On Hilbert-Schmidt operators and determinants corresponding to periodic ODE systems

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In this paper the structure of infinite determinants corresponding to linear periodic ODE systems is investigated. Making use of the theory of Hilbert-Schmidt operators and their determinants it can be shown that the infinite determinant characterizing the stability of such an ODE system has polynomial structure. In the proof we use the fact that the trace of the commutator of two specific operators vanishes. The knowledge of the asymptotic structure of the finite section determinants enables us to improve the convergence of the infinite determinant which is the basis for numerical applications.

1. Introduction

The aim of this paper is to describe the structure and convergence of the infinite determinant corresponding to a linear periodic ODE system of the form

$$P(D,x)\,y(x) = 0 \tag{1.1}$$

with $P(D,x) := D^m + A^{(m-1)}(x)D^{m-1} + \ldots + A^{(1)}(x)D + A^{(0)}(x)$ where $A^{(j)}$ are 1-periodic matrix valued functions and $D := \frac{d}{dx}$. Determinants connected with periodic ordinary differential equations have a long history, starting with the famous work of G. W. HILL [9]. Whereas for Hill's equation a number of results about the structure and convergence of the determinant is known (see [11], [12], [13], for instance), the general form of system (1.1) has not yet been investigated. From an operator theoretical point of view the determinant of system (1.1) appears as a regularized determinant of some Hilbert-Schmidt operator. The close connection between the stability of (1.1) and the regularized determinant is known even for partial differential equations, see [10]. In order to make this connection useful for applications one has to know more about the structure of the regularized determinant. As we will see in Sections 2 and 3, in the case considered here the structure is very simple and allows us to improve the convergence of the corresponding finite section determinants. Thus the determinantal approach is not only interesting in a theoretical sense but also for applications.

In order to simplify the notation we will not consider system (1.1) but the corresponding equivalent first order system which we will write in the form

$$y'(x) = A(x) \cdot y(x) \tag{1.2}$$

where $A(\cdot) \in L^\infty(\mathbb{R}, \mathbb{C}^{n\times n})$ is 1-periodic. (We will return to system (1.1) at the end of Section 2.) First we want to fix some notations. I_n stands for the unit matrix in $\mathbb{C}^{n\times n}$ and $Y(x)$ for the fundamental solution of (1.2), i.e. the matrix solution with the initial value $Y(0) = I_n$. In the following we will deal with functions on the torus $\mathbf{T} := \mathbb{R}/\mathbb{Z}$ with values in $\mathbb{C}^n$, the corresponding L^2-space $L^2(\mathbf{T}, \mathbb{C}^n)$ and the Sobolev spaces $W_p^1(\mathbf{T}, \mathbb{C}^n) := \{f \in L^2(\mathbf{T}, \mathbb{C}^n) : f \text{ absolutely continous}, f' \in L^p(\mathbf{T}, \mathbb{C}^n)\}$. As usual, we set $H_1(\mathbf{T}, \mathbb{C}^n) := W_2^1(\mathbf{T}, \mathbb{C}^n)$.

Instead of $L^2(\mathbf{T}, \mathbb{C}^n)$ we will frequently consider the isometrically isomorphic Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^n) =: H$, making use of the Fourier transform. Operators in H will be written as infinite block matrices (with respect to the standard basis in H). The operator of multiplication with a function Z is denoted by M_Z. For $Z \in L^\infty(\mathbf{T}, \mathbb{C}^{n\times n})$ this operator is an element of the class $\mathcal{L}(H)$ of all bounded operators in H and has the form $M_Z = (Z_{k-l})_{k,l\in\mathbb{Z}}$ with

$$Z_k := \int_{\mathbf{T}} Z(t) \exp(-2\pi \,\mathrm{i}\, kt)\, dt \in \mathbb{C}^{n\times n}\,. \tag{1.3}$$

The symbol $\mathcal{S}_p(H)$ stands for the Neumann-Schatten class of order p in $\mathcal{L}(H)$. For a Hilbert-Schmidt operator $B \in \mathcal{S}_2(H)$ we will consider the regularized determinant $\Delta_2(1 - B)$ (as a standard reference for Hilbert-Schmidt operators and regularized determinants, we mention [8]).

The operator $P_N \in \mathcal{L}(H)$ is defined as the orthogonal projection in H onto the $(2N+1)n$-dimensional subspace

$$\{(c_k)_{k\in\mathbb{Z}} \in H : c_k = 0 \text{ for } |k| > N\}\,.$$

For $B = (B_{kl})_{k,l\in\mathbb{Z}} \in \mathcal{L}(H)$ we set $\det(1-B) := \lim_N \det(P_N(1-B)P_N)$, provided the limit exists.

We now return to equation (1.2). The stability of this equation is characterized by the so-called Floquet exponents which can be defined as the complex numbers ν for which $\exp(\nu)$ is an eigenvalue of $Y(1)$. The following lemma summarizes different possibilities to describe the Floquet exponents.

Lemma 1.1. *For any $\nu \in \mathbb{C}$ the following statements are equivalent:*

(i) *ν is a Floquet exponent of* (1.2).

(ii) *$-\nu$ is an eigenvalue of the operator L in $L^2(\mathbf{T}, \mathbb{C}^n)$ defined by $D(L) := H_1(\mathbf{T}, \mathbb{C}^n)$ and $Lf := f' - Af$.*

(iii) *The regularized determinant $\Delta_2(1 - B_L(\nu))$ is equal to zero, where B_L is defined by*

$$B_L(\nu) := 1 - (L + \nu)F \in \mathcal{S}_2(H)$$

with

$$F := \operatorname{diag}((2\pi \,\mathrm{i}\, l + \delta_{0,l})^{-1} I_n)_{l\in\mathbb{Z}} \in \mathcal{S}_2(H)\,.$$

Whereas the equivalence of (i) and (ii) is obvious, condition (iii) can be derived from the more general considerations in [10], p. 110. Indeed, the explicit formula for F is not crucial for the equivalence but for the following calculations.

Condition (i) of Lemma 1.1 leads in a direct way to a numerical method to compute the Floquet exponents. Here we want to concentrate on condition (iii) and study the properties of the regularized determinant mentioned there. As we will see, $\Delta_2(1 - B_L(\nu))$ has a very simple structure which enables us to use the determinantal approach in applications. Thus generalizations of well-known algorithms in the context of Hill's equation ([4], [11], [12], [16]) are obtained.

2. The structure of the regularized determinant

In order to obtain information about the regularized determinant $\Delta_2(1-B_L(\nu))$ which appears in Lemma 1.1 (iii) we investigate the behaviour of this determinant under transformations of the operator L. First we will prove the following lemma which is connected with the question under which conditions the commutator of a bounded operator and a Hilbert Schmidt operator is a trace class operator and seems to be of interest for itself (cf. also Remark 2.3).

Lemma 2.1. *Let $Z \in W^1_\infty(\mathbf{T}, \mathbb{C}^{n\times n})$ with $\det Z(x) \neq 0$ for all $x \in \mathbf{T}$. Then $F - M_Z F M_Z^{-1}$ is a trace class operator with vanishing trace.*

Proof. It is enough to show that $P_N F M_Z - M_Z P_N F$ converges (for $N \to \infty$) to $C := F M_Z - M_Z F$ in $\mathcal{S}_1$. Indeed, in that case we have $P_N F - M_Z P_N F M_Z^{-1} \to F - M_Z F M_Z^{-1}$ in $\mathcal{S}_1$ ([7], p. 107). But

$$\operatorname{tr}(P_N F - M_Z P_N F M_Z^{-1}) = \operatorname{tr}(P_N F) - \operatorname{tr}(M_Z P_N F M_Z^{-1}) = 0$$

for every N ([8], p. 100) which shows that $F - M_Z F M_Z^{-1}$ has vanishing trace.

Let $C = (C_{kl})_{k,l\in\mathbb{Z}}$. For $k \neq 0 \neq l$ we obtain

$$C_{kl} = \left(\frac{1}{2\pi\,\mathrm{i}\,k} - \frac{1}{2\pi\,\mathrm{i}\,l}\right) Z_{k-l} = -\frac{1}{2\pi\,\mathrm{i}\,l}\cdot\frac{1}{2\pi\,\mathrm{i}\,k}\, Z'_{k-l}$$

where $M_{Z'} = (Z'_{k-l})$ is bounded because the derivative Z' is an element of $L^\infty(\mathbf{T}, \mathbb{C}^{n\times n})$ ([7], p. 567). Therefore, $(1-P_0)C(1-P_0) = -(1-P_0)FM_{Z'}F(1-P_0)$ is an element of $\mathcal{S}_1(H)$. The same is true for C which differs from this operator only by a finite rank operator.

Now consider

$$D^N := (D^N_{kl})_{k,l\in\mathbb{Z}} := C - (P_N F M_Z - M_Z P_N F)\,.$$

We still have to show that $\|D^N\|_{\mathcal{S}_1} \to 0$. To this end we decompose

$$D^N = P_N D^N P_N + (1-P_N)D^N(1-P_N) + P_N D^N(1-P_N) + (1-P_N)D^N P_N \,. \tag{2.1}$$

The first term in this sum is equal to zero for all N. As we have seen above, $C \in \mathcal{S}_1(H)$ and thus $D^N \in \mathcal{S}_1(H)$. Therefore, the second term tends to zero in $\mathcal{S}_1$ ([7], p. 107).

In order to estimate the third term in (2.1) we use ([14], p. 239)

$$\|P_N D^N(1-P_N)\|_{\mathcal{S}_1} \le \sum_{k=-N}^{N} \Big(\sum_{|l|>N} |D^N_{kl}|^2 \Big)^{1/2}. \tag{2.2}$$

We estimate the sum

$$\sum_{k=1}^{N} \left(\sum_{l=N+1}^{\infty} |D^N_{kl}|^2 \right)^{1/2}, \tag{2.3}$$

the remaining parts of the sum in (2.2) can be treated analogously. Direct calculations show that for $|k| \le N$ and $|l| > N$ we have $D^N_{kl} = -(2\pi\,\mathrm{i}\,l)^{-1} Z_{k-l}$. So (2.3) can be estimated by

$$\begin{aligned}
\sum_{k=1}^{N} \left(\sum_{l=N+1}^{\infty} \left| \frac{Z_{k-l}}{2\pi l} \right|^2 \right)^{1/2} &\le \frac{1}{2\pi(N+1)} \sum_{k=1}^{N} \left(\sum_{l=N+1}^{\infty} |Z_{k-l}|^2 \right)^{1/2} \\
&= \frac{1}{2\pi(N+1)} \sum_{k=1}^{N} \left(\sum_{l=N+1-k}^{\infty} \left| \frac{Z'_{-l}}{2\pi l} \right|^2 \right)^{1/2} \\
&\le \left(\frac{1}{2\pi(N+1)} \sum_{k=1}^{N} \frac{1}{2\pi(N+1-k)} \right) \cdot \|Z'\|_{L_2} \\
&\le \frac{1}{2\pi(N+1)} \, \frac{1}{2\pi} (1+\ln N) \cdot \|Z'\|_{L_2} \; \to 0 \,.
\end{aligned}$$

Therefore, also the third term on the right-hand side of (2.1) converges to 0 in $\mathcal{S}_1$. Instead of the last term of this sum we consider the adjoint operator $P_N(D^N)^*(1-P_N)$. The block coefficient at the position $(k,l) \in \mathbb{Z}^2$ of this operator is given by $-(2\pi\,\mathrm{i}\,l)^{-1} Z^*_{l-k}$ if $|k| \le N$ and $|l| > N$ and by 0 else. (Z^*_k denotes the adjoint matrix of Z_k.) From this we see that the estimate for the third term of (2.1) is also valid for the last term if Z is replaced by Z^*. Therefore, we obtain $\|D^N\|_{\mathcal{S}_1} \to 0$ which finishes the proof of this lemma. □

Corollary 2.2. *Let Z be as in Lemma* 2.1. *Then $\Delta_2(1 - B_L(\nu))$ and $\Delta_2(1 - B_{M_Z^{-1} L M_Z}(\nu))$ are equal up to a constant nonvanishing factor which does not depend on ν.*

Proof. Taking the definitions for $B_L(\nu)$ and $B_{M_Z^{-1}LM_Z}(\nu)$, respectively, and applying the product theorem for regularized determinants ([8], p. 169) we immediately obtain

$$(2.4)\qquad \begin{aligned}\Delta_2(1-B_{M_Z^{-1}LM_Z}(\nu)) &= \Delta_2(1-B_L(\nu))\cdot\Delta_2(F^{-1}M_ZFM_Z^{-1})\\ &\cdot\ \exp(-\operatorname{tr}[(1-LF)(1-F^{-1}M_ZFM_Z^{-1})])\\ &\cdot\ \exp(-\operatorname{tr}[\nu(F-M_ZFM_Z^{-1})])\,.\end{aligned}$$

In order to see that the right-hand side of (2.4) is well-defined we note that $1-F^{-1}M_ZFM_Z^{-1}=(M_ZF^{-1}-F^{-1}M_Z)FM_Z^{-1}$. But the commutator of M_Z and F^{-1} is (up to a finite rank operator) equal to $-M_{Z'}$ and thus bounded. So $1-F^{-1}M_ZFM_Z^{-1}\in\mathcal{S}_2(H)$. Lemma 2.1 tells us that the last factor in (2.4) is equal to 1 while the second and third factors do not depend on ν. That the second factor does not vanish follows from the invertibility of $F^{-1}M_ZFM_Z^{-1}$.

□

Remark 2.3. The proof of Lemma 2.1 uses the close connection between F and the derivation. The results of this lemma are obvious if F is replaced by any $\tilde F\in\mathcal{S}_1(H)$. In general, however, it is not clear under which conditions on $\tilde F$ and Z the lemma remains true. At least it is not valid if F is replaced by any $\tilde F\in\mathcal{S}_2(H)$ and M_Z by an arbitrary invertible $B\in\mathcal{L}(H)$, as the following simple example shows. Decompose $H=H_+\oplus H_-$ with $H_+=\ell^2(\mathbb{N}\cup\{0\},\mathbb{C}^n)$, $H_-=\ell^2(-\mathbb{N},\mathbb{C}^n)$ and define B and F with respect to this decomposition as the operator matrices

$$B=\begin{pmatrix}\frac12\mathrm{id}_{H_-} & 0\\ 0 & \mathrm{id}_{H_+}\end{pmatrix},\qquad F=\begin{pmatrix}0 & F_{12}\\ 0 & 0\end{pmatrix}$$

with $F_{12}\in\mathcal{S}_2(H_+,H_-)\backslash\mathcal{S}_1(H_+,H_-)$. Then $F-B^{-1}FB=-F\notin\mathcal{S}_1(H)$.

Now we want to investigate not the regularized determinant $\Delta_2(1-B_L(\nu))$ but the matrix determinant $\det(1-B_L(\nu))$ as defined in Section 1. The well-known formula which connects these determinants ([8], p. 169) leads (after straightforward calculations) to the existence of $\det(1-B_L(\nu))$. We obtain the relation

$$(2.5)\qquad \det(1-B_L(\nu))=\exp(-n(1-\nu)-\operatorname{tr}A_0)\cdot\Delta_2(1-B_L(\nu))$$

where A_0 is defined analogously to (1.3). As an immediate consequence of this equation, the Floquet exponents are exactly the zeros of $\det(1-B_L(\nu))$. The following modification of this infinite determinant will turn out to be useful in the proof of Theorem 2.5 but also in Section 3.

Lemma 2.4. *For* $\nu\in\Lambda:=\{z\in\mathbb{C}:\det\sinh\frac{z-A_0}{2}\neq0\}$ *we set*

$$\overline{B}_L(\nu):=\big((1-\delta_{kl})A_{k-l}(2\pi\,\mathrm{i}\,l+\nu-A_0)^{-1}\big)_{k,l\in\mathbb{Z}}\,.$$

Then $\det(1-\overline{B}_L(\nu))$ *exists for* $\nu\in\Lambda$*, and we obtain* *a)* $\det(1 - B_L(\nu)) = \det(1 - \overline{B}_L(\nu)) \cdot \det(2 \sinh \frac{\nu - A_0}{2})$ *for* $\nu \in \Lambda$,

b) $\det(1 - \overline{B}_L(\nu)) \to 1$ *for* $|\mathrm{Re}\,\nu| \to \infty$.

Proof. Comparing the definitions of $B_L(\nu)$ and $\overline{B}_L(\nu)$ we see for $k, l \in \mathbb{Z}$ and $\nu \in \Lambda$

$$(1 - B_L(\nu))_{kl} = (1 - \overline{B}_L(\nu))_{kl} \cdot \frac{2\pi\,\mathrm{i}\,l + \nu - A_0}{2\pi\,\mathrm{i}\,l + \delta_{0,l}} . \tag{2.6}$$

Therefore the finite section determinants fulfill

$$\begin{aligned} & \det(P_N(1 - B_L(\nu))P_N) \\ = \ & \det(P_N(1 - \overline{B}_L(\nu))P_N) \cdot \prod_{l=-N}^{N} \det\left(\frac{2\pi\,\mathrm{i}\,l + \nu - A_0}{2\pi\,\mathrm{i}\,l + \delta_{0,l}}\right) \\ = \ & \det(P_N(1 - \overline{B}_L(\nu))P_N) \cdot \det\left[(\nu - A_0) \prod_{l=1}^{N} \left(1 + \left(\frac{\nu - A_0}{2\pi l}\right)^2\right)\right] . \end{aligned}$$

For $N \to \infty$ the last determinant converges to $\det(2 \sinh \frac{\nu - A_0}{2}) \neq 0$ as we can see from the product formula for the sinh-function applied to matrices. Thus $\det(1 - \overline{B}_L(\nu))$ exists for $\nu \in \Lambda$ and equality a) holds. To obtain b), we use the estimation

$$\begin{aligned} \|\overline{B}_L(\nu)\|_{\mathcal{S}_2}^2 &= \sum_{k,l} |(\overline{B}_L(\nu))_{kl}|^2 \\ &\leq \sum_{k \neq l} |A_{k-l}|^2 \cdot |(2\pi\,\mathrm{i}\,l + \nu - A_0)^{-1}|^2 \\ &\leq \|A\|_{L_2}^2 \cdot \sum_{l} |(2\pi\,\mathrm{i}\,l + \nu - A_0)^{-1}|^2 \end{aligned}$$

which shows $\overline{B}_L(\nu) \in \mathcal{S}_2(H)$ and $\|\overline{B}_L(\nu)\|_{\mathcal{S}_2} \to 0$ for $|\mathrm{Re}\,\nu| \to \infty$. From the continuity of the regularized determinant we see

$$\det(1 - \overline{B}_L(\nu)) = \Delta_2(1 - \overline{B}_L(\nu)) \to 1 \text{ for } |\mathrm{Re}\,\nu| \to \infty .$$

□

Theorem 2.5. *The determinant* $\det(1 - B_L(\nu))$ *is (up to normalization) a polynomial in* $\exp(\nu)$. *More precisely, the following equality holds for every* $\nu \in \mathbb{C}$ *:*

$$\det(1 - B_L(\nu)) = (-1)^n \exp(-\tfrac{1}{2}(n\nu + \operatorname{tr} A_0)) \cdot \det(Y(1) - \exp(\nu) I_n) .$$

Proof. Due to the theorem of Floquet-Lyapunov there exists a $Z \in W_\infty^1(\mathbf{T}, \mathbb{C}^{n \times n})$ with $\det Z(x) \neq 0$ which transforms (1.2) to a constant system, i.e. we have $(M_Z^{-1} L M_Z) f = f' - K f$ for $f \in H_1(\mathbf{T}, \mathbb{C}^n)$ where $K \in \mathbb{C}^{n \times n}$ is a constant matrix with $\exp K = Y(1)$. From Corollary 2.2 we obtain the existence of some constant $c \neq 0$, not depending on ν, with

$$\begin{aligned}\det(1 - B_L(\nu)) &= \exp(-n(1-\nu) - \operatorname{tr} A_0) \cdot \Delta_2(1 - B_L(\nu)) \\ &= c \cdot \exp(-n(1-\nu) - \operatorname{tr} A_0) \cdot \Delta_2(1 - B_{M_Z^{-1} L M_Z}(\nu)) \\ &= c \cdot \exp(\operatorname{tr} K - \operatorname{tr} A_0) \cdot \det(1 - B_{M_Z^{-1} L M_Z}(\nu))\,.\end{aligned}$$

We calculate the last determinant explicitly ($B_{M_Z^{-1} L M_Z}(\nu)$ is block diagonal). Similarly to the proof of Lemma 2.4 (or using this lemma) we get

$$\begin{aligned}\det(1 - B_{M_Z^{-1} L M_Z}(\nu)) &= \det(2 \sinh \tfrac{\nu - K}{2}) \\ &= \det\left[\exp(-\tfrac{\nu + K}{2}) \cdot (\exp(\nu) I_n - \exp K)\right] \\ &= (-1)^n \exp(-\tfrac{1}{2}(n\nu + \operatorname{tr} K)) \cdot \det(Y(1) - \exp(\nu) I_n)\,,\end{aligned}$$

and therefore

$$\begin{aligned}&\exp(\tfrac{1}{2}(n\nu + \operatorname{tr} A_0)) \cdot \det(1 - B_L(\nu)) \\ &= (-1)^n\, c \exp(\tfrac{1}{2}(\operatorname{tr} K - \operatorname{tr} A_0)) \cdot \det(Y(1) - \exp(\nu) I_n) \\ &= (-1)^n\, \tilde{c} \cdot \det(Y(1) - \exp(\nu) I_n)\,.\end{aligned} \tag{2.7}$$

Here $\tilde{c} := c \exp(\frac{1}{2}(\operatorname{tr} K - \operatorname{tr} A_0))$ is independent of ν.

It remains to compute $\tilde{c}$. The left-hand side of (2.7) can be written as (cf. Lemma 2.4 a))

$$\begin{aligned}&\exp(\tfrac{1}{2}(n\nu + \operatorname{tr} A_0)) \cdot \det(2 \sinh \tfrac{\nu - A_0}{2}) \cdot \det(1 - \overline{B}_L(\nu)) \\ &= (-1)^n \det(\exp A_0 - \exp(\nu) I_n) \cdot \det(1 - \overline{B}_L(\nu))\,.\end{aligned}$$

Due to Lemma 2.4 b) this expression tends to $(-1)^n \det(\exp A_0)$ for $\operatorname{Re} \nu \to -\infty$. Using the formula of Liouville we get

$$\det(\exp A_0) = \exp\left(\operatorname{tr} \int_0^1 A(t)\, dt\right) = \det Y(1)\,.$$

Comparing the limits of both sides of (2.7) for $\operatorname{Re} \nu \to -\infty$ the constant $\tilde{c}$ is seen to be equal to 1 which finishes the proof of the theorem. □

Remark 2.6. a) In the proof of Theorem 2.5 the equivalence of (i) and (iii) in Lemma 1.1 was not used. On the other hand, this equivalence follows immediately from the formula of Theorem 2.5.
b) Due to Theorem 2.5 the Floquet exponents of (1.2) can be computed if the value of $\det(1 - B_L(\nu))$ is known for $n - 1$ different values of ν. (The leading coefficient and the constant term of the polynomial appearing at the right-hand side of Theorem 2.5 are known.) Therefore, it is important to investigate the

convergence of this infinite determinant for fixed $\nu \in \mathbb{C}$. This will be done in Section 3. In particular, in the case of Hill's equation where $n = 2$ we obtain the classical result that the Floquet exponents can be computed from the value of $\det(1 - B_L(0))$, for instance.

We now return to equation (1.1) and assume the dimension of the matrices $A^{(j)}(x)$ to be equal to $\tilde{n}$. If we transform this system to the equivalent first order system (1.2) and apply the results above we obtain the determinant of an infinite block matrix whose coefficients have dimension $m\tilde{n}$. This dimension, however, can be reduced to $\tilde{n}$ (what means an important improvement with respect to computational aspects) as we can see from the following lemma. In this lemma the Fourier coefficients $P_k(t)$ of the polynomial $P(D, x)$ (cf. equation (1.1)) are defined by

$$P_k(t) := \delta_{0,k} I_n t^m + A_k^{(m-1)} t^{m-1} + \ldots + A_k^{(1)} t + A_k^{(0)} \qquad (k \in \mathbb{Z}) .$$

Lemma 2.7. *Define* $1 - B_L^{(m)}(\nu) := \big((2\pi\,\mathrm{i}\,l + \delta_{0,l})^{-m} P_{k-l}(2\pi\,\mathrm{i}\,l + \nu)\big)_{k,l\in\mathbb{Z}}$. *Then the Floquet exponents of* (1.1) *are exactly the zeros of* $\det(1 - B_L^{(m)}(\nu))$. *The function* $\exp(\frac{1}{2} m\tilde{n}\nu) \cdot \det(1 - B_L^{(m)}(\nu))$ *is a polynomial in* $\exp(\nu)$ *of degree* $m\tilde{n}$ *with constant term* $(-1)^{m\tilde{n}} \exp(\frac{1}{2}\operatorname{tr} A_0^{(m-1)})$ *and leading coefficient* $\exp(-\frac{1}{2}\operatorname{tr} A_0^{(m-1)})$.

Proof. Transforming (1.1) to a first order system and applying Theorem 2.5 we obtain an infinite block matrix whose (k, l)-coefficient is equal to

$$\frac{1}{2\pi\,\mathrm{i}\,l + \delta_{0,l}} \begin{bmatrix} \alpha_l \delta_{kl} I_n & -\delta_{kl} I_n & & \\ & \ddots & \ddots & \\ & & \alpha_l \delta_{kl} I_n & -\delta_{kl} I_n \\ A_{k-l}^{(0)} & A_{k-l}^{(1)} & \cdots & \alpha_l \delta_{kl} I_n + A_{k-l}^{(m-1)} \end{bmatrix}$$

where we have set $\alpha_l := 2\pi\,\mathrm{i}\,l + \nu$. Straightforward calculations show that this determinant can be reduced by elementary column transformations to $\det(1 - B_L^{(m)}(\nu))$ as defined in the lemma. □

The determinant of Lemma 2.7 is important for applications of the determinantal approach to the mechanics of vibrations ([1], [2]). For classical Hill systems we have $A^{(1)}(x) = 0$ and no complex computation is necessary (if the input data are real) because in this case

$$1 - B_L^{(2)}(\nu) = \left(\frac{1}{(2\pi l)^2 - \delta_{0,l}} \left[(2\pi l + \mathrm{i}\,\nu)^2 \delta_{kl} I_n - A_{k-l}^{(0)} \right] \right)_{k,l\in\mathbb{Z}}$$

which is a real matrix function for $\nu \in \mathrm{i}\,\mathbb{R}$ (for a more detailed analysis of Hill systems, cf. also [5]).

3. On the convergence of the infinite determinant

In Section 2 the calculation of the Floquet exponents of (1.2) was reduced to the evaluation of $\det(1 - B_L(\nu))$ for a finite number of different $\nu \in \mathbb{C}$. In this section we want to investigate the convergence of the finite section determinants appearing in the definition of $\det(1 - B_L(\nu))$. From now on we will restrict ourselves to the case where the matrix function $A(\cdot)$ is a trigonometric polynomial, i.e. we have $A_k = 0$ for $|k| > b$ with some $b \in \mathbb{N}_0$. In the following let $\nu \in \mathbb{C}$ be fixed. We tacitly assume that all factors and determinants which appear in the formulas below are different from zero. We will use the abbreviations B_{kl} and $\overline{B}_{kl}$ instead of $(B_L(\nu))_{kl}$ and $(\overline{B}_L(\nu))_{kl}$, respectively, and set $\delta_N := \det(P_N(1 - B_L(\nu))P_N)$ and $\overline{\delta}_N := \det(P_N(1-\overline{B}_L(\nu))P_N)$. The first and second lemma of this section deal with the asymptotic behaviour of the sequence $(\overline{\delta}_N)_N$ and $(\delta_N)_N$, respectively.

Lemma 3.1. *Define the complex numbers $\overline{\gamma}_N$ for $N \in \mathbb{N}$ by*

$$\begin{aligned} \overline{\gamma}_N \;:=\; & \det\Bigg[I_n - \sum_{p=1}^{b} \overline{B}_{-N,-N+p}\overline{B}_{-N+p,-N} \\ & - \sum_{\substack{p,q=1\\ p\neq q}}^{b} \overline{B}_{-N,-N+p}\overline{B}_{-N+p,-N+q}\overline{B}_{-N+q,-N}\Bigg] \\ \cdot\; & \det\Bigg[I_n - \sum_{p=1}^{b} \overline{B}_{N,N-p}\overline{B}_{N-p,N} \\ & - \sum_{\substack{p,q=1\\ p\neq q}}^{b} \overline{B}_{N,N-p}\overline{B}_{N-p,N-q}\overline{B}_{N-q,N}\Bigg]. \end{aligned} \tag{3.1}$$

Then $\overline{\delta}_N - \overline{\gamma}_N\overline{\delta}_{N-1} = \mathrm{O}(N^{-4})$ for $N \to \infty$.

Proof. We make use of the transformation of $\overline{B}_L(\nu)$ to a onesided infinite matrix $C := (C_{kl})_{k,l=0}^{\infty}$ given by

$$C_{kl} := \begin{pmatrix} \overline{B}_{-k,-l} & \overline{B}_{-k,l} \\ \overline{B}_{k,-l} & \overline{B}_{kl} \end{pmatrix} \qquad (k, l \in \mathbb{N})$$

(and obvious modifications for $k = 0$ or $l = 0$), cf. [12], p. 16. Now we will use the fact that

$$\det(1-C)_N - \det\Big(I_{2n} - \sum_{p=1}^{b} C_{N,N-p}C_{N-p,N} - \sum_{\substack{p,q=1\\ p\neq q}}^{b} C_{N,N-p}C_{N-p,N-q}C_{N-q,N}\Big)\det(1-C)_{N-1} = \mathrm{O}(N^{-4})\,, \tag{3.2}$$

where $(1-C)_N := (\delta_{kl}I_n - C_{kl})_{k,l=0}^{N}$. To prove (3.2), one has to generalize Satz 6.11 in [13] where the analogue of (3.2) for scalar-valued C_{kl} can be found. The generalization to matrix-valued C_{kl} can be made using the main ideas from [13] and some technical estimates for submatrices and subdeterminants of $(1-C)_N$. We want to omit the complicated but straightforward calculations; the details can be found in [3]. From (3.2) the desired result follows, because $\det(1-C)_N = \overline{\delta}_N$ for all N, and for N large enough the second determinant in (3.2) is equal to $\overline{\gamma}_N$.
□

Remark 3.2. The convergence order of N^{-4} appearing in Lemma 3.1 can be improved if the definition of $\overline{\gamma}_N$ is modified by additional sums. In principle it is possible to describe the asymptotics of the sequence $(\overline{\delta}_N)$ up to an arbitrary order. This can be seen from a generalization of Satz 5.11 in [13] to the matrix case; again we refer to [3] for the details. For the application to the methods of convergence improvement which will be discussed later in this section, the order given in Lemma 3.1 is sufficient.

Lemma 3.3. *We have* $\delta_N - \gamma_N\delta_{N-1} = \mathrm{O}(N^{-4})$ *for*

$$\gamma_N := 1 + \frac{\mathrm{tr}\,(\nu - A_0)^2}{(2\pi N)^2} + 2\sum_{p=1}^{b}\frac{\mathrm{tr}\,(A_pA_{-p})}{(2\pi)^2N(N-p)}\,.$$

P r o o f. Substituting the definition of $\overline{B}_{kl}$ into the expression for $\overline{\gamma}_N$ as given in Lemma 3.1 we see that the first and second factor in (3.1) is equal to

$$\det\left(\frac{\mp 2\pi\,\mathrm{i}\,N + \nu - A_0}{\mp 2\pi\,\mathrm{i}\,N}\right)^{-1}\cdot\det\Bigg[\frac{\mp 2\pi\,\mathrm{i}\,N + \nu - A_0}{\mp 2\pi\,\mathrm{i}\,N} - \sum_p \frac{A_{\mp p}}{\mp 2\pi\,\mathrm{i}\,N}\left(\frac{\mp 2\pi\,\mathrm{i}\,(N-p) + \nu - A_0}{\mp 2\pi\,\mathrm{i}\,(N-p)}\right)^{-1}\frac{A_{\pm p}}{\mp 2\pi\,\mathrm{i}\,(N-p)} + \sum_{p,q}\frac{A_{\mp p}}{\mp 2\pi\,\mathrm{i}\,N}\left(\frac{\mp 2\pi\,\mathrm{i}\,(N-p) + \nu - A_0}{\mp 2\pi\,\mathrm{i}\,(N-p)}\right)^{-1}\frac{A_{\mp(q-p)}}{\mp 2\pi\,\mathrm{i}\,(N-p)}\left(\frac{\mp 2\pi\,\mathrm{i}\,(N-q) + \nu - A_0}{\mp 2\pi\,\mathrm{i}\,(N-q)}\right)^{-1}\frac{A_{\pm q}}{\mp 2\pi\,\mathrm{i}\,(N-q)}\Bigg]\,, \tag{3.3}$$

where the upper sign corresponds to the first and the lower sign to the second factor in (3.1). First we want to rewrite the product of the second factors in (3.3) with different signs up to an accuracy of $\mathrm{O}(N^{-4})$. We make use of $\det(I_n + A) = 1 + \operatorname{tr} A + \mathrm{O}(N^{-4})$ for $A = \mathrm{O}(N^{-2})$ and of

$$\Big(I_n \mp \frac{\nu - A_0}{2\pi\,\mathrm{i}\,(N-p)}\Big)^{-1} = I_n \pm \frac{\nu - A_0}{2\pi\,\mathrm{i}\,(N-p)} + \mathrm{O}(N^{-2})\,.$$

Elementary calculations show

$$(3.4) \qquad \overline{\gamma}_N = \det\Big(I_n + \Big(\frac{\nu - A_0}{2\pi N}\Big)^2\Big)^{-1} \cdot \Bigg[1 + \frac{\operatorname{tr}(\nu - A_0)^2}{(2\pi N)^2} + \sum_p \frac{2\operatorname{tr}(A_p A_{-p})}{2\pi N \cdot 2\pi(N-p)} + \sum_p \Big(\frac{1}{(2\pi\,\mathrm{i}\,N)^2 2\pi\,\mathrm{i}\,(N-p)} - \frac{1}{2\pi i N(2\pi\,\mathrm{i}\,(N-p))^2}\Big) \cdot \operatorname{tr}\Big[(\nu - A_0)(A_p A_{-p} - A_{-p} A_p)\Big] + \sum_{p\neq q} \frac{\operatorname{tr}(A_p A_{q-p} A_{-q}) - \operatorname{tr}(A_{-p} A_{p-q} A_q)}{2\pi\,\mathrm{i}\,N \cdot 2\pi\,\mathrm{i}\,(N-p) \cdot 2\pi\,\mathrm{i}\,(N-q)}\Bigg] + \mathrm{O}(N^{-4})\,.$$

For $\alpha, \beta \in \mathbb{C}$ one obviously has

$$\frac{1}{N(N-\alpha)(N-\beta)} - \frac{1}{N^3} = \mathrm{O}(N^{-4})\,.$$

Therefore, the first sum in (3.4) is of order N^{-4} and can be omitted. The second sum is equal to

$$\frac{1}{(2\pi\,\mathrm{i}\,N)^3} \sum_{p\neq q} \Big[\operatorname{tr}(A_p A_{q-p} A_{-q}) - \operatorname{tr}(A_{-p} A_{p-q} A_q)\Big] + \mathrm{O}(N^{-4})\,,$$

and a simple change of the summation index shows that the value of this sum is equal to zero. So we have shown

$$\overline{\gamma}_N = \det\Big(I_n + \Big(\frac{\nu - A_0}{2\pi N}\Big)^2\Big)^{-1} \cdot (\gamma_N + \mathrm{O}(N^{-4}))\,.$$

But from the connection between δ_N and $\overline{\delta}_N$ (see (2.6)) and Lemma 3.1 we can conclude $\delta_N - \gamma_N \delta_{N-1} = \mathrm{O}(N^{-4})$, and the proof is complete. □

The preceding lemmas allow us to describe the order of convergence for the determinants $\det(1 - B_L(\nu))$ and $\det(1 - \overline{B}_L(\nu))$ (Theorem 3.4) and to improve this order (Theorem 3.5).

Theorem 3.4. *For $N \to \infty$ we have $\delta_N - \delta_{N-1} = \mathrm{O}(N^{-2})$ and $\overline{\delta}_N - \overline{\delta}_{N-1} = \mathrm{O}(N^{-2})$. In general, the exponent -2 cannot be replaced by any smaller number.*

Proof. From Lemma 3.3 and $1-\gamma_N = \mathrm{O}(N^{-2})$ we know $\delta_N - \delta_{N-1} = (\delta_N - \gamma_N\delta_{N-1}) - (1-\gamma_N)\delta_{N-1} = \mathrm{O}(N^{-2})$. We write

$$\overline{\delta}_N - \overline{\delta}_{N-1} = \prod_{l=-N}^{N} \det\Big(\frac{2\pi\,\mathrm{i}\,l + \nu - A_0}{2\pi\,\mathrm{i}\,l + \delta_{0,l}}\Big)^{-1} \cdot \Big[\delta_N - \det\Big(I_n + \Big(\frac{\nu - A_0}{2\pi N}\Big)^2\Big)\delta_{N-1}\Big].$$

Due to the general asumptions at the beginning of this section, the product $\prod_{l=-N}^{N}\cdots$ remains bounded for $N \to \infty$ whereas the last factor is equal to $\delta_N - \delta_{N-1} + \mathrm{O}(N^{-2})$ and thus of order N^{-2}.

The following examples show that the estimation of the theorem cannot be improved without additional asumptions. If $B_L(\nu)$ is block diagonal, i.e. $A_k = 0$ for $k \neq 0$ then

$$\delta_N - \delta_{N-1} = \det\Big[(\nu - A_0)\prod_{l=1}^{N-1}\Big(I_n + \Big(\frac{\nu - A_0}{2\pi l}\Big)^2\Big)\Big] \cdot \Big(\det\Big(I_n + \Big(\frac{\nu - A_0}{2\pi N}\Big)^2\Big) - 1\Big)$$

has exactly convergence order N^{-2}. To see the same for $\overline{\delta}_N$ is more complicated. We take $n = 1$, $A_0 = 0$, $A_1 = A_{-1} = -1$. Then direct calculations show

$$\begin{aligned}\overline{\delta}_N = \overline{\delta}_{N-1} - \Big(&\frac{1}{2\pi\,\mathrm{i}\,N - \nu}\,\frac{1}{2\pi\,\mathrm{i}\,(N-1) - \nu} \\ + \quad &\frac{1}{2\pi\,\mathrm{i}\,N + \nu}\,\frac{1}{2\pi\,\mathrm{i}\,(N-1) + \nu}\Big)\overline{\delta}_{N-2} + \mathrm{O}(N^{-3}) \\ = \quad &\overline{\delta}_{N-1} + \frac{1}{2\pi^2 N(N-1)}\,\overline{\delta}_{N-2} + \mathrm{O}(N^{-3})\,.\end{aligned}$$

If we take ν with $|\det(1 - \overline{B}_L(\nu))| > 2\varepsilon$ the finite section determinant $\overline{\delta}_{N-2}$ fulfills $|\overline{\delta}_{N-2}| > \varepsilon$ for N large enough and thus

$$|\overline{\delta}_N - \overline{\delta}_{N-1}| \geq \frac{\varepsilon}{2\pi^2 N(N-1)} + \mathrm{O}(N^{-3})$$

which shows that the estimation for $\overline{\delta}_N$ cannot be improved. □

Theorem 3.5. *Let $f_0(\operatorname{tr}(\nu - A_0)^2) \neq 0$ and $f_p(2\operatorname{tr}(A_pA_{-p})) \neq 0$ for $p = 1, \dots, b$ where the auxiliary functions f_p are defined by*

$$f_p(z) := \begin{cases} \sinh\Big(\frac{\sqrt{z}}{2}\Big)\cdot\Big(\frac{2}{\sqrt{z}}\Big) & \text{if } p \text{ even,} \\ \cosh\Big(\frac{\sqrt{z}}{2}\Big) & \text{if } p \text{ odd.}\end{cases}$$

Let the modified sequence $(\tilde{\delta}_N)_N$ *be given by*

$$\tilde{\delta}_N := \delta_N \cdot \prod_{m=1}^{N} \left[\left(1 + \frac{\operatorname{tr}(\nu - A_0)^2}{(2\pi m)^2}\right) \prod_{\substack{p=1 \\ p<2m}}^{b} \left(1 + \frac{2\operatorname{tr}(A_p A_{-p})}{\pi^2(2m-p)^2}\right) \right]^{-1} .$$

Then $\tilde{\delta}_N - \tilde{\delta}_{N-1} = \mathrm{O}(N^{-4})$ *and*

$$\det(1 - B_L(\nu)) = \lim_{N\to\infty} \tilde{\delta}_N \cdot f_0(\operatorname{tr}(\nu - A_0)^2) \prod_{p=1}^{b} f_p(2\operatorname{tr}(A_p A_{-p})) .$$

Proof. From the definition of $\tilde{\delta}_N$ we immediately see $\tilde{\delta}_N - \tilde{\delta}_{N-1} = (\delta_N - \tilde{\gamma}_N \delta_{N-1}) \prod_{m=1}^{N}[\ldots]^{-1}$ where the product is the same as in the theorem and

$$\tilde{\gamma}_N := \left(1 + \frac{\operatorname{tr}(\nu - A_0)^2}{(2\pi N)^2}\right) \prod_{p=1}^{b} \left(1 + \frac{2\operatorname{tr}(A_p A_{-p})}{\pi^2(2N-p)^2}\right) .$$

It is easy to see that $\tilde{\gamma}_N - \gamma_N = \mathrm{O}(N^{-4})$ and thus $\tilde{\delta}_N - \tilde{\delta}_{N-1} = \mathrm{O}(N^{-4})$. On the other hand,

$$\det(1 - B_L(\nu)) = \lim_N \delta_N = \lim_N \tilde{\delta}_N \cdot \prod_{m=1}^{\infty} \tilde{\gamma}_m ,$$

and the value for the infinite product can be calculated from the well-known product formulas for the sinh- and cosh-function. □

Remark 3.6. The possibility to apply the determinantal approach to numerical problems always depends on methods of convergence acceleration. From this point of view Theorem 3.5 is important. While in this theorem the convergence order is N^{-4}, for Hill systems the order can be improved up to N^{-8} and in special cases even N^{-12} ([5], [15], [16]). In this sense Hill's equation was not only the first equation for which infinite determinants have been defined but also the equation for which this method works best. But even the order N^{-4} which can be achieved for general systems of the form (1.1) is enough to ensure the comparability of the determinantal method with standard methods. This can be seen from numerical examples. Because we do not want to go into details concerning numerical aspects we just want to state one typical result. The following table contains a comparison of computing time and relative error for the determinantal method with and without acceleration of convergence. The system considered in this example was some model problem of the form (1.1) with m equal to 2 and the dimension of the matrices $A^{(j)}(x)$ equal to 2. So there are four essentially different Floquet exponents, and the relative error stated in the table is the maximum of the relative errors of these exponents. The calculation was done in Fortran 77 on a SUN workstation, and the computing time is given in CPU-seconds.

Block dimension N	Using (δ_N)		Using $(\tilde{\delta}_N)$	
	error	time	error	time
5	$1.6 \cdot 10^{-1}$	0.03	$2.2 \cdot 10^{-3}$	0.03
10	$8.7 \cdot 10^{-2}$	0.05	$2.9 \cdot 10^{-4}$	0.05
20	$4.5 \cdot 10^{-2}$	0.10	$3.7 \cdot 10^{-5}$	0.11
40	$2.3 \cdot 10^{-2}$	0.27	$4.6 \cdot 10^{-6}$	0.27

Table 1: Relative error and computing time for the determinantal method.

As we can see from the table, the acceleration of convergence as described in Theorem 3.5 has almost no influence on the computational time but is crucial for the accuracy of the method. That the determinantal approach is considerably faster than numerical integration can be seen from the corresponding data for the solution of the initial value problem: The computing time needed to obtain a relative error of $2.7 \cdot 10^{-2}$, $2.4 \cdot 10^{-3}$ and $1.0 \cdot 10^{-5}$ was in this example 1.55, 2.02 and 3.13 CPU-seconds, respectively! Thus using infinite determinants is more than ten times faster than solving the initial value problem. This comparison confirms earlier results on the determinantal method vs. numerical integration, see [5], [15] and others. Finally, we want to remark that the determinantal approach which was discussed here can be used as a first step in a two-step algorithm (where the second step is an eigenvalue method). For Hill systems, first results in this direction can be found in [6].

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