y}\right|_V^{\zeta}\right)^2\left|\delta\right|_V^{2\eta}\le\Theta_2\left|\delta\right|_V^2 \tag{4.14a}$$

with

$$\Theta_2 := 2C_{\mathcal{N}}^2\left(\left|\widetilde{y}\right|_V^{2\zeta}+\left|\widehat{y}\right|_V^{2\zeta}\right)\left|\delta\right|_V^{2(\eta-1)}. \tag{4.14b}$$

Combining (4.11), (4.12), (4.13), (4.14), and (2.10e), and using the Young inequality, we can conclude that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\left|w\right|_H^2+\left|\delta\right|_V^2\right)&\le -2\varepsilon\left|\delta\right|_V^2-2(\varsigma-\varepsilon)\left|w\right|_H^2+\varepsilon(\varsigma-\varepsilon)\alpha_1^{-\frac{1}{2}}\left(\left|\delta\right|_V^2+\left|w\right|_H^2\right)\\ &\quad+\alpha_1^{-1}C_{\mathrm{r}}^2\left|\delta\right|_V^2+\left|w\right|_H^2+C_P^2\alpha_1^{-1}(\alpha_{\mathbb{M}_\sigma}+\varsigma\varepsilon)^2\left|\delta\right|_V^2+\left|w\right|_H^2\\ &\quad+2C_P\varsigma\left|w\right|_H^2+C_{\mathrm{r}}^2C_P^2\alpha_1^{-1}\left|\delta\right|_V^2+\left|w\right|_H^2\\ &\quad+C_P^2\left|\mathcal{F}_{\mathbb{M}_\sigma}\right|_{\mathcal{L}(H\times H,H)}^2\Theta_1\left|\delta\right|_V^2+2C_P\left|\mathcal{F}_{\mathbb{M}_\sigma}\right|_{\mathcal{L}(H\times H,H)}\left|w\right|_H^2\\ &\quad+\left|w\right|_H^2+\Theta_2\left|\delta\right|_V^2+\left|w\right|_H^2.\end{aligned} \tag{4.15}$$

Therefore, we arrive at

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\left|w\right|_H^2+\left|\delta\right|_V^2\right)\le\Theta_3\left|w\right|_H^2+\Theta_4\left|\delta\right|_V^2\le\max\{\Theta_3,\Theta_4\}\left(\left|w\right|_H^2+\left|\delta\right|_V^2\right),$$

with

$$\begin{aligned}\Theta_3 &:= -2(\varsigma-\varepsilon)+\varepsilon(\varsigma-\varepsilon)\alpha_1^{-\frac{1}{2}}+2C_P\left(\varsigma+\left|\mathcal{F}_{\mathbb{M}_\sigma}\right|_{\mathcal{L}(E_{\mathbb{M}_\sigma}\times E_{\mathbb{M}_\sigma},E_{\mathbb{M}_\sigma})}\right)+5,\\ \Theta_4 &:= -2\varepsilon+\varepsilon(\varsigma-\varepsilon)\alpha_1^{-\frac{1}{2}}+\alpha_1^{-1}\left(C_{\mathrm{r}}^2+C_{\mathrm{r}}^2C_P^2+C_P^2(\alpha_{\mathbb{M}_\sigma}+\varsigma\varepsilon)^2\right)\\ &\quad+C_P^2\left|\mathcal{F}_{\mathbb{M}_\sigma}\right|_{\mathcal{L}(E_{\mathbb{M}_\sigma}\times E_{\mathbb{M}_\sigma},E_{\mathbb{M}_\sigma})}^2\Theta_1+\Theta_2.\end{aligned}$$

Since $\overline{\Theta}:=\max\{\Theta_3,\Theta_4\}$ is locally integrable, by the Gronwall inequality we obtain

$$\left|w(t)\right|_H^2+\left|\delta(t)\right|_V^2\le\mathrm{e}^{\int_s^t\overline{\Theta}(\tau),\mathrm{d}\tau}\left(\left|w(s)\right|_H^2+\left|\delta(s)\right|_V^2\right)=0,\qquad 0<s\le t\in\mathbb{R}.$$

which implies that $(\delta(t),\dot{\delta}(t))=0$, for all $t\ge s$, and thus also

$$\widetilde{\mathbf{y}}(t)=(\widetilde{y}(t),\dot{\widetilde{y}}(t))=(\widehat{y}(t),\dot{\widehat{y}}(t))=\widehat{\mathbf{y}}(t),\quad\text{for all}\quad t\ge s.$$

This finishes the proof. □

We are now ready to prove Theorem 1.1.

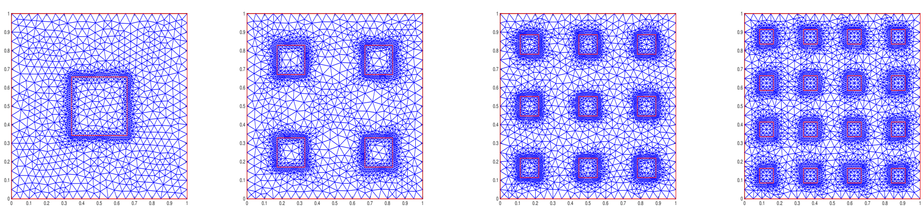
Proof of Theorem 1.1. Observe that we can write (1.6) as (4.1). Since we already know, from Section 3.2, that Assumptions 2.1–2.3 and (2.10a)–(2.10c) in Assumption 2.6 are satisfied, it remains to check that also Assumption 2.4 holds. Observe that (see, e.g., [32, Sect. 5.2.1]), for a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$ we have that

$$\begin{aligned}\left| |y|_{\mathbb{R}}^{r-1} y - |z|_{\mathbb{R}}^{r-1} z \right|_H &= \left| |y|_{\mathbb{R}}^{r-1} y - |z|_{\mathbb{R}}^{r-1} z \right|_{L^2} \le C_1 \left(|y|_{L^{2r}}^{r-1} + |y|_{L^{2r}}^{r-1} \right) |y - z|_{L^{2r}} \\ &\le C_2 \left(|y|_{L^6}^{r-1} + |y|_{L^6}^{r-1} \right) |y - z|_{L^6} \le C_3 \left(|y|_V^{r-1} + |y|_V^{r-1} \right) |y - z|_V .\end{aligned}$$

Assumption 2.4 follows with $\zeta = r-1$ and $\eta = 1$. Note that $\zeta + \eta = r > 1$ and this completes the proof. ☐

5. Numerical simulations

In this section, we report on the numerical experiments which validate our theoretical findings in the previous sections. We investigate the performance of the oblique feedback control for both of the linear system (1.8) and the nonlinear system (1.6), both under homogeneous Dirichlet boundary conditions. For the discretization of the state we write the equation as a system of first order equations in time. The spatial domain $\Omega = (0,1) \times (0,1)$ was discretized by a conforming linear finite element scheme using continuous piecewise linear basis functions over the triangulations given by Figure 1.



(a) 1 actuator (b) 4 actuators (c) 9 actuators (d) 16 actuators

Figure 1. Geometry of the domain and actuators.

This figure illustrates the meshes and also the placements and supports of the actuators for the cases in which 1, 4, 9, and 16 actuators are used. The actuators are indicator functions, $U_{\#\mathbb{M}_\sigma} = \operatorname{span}\{1_{\omega_i} \mid 1 \le i \le \#\mathbb{M}_\sigma\}$, whose supports are the (interior) rectangular domains ω_is depicted in Figure 1. For the temporal discretization, we applied the Crank-Nicolson time stepping method with step-size 10^{-3}. For both systems (1.8) and (1.6), we have set

$$\nu = \varsigma = 1,$$

and chosen, for $x = (x_1, x_2) \in \Omega$,

$$a(t,x) := -30 - 30|\sin(t + x_1)|, \tag{5.1a}$$

$$(y_0(x), y_1(x)) := \delta(e^{(-20((x_1-0.5)^2 + (x_2-0.5)^2))}, 0), \tag{5.1b}$$

where the scaling parameter $\delta > 0$ will be specified in each example below.

For the set of eigenfunctions $E_{\mathbb{M}_\sigma}$ we have chosen

$$E_{\mathbb{M}_\sigma} := \operatorname{span}\{e_{\mathbf{i}} \mid \mathbf{i} = (\mathbf{i}_1, \mathbf{i}_2) \in \{1,2,\dots,M\} \times \{1,2,\dots,M\}\}, \quad (\#\mathbb{M}_\sigma = M^2),$$

with

$$e_{\mathbf{i}}(x) := 2\sin(\pi \mathbf{i}_1 x_1)\sin(\pi \mathbf{i}_2 x_2).$$

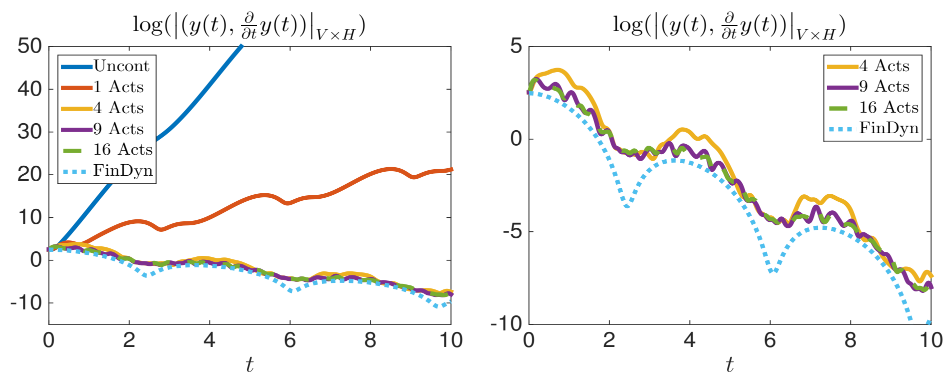
These functions are normalized eigenfunctions of the Laplacian with homogeneous Dirichlet boundary conditions on the unit square (cf. [20, Sect. 4.8.1]). Further, the operator $\mathcal{F}_{\mathbb{M}_\sigma}$ within the feedback control $\mathbf{K}_{U_{\#\mathbb{M}_\sigma}}^{\mathcal{F}_{\mathbb{M}_\sigma}}$ (see (2.8)) have been chosen as in (2.12) with

$$\lambda_1 = \lambda_2 = 1.$$

For the discretization and implementation of the oblique projection based feedback $\mathbf{K}_{U_{\#\mathbb{M}_\sigma}}^{\mathcal{F}_{\mathbb{M}_\sigma}}$ we followed the approach given in [33, Sect. 8.1]. We have run every numerical simulation for time $t \in [0, 10]$. All numerical simulations have been carried out on the MATLAB platform.

Remark 5.1. The initial condition in (5.1b) is not in $V \times H = H_0^1(\Omega) \times L^2(\Omega)$ as required in the theoretical results. However, the Dirichlet boundary trace has relatively small values $y_0(\overline{x})|_{\partial\Omega} \leq \mathrm{e}^{-5} \approx 0.007$. We have neglected these values in the simulations by setting the initial numerical data $\overline{y}_0$ as $\overline{y}_0(p_i) \coloneqq y_0(p_i)$ at interior mesh points $p_i \in \Omega$, and $\overline{y}_0(p_i) \coloneqq 0$ at boundary mesh points $p_i \in \partial\Omega$.

Example 5.2 (Linear case)**.** We deal with the linear system (1.8). We set $\delta = 2$ in (5.1). For this choice the uncontrolled state y^{un} is exponentially unstable with approximately rate 10. This fact is illustrated in Figure 2(a), where the evolution of $\log(|(y^{un}(t), \frac{\partial}{\partial t} y^{un}(t))|_{V\times H})$ is shown by the curve labeled "Uncont".



(a) (b)

FIGURE 2. Example 5.2: Logarithmic evolution of energy for states.

We have applied the oblique feedback control to stabilize the system around zero with different numbers of actuators. The performance of the feedback controls is reported in Figure 2(a), which demonstrates the logarithmic evolution of the energy for the controlled states. In Figure 2(b), we replotted the curves corresponding to the cases 4, 9, and 16 actuators in order to clearly see the behaviors of the corresponding energies and to make comparison to the energy of the finite-dimensional component of the solution, $(P_{E_{\mathbb{M}_\sigma}} y(t), P_{E_{\mathbb{M}_\sigma}} \frac{\partial}{\partial t} y(t))$. The evolution of $\log(|(P_{E_{\mathbb{M}_\sigma}} y(t), P_{E_{\mathbb{M}_\sigma}} \frac{\partial}{\partial t} y(t))|_{V\times H})$, depicted in Figure 2, corresponds to the dotted curve assigned with "FinDyn". Recall that the finite-dimensional component satisfies the dynamical system (2.13).

It can be observed from Figure 2, that the feedback control is able to steer the system exponentially to zero for the cases we take 4, 9, and 16 actuators, whereas for the case of 1 actuator (as in Figure 1) the feedback control fails to stabilize the system and the corresponding state is exponentially unstable with approximately rate 1.8. Clearly, for the larger number of actuators, a better stabilization rate was obtained. This is related to (2.10c), estimate (3.6b), and the fact that the norm of the oblique projection is bounded. This boundedness follows from [20, Lem. 4.4 and Cor. 2.9] and [33, Thm. 4.4]. On the other hand, we can see from Figure 2(b) that the curves corresponding to 9 and 16 actuators almost overlap each other. This means that the rate of stabilization is bounded from above and at some point it cannot be further improved by using more actuators. This observation is in accordance with the statement of Theorem 3.5. Observe that $|(y(t), \frac{\partial}{\partial t} y(t))|_{V\times H} \geq |(P_{E_{\mathbb{M}_\sigma}} y(t), P_{E_{\mathbb{M}_\sigma}} \frac{\partial}{\partial t} y(t))|_{V\times H}$ for all of the cases. In particular, $|(y(t), \frac{\partial}{\partial t} y(t))|_{V\times H}$ can not decrease exponentially faster than $|(P_{E_{\mathbb{M}_\sigma}} y(t), P_{E_{\mathbb{M}_\sigma}} \frac{\partial}{\partial t} y(t))|_{V\times H}$.

The evolution of the values of the feedback control components for the choice of 9 actuators is plotted in Figure 3.

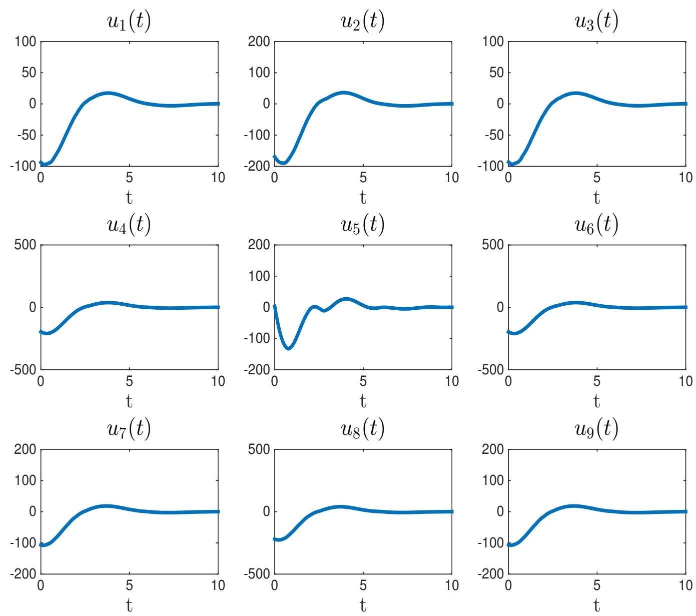
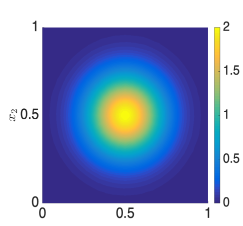
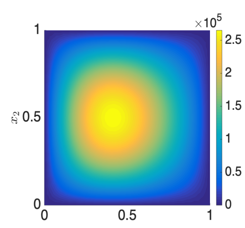


FIGURE 3. Example 5.2: Evolution of the $u_i(t)$s, for $i = 1, \ldots, 9$, corresponding to 9 actuators

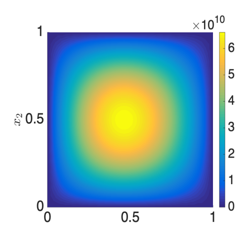
Next, we present snapshots of the state y. Figure 4 depicts some snapshots of the uncontrolled state y^{un}. Figure 5 shows the controlled state at different times for the choice of 9 actuators.



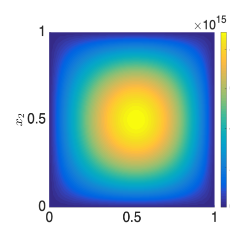
(a) $t = 0$



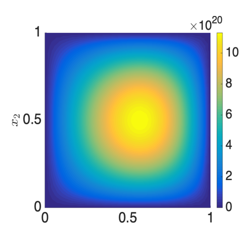
(b) $t = 2.5$



(c) $t = 5$

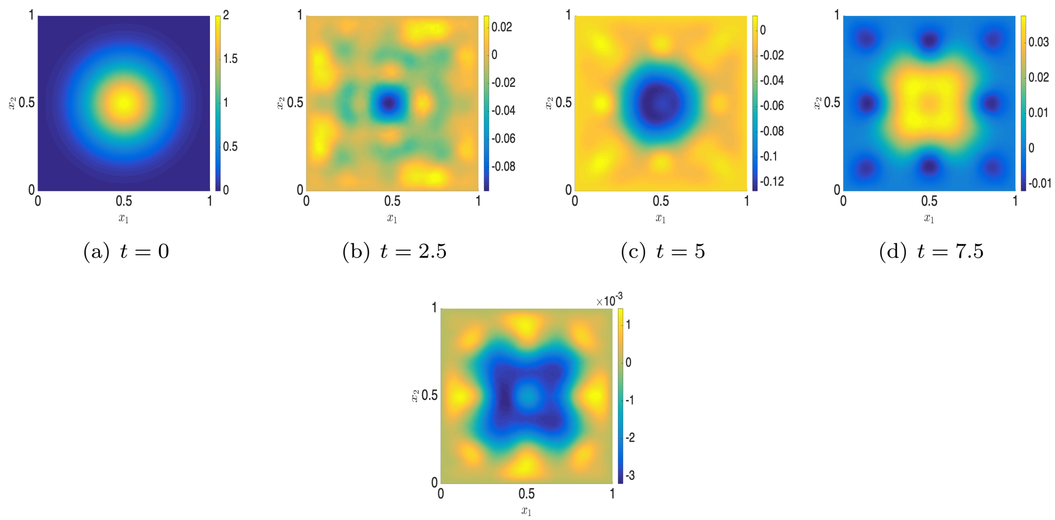


(d) $t = 7.5$



(e) $t = 10$

FIGURE 4. Example 5.2: Snapshots of the uncontrolled state.

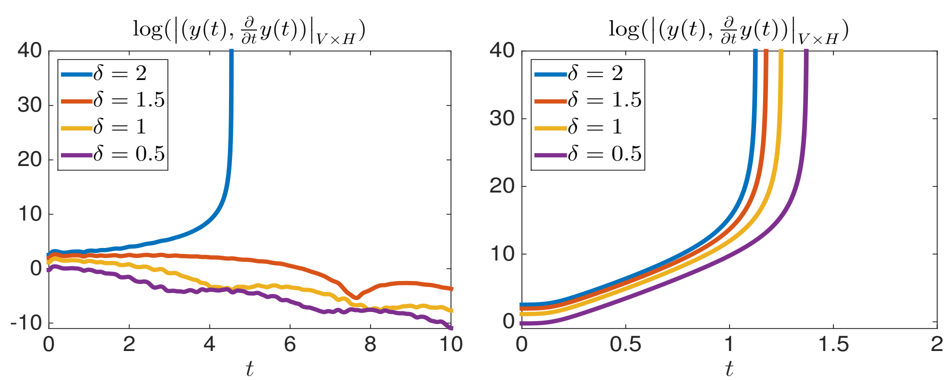


(a) $t = 0$ (b) $t = 2.5$ (c) $t = 5$ (d) $t = 7.5$

(e) $t = 10$

FIGURE 5. Example 5.2: Snapshots of the controlled state with 9 actuators.

Example 5.3 (Nonlinear case)**.** Here we are concerned with the nonlinear system (1.6) where $r = 2$ has been chosen in the nonlinearity. We investigate the performance of the oblique feedback control with 16 actuators for the different values of δ in (5.1). The logarithmic evolution of the uncontrolled states are depicted in Figure 6(b) for the choice of $\delta \in \{0.5, 1, 1.5, 2\}$.



(a) Controlled (b) Uncontrolled

FIGURE 6. Example 5.3 : Logarithmic evolution of energy for states for different values of δ.

As can be seen from Figure 6(b), the energy of the uncontrolled states blows up in finite time for all $\delta \in \{0.5, 1, 1.5, 2\}$. After applying the feedback control we observe that the control is not able to stabilize the system for large initial condition ($\delta = 2$). This fact is depicted by Figure (6(a)), which presents the evolution of the controlled $|(y(t), \frac{\partial}{\partial t}y(t))|_{V\times H}$ for the same values of δ. However, we can also see that for smaller $\delta \in \{0.5, 1, 1.5\}$ the feedback control is able to exponentially stabilize the system to zero. These observations agree with (1.7) in Theorem 1.1. That is, stability holds for small enough initial conditions.

6. Final remarks

We have shown that an explicit oblique projection feedback *globally* stabilizes weakly damped linear wave equations, and *locally* stabilizes weakly damped semilinear wave equations. The recent work [32] suggests that it may be possible to improve the latter *local* stability result. Namely, by using an appropriate *nonlinear* explicit oblique projection feedback and a larger number of actuators, $M = M(|(y_0, y_1)|_{V\times H})$, then we will likely be able to get rid of the smallness condition $|(y_0, y_1)|_{V\times H} < \epsilon$ in (1.7). There are however nontrivial details that must be checked. The possibility of getting rid of the smallness condition is of course of paramount importance in applications, since ε can be very small for linear feedbacks, as we have seen in Figure 6(a). Thus such possibility is an interesting subject for a future work.

Existence and regularity of solutions for (damped) wave like equations have been extensively studied by many researchers and for several types of nonlinearities, see e.g., [34, 16, 15, 40, 25]. Here it is not our goal to investigate the performance of the oblique projection under all possible nonlinearities. We are rather interested in giving some examples where the free dynamics is unstable and even blowing up in finite time. These situations are natural challenges for any given stabilizing feedback operator.

Appendix

A.1. **Proof of Proposition 3.2.** We split the proof into the cases $\mu_0 < \min\{\mu_1, \mu_2\}$ and $\mu_0 = \min\{\mu_1, \mu_2\}$.

We consider first the case $0 \le \mu_0 < \min\{\mu_1, \mu_2\}$. Direct computations give us

$$\begin{aligned}\mathrm{e}^{-\mu_1(t-\tau)}\mathrm{e}^{-\mu_2(\tau-s)} &= \mathrm{e}^{-\mu_1(t-\tau)}\mathrm{e}^{\mu_0(t-\tau)}\mathrm{e}^{-\mu_0(t-\tau)}\mathrm{e}^{-\mu_0(\tau-s)}\mathrm{e}^{\mu_0(\tau-s)}\mathrm{e}^{-\mu_2(\tau-s)}\\ &= \mathrm{e}^{-(\mu_1-\mu_0)(t-\tau)}\mathrm{e}^{-\mu_0(t-s)}\mathrm{e}^{-(\mu_2-\mu_0)(\tau-s)}\end{aligned}$$

and find

$$\begin{aligned}&\int_s^t \mathrm{e}^{-\mu_1(t-\tau)}\mathrm{e}^{-\mu_2(\tau-s)}\,\mathrm{d}\tau = \mathrm{e}^{-\mu_0(t-s)}\int_s^t \mathrm{e}^{-(\mu_1-\mu_0)(t-\tau)}\mathrm{e}^{-(\mu_2-\mu_0)(\tau-s)}\,\mathrm{d}\tau\\ &\le \mathrm{e}^{-\mu_0(t-s)}\min\left\{\tfrac{1-\mathrm{e}^{-(\mu_1-\mu_0)(t-s)}}{\mu_1-\mu_0}, \tfrac{1-\mathrm{e}^{-(\mu_2-\mu_0)(t-s)}}{\mu_2-\mu_0}\right\}\\ &= \overline{C}_{\left[\min\left\{\frac{1}{\mu_1-\mu_0}, \frac{1}{\mu_2-\mu_0}\right\}\right]}\mathrm{e}^{-\mu_0(t-s)}\\ &= \overline{C}_{\left[\frac{1}{\max\{\mu_1,\mu_2\}-\mu_0}\right]}\mathrm{e}^{-\mu_0(t-s)}, \quad \text{in case} \quad 0 \le \mu_0 < \min\{\mu_1, \mu_2\}.\end{aligned}$$

Finally we consider the case $0 \le \mu_0 = \min\{\mu_1, \mu_2\}$, and $\mu_1 \ne \mu_2$. Without lack of generality, we assume that $0 < \mu_1 < \mu_2$. That is, $\mu_0 = \mu_1$ and we find

$$\begin{aligned}&\int_s^t \mathrm{e}^{-\mu_1(t-\tau)}\mathrm{e}^{-\mu_2(\tau-s)}\,\mathrm{d}\tau = \mathrm{e}^{-\mu_1(t-s)}\int_s^t \mathrm{e}^{-\mu_1(s-\tau)}\mathrm{e}^{-\mu_2(\tau-s)}\,\mathrm{d}\tau\\ &= \mathrm{e}^{-\mu_1(t-s)}\int_s^t \mathrm{e}^{-(\mu_2-\mu_1)(\tau-s)}\,\mathrm{d}\tau = \tfrac{1-\mathrm{e}^{-(\mu_2-\mu_1)(t-s)}}{\mu_2-\mu_1}\mathrm{e}^{-\mu_1(t-s)}\\ &= \overline{C}_{\left[\frac{1}{\max\{\mu_1,\mu_2\}-\mu_0}\right]}\mathrm{e}^{-\mu_0(t-s)}, \quad \text{in case} \quad 0 \le \mu_0 = \mu_1 < \mu_2.\end{aligned}$$

This finished the proof of Proposition 3.2. □

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