Separation and duality in locally L^0–convex modules*

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Abstract

Motivated by financial applications, we study convex analysis for modules over the ordered ring L^0 of random variables. We establish a module analogue of locally convex vector spaces, namely locally L^0–convex modules. In this context, we prove hyperplane separation theorems. We investigate continuity, subdifferentiability and dual representations of Fenchel–Moreau type for L^0–convex functions from L^0–modules into L^0. Several examples and applications are given.

Key words: L^0–Modules, Hahn–Banach Extension, Hyperplane Separation, Locally L^0–Convex Modules, L^0–Convex Functions, Lower Semi Continuity, Subdifferentiability, Fenchel–Moreau Duality

Contents

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1 Introduction

Various fundamental results in mathematical finance draw from convex analysis. For instance, arbitrage theory or duality of risk and utility functions are concepts built on the Hahn–Banach extension theorem and its consequences for hyperplane separation in locally convex vector spaces, cf. [7, 10].

The simplest situation is a one period setup:

$$\mathbb{R} \underset{p}{\overset{\pi,\rho,u}{\leftleftarrows}} E \tag{1.1}$$

$$0 \longrightarrow T$$

Random future (date T) payments are modeled as elements of a locally convex vector space E endowed with semi norms p. Price, risk or utility assessments π, ρ, or u, map E linearly, convexly, or concavely, into the real line $\mathbb{R}$, respectively.

However, the idea of hedging random future payments develops its power in a multi period setting. We therefore randomize the initial data, and let $\pi = \pi(\omega, \cdot)$, $\rho = \rho(\omega, \cdot)$, or $u = u(\omega, \cdot)$, be ω dependent, where $\omega \in \Omega$ denotes the initial states modeled by a probability space $(\Omega, \mathcal{F}, P)$. Here $\mathcal{F}$ is understood as the information available at some future initial date $t < T$.

While classical convex analysis perfectly applies in the one period model (1.1), its application in a multi period framework is rather delicate. Take, for instance, the convexity properties of the risk measure ρ. These properties have to be extended to ω wise convexity properties of $\rho(\omega, \cdot)$ for almost all $\omega \in \Omega$. But ω–wise convex duality correspondences for $\rho(\omega, \cdot)$ have to be made measurable in ω to assert intertemporal consistency in a recursive multi period setup. This would require heavy measurable selection criteria.

We propose instead to consider $\pi = \pi(\omega, \cdot)$, $\rho = \rho(\omega, \cdot)$, or $u = u(\omega, \cdot)$, as maps into $L^0 = L^0(\Omega, \mathcal{F}, P)$, the ordered ring of (equivalence classes of) random variables:

$$\begin{tikzcd} \mathbb{R} & L^0 \arrow[l, phantom] & E \arrow[l, "\pi,\rho,u"'] \arrow[ll, bend left, "p"] \end{tikzcd}$$

$$0 \longrightarrow t \longrightarrow T$$

The space E, in turn, is considered as module over L^0.

This requires hyperplane separation and convex duality results on topological modules, which seem to be new in the literature. In this paper, we provide a comprehensive treatment of convex analysis for topological L^0-modules. While our emphasis is on financial applications as outlined above, the results in this paper are of theoretical nature. We illustrate the scope of applications that can be covered by our results in Section 3.2 below.

The paper is divided into two parts. The first part covers Hahn–Banach extension and hyperplane separation theorems. In the second part, as an application of the first, duality results are established. The related literature is discussed in the course of the text. The remainder of the paper is as follows:

Part I. In Section 2.1 we state the main results on locally L^0–convex topologies and hyperplane separation in locally L^0–convex modules. For the sake of readability, all proofs are postponed to the subsequent respective sections. In Section 2.2 we prove a Hahn–Banach type extension theorem in the context of L^0–modules. Instead of sublinear and linear functions on a vector space we study L^0–sublinear and L^0–linear functions on an L^0–module. In Section 2.3 we characterize a class of topological L^0–modules, namely locally L^0–convex modules. An important feature of a locally L^0–convex module E is that the neighborhoods of 0 absorb E over L^0. This is the key difference to the notion of a locally convex module which is merely absorbent over the real line, cf. [13, 20, 23]. The neighborhood base of a locally L^0–convex module is constructed by means of L^0–semi norms. Such vector valued, or vectorial, norms go back to [14]. In Section 2.4 we establish some preliminary results for L^0–valued gauge functions. In Section 2.5 we prove the hyperplane separation theorems in locally L^0–convex modules. We separate a non empty open L^0–convex set from an L^0–convex set and we strictly separate a point from a non empty closed L^0–convex set by means of continuous L^0–linear functions.

Part II. In Section 3.1 we state the main Fenchel–Moreau type duality results in locally L^0–convex modules. Section 3.2 illustrates the scope of financial applications. As in part one, all proofs are postponed to the subsequent respective sections. In Section 3.3 we prove that L^0–convex functions share a certain local property. In Section 3.4 we characterize lower semi continuous functions. In Section 3.5 we establish continuity results for L^0–convex functions. For instance, under topological assumptions on E, proper L^0–convex functions are automatically continuous on the interior of their effective domain. In Section 3.6 we prove that proper lower semi continuous L^0–convex functions are subdifferentiable on the interior of their effective domain. In Section 3.7 we prove our Fenchel–Moreau type dual representation for proper lower semi continuous L^0–convex functions.

2 Part I. Separation in locally L^0–convex modules

2.1 Main results

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by L^0 the ring of real valued $\mathcal{F}$–measurable random variables. Random variables and sets which coincide almost surely are identified. Recall that L^0 equipped with the order of almost sure dominance is a lattice ordered ring. Throughout, the strict inequality $X > Y$ between two random variables is to be understood as point–wise almost surely (in other texts, "$X > Y$" is sometimes interpreted as "$X \geq Y$ and $X \neq Y$"). Define $L^0_+ := \{Y \in L^0 \mid Y \geq 0\}$ and $L^0_{++} := \{Y \in L^0 \mid Y > 0\}$. By $\bar{L}^0$ we denote the space of all $\mathcal{F}$–measurable random variables which take values in $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and we define $\bar{L}^0_+ := \{Y \in \bar{L}^0 \mid Y \geq 0\}$. Throughout, we follow the convention $0 \cdot (\pm\infty) := 0$.

The order of almost sure dominance allows to define the following topology on L^0. We let

$$B_\varepsilon := \{Y \in L^0 \mid |Y| \leq \varepsilon\}$$

denote the ball of radius $\varepsilon \in L^0_{++}$ centered at $0 \in L^0$. A set $V \subset L^0$ is a neighborhood of $Y \in L^0$ if there is $\varepsilon \in L^0_{++}$ such that $Y + B_\varepsilon \subset V$. A set $V \subset L^0$ is open if it is a neighborhood of all $Y \in V$. Inspection shows that the collection of all open sets is a topology on L^0, which is referred to as topology induced by $|\cdot|$. By construction, $\mathcal{U} := \{B_\varepsilon \mid \varepsilon \in L^0_{++}\}$ is a neighborhood base of $0 \in L^0$. Throughout, we make the convention that $L^0 = (L^0, |\cdot|)$ is endowed with this topology.

Notice that $(L^0, |\cdot|)$ is not a real topological vector space, in general. Indeed, suppose $(\Omega, \mathcal{F}, P)$ is atom–less. Then the scalar multiplication $\mathbb{R} \to L^0$, $\alpha \mapsto \alpha \cdot 1$ is not continuous at $\alpha = 0$. The topology on L^0 induced by $|\cdot|$ is finer than the topology of convergence in probability, which is often used in convex analysis on L^0, such as in [3]. For example, L^0_{++} is open in $(L^0, |\cdot|)$ but not in the topology of convergence in probability.

However, it follows from Theorem 2.4 below that $(L^0, |\cdot|)$ is a topological ring or, equivalently, a topological L^0–module in the following sense:

Definition 2.1 *A topological L^0–module $(E, \mathcal{T})$ is an L^0–module E endowed with a topology $\mathcal{T}$ such that the module operations*

(i) $(E, \mathcal{T}) \times (E, \mathcal{T}) \to (E, \mathcal{T}), (X_1, X_2) \mapsto X_1 + X_2$ *and*

(ii) $(L^0, |\cdot|) \times (E, \mathcal{T}) \to (E, \mathcal{T}), (Y, X) \mapsto YX$

are continuous w.r.t. the corresponding product topologies.

Locally convex topologies in our framework are defined as follows:

Definition 2.2 *A topology $\mathcal{T}$ on E is locally L^0–convex if $(E, \mathcal{T})$ is a topological L^0–module and there is a neighborhood base $\mathcal{U}$ of $0 \in E$ for which each $U \in \mathcal{U}$ is*

(i) *L^0–convex: $YX_1 + (1 - Y)X_2 \in U$ for all $X_1, X_2 \in U$ and $Y \in L^0$ with $0 \leq Y \leq 1$,*

(ii) *L^0–absorbent: for all $X \in E$ there is $Y \in L^0_{++}$ such that $X \in YU$,*

(iii) *L^0–balanced: $YX \in U$ for all $X \in U$ and $Y \in L^0$ with $|Y| \leq 1$.*

In this case, $(E, \mathcal{T})$ is a locally L^0–convex module.

Note that an L^0–convex set $K \subset E$ with $0 \in K$ satisfies $YK \subset K$ for all $Y \in L^0$ with $0 \leq Y \leq 1$; in particular, $1_A K \subset K$ for all $A \in \mathcal{F}$.

Next we show how to construct, and actually characterize all, locally L^0–convex modules. Let E be an L^0–module.

Definition 2.3 *A function $\|\cdot\| : E \to L^0_+$ is an L^0–semi norm on E if:*

(i) $\|YX\| = |Y|\|X\|$ *for all* $Y \in L^0$ *and* $X \in E$,

(ii) $\|X_1 + X_2\| \leq \|X_1\| + \|X_2\|$ *for all* $X_1, X_2 \in E$.

If, moreover,

(iii) $\|X\| = 0$ *implies* $X = 0$,

then $\|\cdot\|$ is an L^0–norm on E.

Any family $\mathcal{P}$ of L^0–semi norms on E induces a topology in the following way. For finite $\mathcal{Q} \subset \mathcal{P}$ and $\varepsilon \in L^0_{++}$ we define

$$U_{\mathcal{Q},\varepsilon} := \left\{ X \in E \mid \sup_{\|\cdot\| \in \mathcal{Q}} \|X\| \leq \varepsilon \right\}$$

and

$$\mathcal{U} := \{U_{\mathcal{Q},\varepsilon} \mid \mathcal{Q} \subset \mathcal{P} \text{ finite and } \varepsilon \in L^0_{++}\}. \tag{2.2}$$

We then proceed as for $(L^0, |\cdot|)$ above and define a topology, referred to as topology induced by $\mathcal{P}$, on E with neighborhood base $\mathcal{U}$ of 0. We thus obtain a locally L^0-convex module, as the following theorem states:

Theorem 2.4 *A topological L^0–module $(E, \mathcal{T})$ is locally L^0–convex if and only if $\mathcal{T}$ is induced by a family of L^0–semi norms.*

Proof. This follows from Lemma 2.16 and Corollary 2.24. □

By convention, an L^0–normed module $(E, \|\cdot\|)$ is always endowed with the locally L^0–convex topology induced by $\|\cdot\|$. Notice that any L^0–norm $\|\cdot\|$ on $E = L^0$ satisfies $\|1\| > 0$ and $\|\cdot\| = \|1\| \, |\cdot|$.

An important L^0–normed module is given in the following example. Recall that a function $\mu : E \to L^0$ is L^0–linear if $\mu(Y_1X_1 + Y_2X_2) = Y_1\mu(X_1) + Y_2\mu(X_2)$ for all $X_1, X_2 \in E$ and $Y_1, Y_2 \in L^0$.

Example 2.5 *Let $(\Omega, \mathcal{E}, P)$ be a probability space with $\mathcal{F} \subset \mathcal{E}$, and let $p \in [1, +\infty]$. We define the function $\|\cdot\|_p : \bar{L}^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by*

$$\|X\|_p := \begin{cases} \lim_{n\to\infty} E[|X|^p \wedge n \mid \mathcal{F}]^{1/p} & \text{if } p < +\infty, \\ \text{ess.inf}\{Y \in \bar{L}^0(\mathcal{F}) \mid Y \geq |X|\} & \text{if } p = +\infty, \end{cases} \tag{2.3}$$

and denote

$$L^p_{\mathcal{F}}(\mathcal{E}) := \left\{X \in L^0(\mathcal{E}) \mid \|X\|_p \in L^0(\mathcal{F})\right\}.$$

In [15], it is shown that $(L^p_{\mathcal{F}}(\mathcal{E}), \|\cdot\|_p)$ is an $L^0(\mathcal{F})$–normed module, which is complete in the sense that any Cauchy net in $L^p_{\mathcal{F}}(\mathcal{E})$ has a limit in $L^p_{\mathcal{F}}(\mathcal{E})$. Moreover, for $p < \infty$, the $L^0(\mathcal{F})$–module of all continuous $L^0(\mathcal{F})$–linear functions $\mu : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ can be identified with $L^q_{\mathcal{F}}(\mathcal{E})$, where $q := p/(p-1)$ ($:= +\infty$ if $p = 1$).

Since $X/\|X\|_p \in L^p(\mathcal{E})$ (with the convention $0/0 := 0$) for $X \in L^p_{\mathcal{F}}(\mathcal{E})$, we conclude that $L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F}) \cdot L^p(\mathcal{E})$ as sets. In particular, for $\mathcal{F} = \{\emptyset, \Omega\}$ the function $\|\cdot\|_p$ can be identified with the classical L^p–norm. In turn $L^p_{\{\emptyset,\Omega\}}(\mathcal{E})$ can be identified with the classical L^p space $L^p(\mathcal{E})$. In fact, whenever $\mathcal{F} = \sigma(A_1, \ldots, A_n)$ is finitely generated, we can identify $L^p_{\sigma(A_1,\ldots,A_n)}(\mathcal{E})$ with $L^p(\mathcal{E})$.

Hahn–Banach type extension theorems for modules appear already in the fifties. This started with [11], where modules over totally ordered rings were considered. Modules over rings which are algebraically and topologically isomorphic to the space of essentially bounded measurable functions on a finite measure space were considered in [12, 21, 19]. Nowadays, it is well known, cf. [4, 22], that a Hahn–Banach type extension theorem for modules over more general ordered rings can be established. In particular, this is the case for L^0–modules.

However, to our knowledge, the following hyperplane separation theorems for L^0-modules are new in the literature. The proofs are given in Section 2.5 below.

Theorem 2.6 (Hyperplane Separation I) *Let E be a locally L^0–convex module and let $K, M \subset E$ be L^0–convex, K open and non empty. If $1_A M \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$ then there is a continuous L^0–linear function $\mu : E \to L^0$ such that*

$$\mu Y < \mu Z \text{ for all } Y \in K \text{ and } Z \in M.$$

For the second hyperplane separation theorem we need to impose some technical assumption on the topology.

Definition 2.7 *A topological L^0–module has the countable concatenation property if for every countable collection (U_n) of neighborhoods of $0 \in E$ and for every countable partition $(A_n) \subset \mathcal{F}$ ($A_n \cap A_m = \emptyset$ for $n \neq m$ and $\bigcup_{n\in\mathbb{N}} A_n = \Omega$) the set*

$$\sum_{n\in\mathbb{N}} 1_{A_n} U_n$$

again is a neighborhood of $0 \in E$.

Notice that any L^0–normed module has the countable concatenation property.

Theorem 2.8 (Hyperplane Separation II) *Let E be a locally L^0–convex module that has the countable concatenation property and let $K \subset E$ be closed L^0–convex and non empty. If $X \in E$ satisfies $1_A\{X\} \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$ then there is $\varepsilon \in L^0_{++}$ and a continuous L^0–linear function $\mu : E \to L^0$ such that*

$$\mu Y + \varepsilon < \mu X \text{ for all } Y \in K.$$

2.2 Hahn–Banach extension theorem

In this section, we establish a Hahn–Banach type extension theorem. We recall that the main result of this section, Theorem 2.14, is already contained in [4, 22]. Nevertheless, for the sake of completeness, we provide a self contained proof which is tailored to our setup. The fact that not all elements in L^0 possess a multiplicative inverse leads to difficulties in showing that the "one step extension" from the proof of the classical Hahn–Banach theorem is well defined in our framework. For this reason, we derive some preliminary results first.

The following lemma recalls that $\mathcal{F}$ is a complete lattice w.r.t. the partial order of almost sure set inclusion.

Lemma 2.9 *Every non empty collection $\mathcal{D} \subset \mathcal{F}$ has a supremum denoted by* $\text{ess.sup}\mathcal{D}$ *and called essential supremum of $\mathcal{D}$. Further, if $\mathcal{D}$ is directed upwards ($A \cup B \in \mathcal{D}$ for all $A, B \in \mathcal{D}$) there is an increasing sequence (A_n) in $\mathcal{D}$ such that* $\text{ess.sup}\mathcal{D} = \bigcup_{n\in\mathbb{N}} A_n$.

If $\mathcal{D} \subset \mathcal{F}$ is empty we set $\text{ess.sup}\mathcal{D} := \emptyset$.

Proof. For a countable set $\mathcal{C} \subset \mathcal{D}$ define $A_{\mathcal{C}} := \bigcup_{A\in\mathcal{C}} A$. Then $A_{\mathcal{C}} \in \mathcal{F}$ and the upper bound

$$c := \sup\{P[A_{\mathcal{C}}] \mid \mathcal{C} \subset \mathcal{D} \text{ countable}\}$$

is attained by some $\mathcal{C}_{\text{sup}}$; indeed, take a sequence $(\mathcal{C}_n)$ in $\mathcal{D}$ with $P[A_{\mathcal{C}_n}] \to c$ and $\mathcal{C}_{\text{sup}} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$. Then, $\mathcal{C}_{\text{sup}} \in \mathcal{F}$ and $P[A_{\mathcal{C}_{\text{sup}}}] = c$. We conclude that $\text{ess.sup}\mathcal{D} := A_{\mathcal{C}_{\text{sup}}}$ is as required. Indeed, $\text{ess.sup}\mathcal{D}$ is an upper bound of $\mathcal{D}$, otherwise there would be $A \in \mathcal{D}$ with $P[A \setminus \text{ess.sup}\mathcal{D}] > 0$ and in turn $P[A_{\mathcal{C}_{\text{sup}}\cup\{A\}}] > P[A_{\mathcal{C}_{\text{sup}}}] = c$. To see that $\text{ess.sup}\mathcal{D}$ is a least upper bound, observe $\text{ess.sup}\mathcal{D} \subset A'$ whenever $A' \in \mathcal{F}$ with $A \subset A'$ for all $A \in \mathcal{D}$. By construction, there is an increasing sequence approximating $\text{ess.sup}\mathcal{D}$ if $\mathcal{D}$ is directed upwards. □

Let E be an L^0–module. For a set $C \subset E$, we define the map $M(\cdot \mid C) : E \to \mathcal{F}$,

$$M(Z \mid C) := \text{ess.sup}\{A \in \mathcal{F} \mid 1_A Z \in C\}. \tag{2.4}$$

If C is an L^0–submodule of E the collection $\{A \in \mathcal{F} \mid 1_A Z \in C\}$ is directed upwards for all $Z \in E$ and hence there exists an increasing sequence $(M_n) \subset \mathcal{F}$ such that

$$M(Z \mid C) = \bigcup_{n\in\mathbb{N}} M_n. \tag{2.5}$$

Definition 2.10 *A set $C \subset E$ has the closure property if*

$$1_{M(Z|C)} Z \in C \textit{ for all } Z \in E. \tag{2.6}$$

By $\hat{C}$ we denote the smallest subset of E that has the closure property and contains C.

Note that $\hat{C}$ is given by
$$\hat{C} = \{1_{M(Z|C)}Z \mid Z \in E\}$$
and therefore $\hat{C}$ always exists and is well defined. By definition, the closure property is a property in reference to E. In particular, E has the closure property.

Lemma 2.11 *Let $C \subset E$ be an L^0–submodule. Then $\hat{C}$ is again an L^0–submodule.*

Proof. Let $X \in \hat{C}$ and $Y \in L^0$. Denote $Z = YX$. By definition, there exists some $X' \in E$ with $X = 1_{M(X'|C)}X'$. Since C is an L^0–submodule of E there exist an increasing sequence $(M_n) \subset \mathcal{F}$ with $M_n \nearrow M(X' \mid C)$ such that $1_{M_n}X' \in C$. Hence $1_{M_n}Z = Y1_{M_n}X' \in C$, and thus $M_n \subset M(Z \mid C)$, for all $n \in \mathbb{N}$. We conclude that $M(X' \mid C) \subset M(Z \mid C)$ and thus
$$YX = Y1_{M(X'|C)}X' = 1_{M(Z|C)}Z \in \hat{C}.$$

Now let $X = 1_A X'$, $Y = 1_B Y' \in \hat{C}$ where $A := M(X' \mid C)$ and $B := M(Y' \mid C)$, for some X', $Y' \in E$. Denote
$$Z = X + Y = 1_{A\setminus B}X + 1_{A\cap B}(X + Y) + 1_{B\setminus A}Y.$$
As above there exist increasing sequences (A_n), $(B_n) \subset \mathcal{F}$ with $A_n \nearrow A$ and $B_n \nearrow B$ such that $1_{A_n}X'$, $1_{B_n}Y' \in C$ and thus
$$\begin{aligned} 1_{A_n\setminus B}X &= 1_{A\setminus B}1_{A_n}X' \in C \\ 1_{A_n\cap B_n}(X + Y) &= 1_{B_n}1_{A_n}X' + 1_{A_n}1_{B_n}Y' \in C \\ 1_{B_n\setminus A}Y &= 1_{B\setminus A}1_{B_n}Y' \in C. \end{aligned}$$
Define the disjoint union $M_n = (A_n \setminus B) \cup (A_n \cap B_n) \cup (B_n \setminus A)$. We obtain
$$1_{M_n}Z = 1_{A_n\setminus B}X + 1_{A_n\cap B_n}(X + Y) + 1_{B_n\setminus A}Y \in C,$$
and thus $M_n \subset M(Z \mid C)$, for all $n \in \mathbb{N}$. Since $M_n \nearrow A \cup B$, we conclude that $A \cup B \subset M(Z \mid C)$ and thus
$$X + Y = 1_{M(Z|C)}Z \in \hat{C}.$$
Hence the lemma is proved. □

For a set $C \subset E$ we denote by
$$\operatorname{span}_{L^0}(C) := \left\{ \sum_{i=1}^{n} Y_i X_i \mid X_i \in C, Y_i \in L^0, 0 \le i \le n, n \in \mathbb{N} \right\}$$
the L^0–submodule of E generated by C. The next example illustrates the situation where an L^0–submodule C of E does not have the closure property.

Example 2.12 *Consider the probability space $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}[0,1]$ the Borel σ–algebra and P the Lebesgue measure on $[0,1]$. Let $E = L^0$, and define*

$$C := \operatorname{span}_{L^0}\{1_{[1-2^{-(n-1)},1-2^{-n}]} \mid n \in \mathbb{N}\}.$$

Then, $1 \notin C$ but $1 \in \hat{C}$.

Proposition 2.13 *Let $C \subset E$ be an L^0–submodule of E, $Z' \in E$ and $Z := 1_{M(Z'|C)^c} Z'$. Then*

(i) *$M(Z' \mid C) = M(Z \mid C)$,*

(ii) *$X = X'$ and $Y = Y'$ on $M(Z \mid C)^c$ whenever $X + YZ = X' + Y'Z$ for $X, X' \in C$ and $Y, Y' \in L^0$, and*

(iii) *for $W \in 1_{M(Z|C)^c} L^0$ and an L^0–linear function $\mu : C \to L^0$*

$$\bar{\mu}(X + YZ) := \mu X + YW \text{ for all } X \in C \text{ and } Y \in L^0 \tag{2.7}$$

defines the unique L^0–linear extension of μ to $\operatorname{span}_{L^0}(C, Z)$ which satisfies $\bar{\mu} Z = W$.

If in addition to this C has the closure property,

(iv) *$\operatorname{span}_{L^0}(C, Z') = \operatorname{span}_{L^0}(C, Z)$.*

Proof. (i) By definition of Z, $M(Z' \mid C) \subset M(Z \mid C)$, and since $P[M(Z \mid C) \setminus M(Z' \mid C)] > 0$ would contradict the definition of $M(Z' \mid C)$ we have $M(Z' \mid C) = M(Z \mid C)$.

(ii) $X + YZ = X' + Y'Z$ is equivalent to $X - X' = (Y' - Y)Z$. If $B := \{Y' - Y \neq 0\} \cap M(Z \mid C)^c$ had positive measure then on B, $Z = (X - X')/(Y' - Y) \in C$ in contradiction to the definition of $M(Z \mid C)$. Hence $Y = Y'$ and in turn $X = X'$ on $M(Z \mid C)^c$.

(iii) This is an immediate consequence of (ii).

(iv) By definition of Z, $\operatorname{span}_{L^0}(C, Z) \subset \operatorname{span}_{L^0}(C, Z')$. Since C has the closure property, $1_{M(Z'|C)} Z' \in C$ and hence $\operatorname{span}_{L^0}(C, Z) = \operatorname{span}_{L^0}(C, Z')$. □

A function $p : E \to L^0$ is L^0–sublinear if $p(YX) = Y p(X)$ for all $X \in E$ and $Y \in L^0_+$ and $p(X_1 + X_2) \leq p(X_1) + p(X_2)$ for all $X_1, X_2 \in E$. We can now state and prove the main result of this section.

Theorem 2.14 (Hahn–Banach) *Consider an L^0–sublinear function $p : E \to L^0$, an L^0–submodule C of E and an L^0–linear function $\mu : C \to L^0$ such that*

$$\mu X \leq p(X) \text{ for all } X \in C.$$

Then μ extends to an L^0–linear function $\bar{\mu} : E \to L^0$ such that $\bar{\mu} X \leq p(X)$ for all $X \in E$.

Proof. **Step 1:** In view of Lemma 2.15 below we can assume that C has the closure property and that there exists $Z' \in E \setminus C$. Then $Z := 1_{M(Z'|C)^c} Z' \notin C$ and $Z \neq 0$. We will show that μ extends L^0–linearly to $\bar{\mu} : \mathrm{span}_{L^0}(C, Z) \to C$, such that

$$\bar{\mu} X \leq p(X) \text{ for all } X \in \mathrm{span}_{L^0}(C, Z). \tag{2.8}$$

More precisely, we claim that

$$W := 1_{M(Z|C)^c} \operatorname*{ess.sup}_{X \in C} (\mu X - p(X - Z))$$

and $\bar{\mu}$ defined as in (2.7) satisfies

$$\mu X + YW \leq p(X + YZ) \text{ for all } X \in C \text{ and } Y \in L^0 \tag{2.9}$$

which, apparently, is equivalent to (2.8). To verify this claim, let $X, X' \in C$ and observe

$$\begin{aligned} \mu X + \mu X' &= \mu(X + X') \\ &\leq p(X + X') \\ &= p(X' + Z + X - Z) \\ &\leq p(X' + Z) + p(X - Z). \end{aligned}$$

Hence,

$$\mu X - p(X - Z) \leq p(X' + Z) - \mu X' \text{ for all } X, X' \in C. \tag{2.10}$$

Since $Z = 0$ on $M(Z \mid C)$ we have $\mu X - p(X - Z) \leq 0$ on $M(Z \mid C)$ as well as $p(X' + Z) - \mu X' \geq 0$ on $M(Z \mid C)$ for all $X, X' \in C$. Hence, (2.10) implies

$$\mu X - p(X - Z) \leq W \leq p(X' + Z) - \mu X' \text{ for all } X, X' \in C \tag{2.11}$$

and in turn

$$\mu X \pm W \leq p(X \pm Z) \text{ for all } X \in C.$$

From this we derive

$$1_A(\mu X + W) \leq 1_A p(X + Z) = 1_A p(X + 1_A Z) \tag{2.12}$$

$$1_{A^c}(\mu X - W) \leq 1_{A^c} p(X - Z) = 1_{A^c} p(X - 1_{A^c} Z) \tag{2.13}$$

for all $A \in \mathcal{F}$. Adding up the inequalities in (2.12) and (2.13) yields

$$\mu X + (1_A - 1_{A^c})W \leq p(X + (1_A - 1_{A^c})Z) \text{ for all } X \in C \text{ and } A \in \mathcal{F}. \tag{2.14}$$

Further, for all $Y \in L^0$ with $P[Y \neq 0] = 1$ we have $Y/|Y| = 1_A - 1_{A^c}$, where $A := \{Y > 0\} \in \mathcal{F}$. Thus, (2.14) implies

$$|Y| \left(\mu \left(\frac{X}{|Y|} \right) + \frac{Y}{|Y|} W \right) \leq |Y| p \left(\frac{X}{|Y|} + \frac{Y}{|Y|} Z \right).$$

for all $X \in C$ and $Y \in L^0$ with $P[Y \neq 0] = 1$. From this we derive

$$\mu X + YW \leq p(X + YZ) \text{ for all } X \in C \text{ and } Y \in L^0 \text{ with } P[Y \neq 0] = 1. \tag{2.15}$$

But this already implies the required inequality in (2.9). Indeed, for $X \in C$ and arbitrary $Y \in L^0$ we define $Y' := Y1_A + 1_{A^c}$, where $A := \{Y \neq 0\}$, and derive from (2.15)

$$1_A(\mu X + YW) = 1_A(\mu X + Y'W) \leq 1_A p(X + Y'Z) = 1_A p(X + YZ) \tag{2.16}$$

$$1_{A^c}(\mu X + YW) = 1_{A^c}(\mu X) \leq 1_{A^c} p(X) = 1_{A^c} p(X + YZ). \tag{2.17}$$

Adding up (2.16) and (2.17), we see that (2.15) implies (2.9) and complete this step.

Step 2: The set

$$\mathcal{I} := \left\{ (D, \bar{\mu}) \mid \begin{array}{c} C \subset D \overset{L^0\text{–linear}}{\subset} E, D \text{ has the closure property} \\ \bar{\mu} : D \overset{L^0\text{–linear}}{\rightarrow} L^0, \bar{\mu}|_C = \mu \text{ and } \bar{\mu}X \leq p(X) \text{ for all } X \in D \end{array} \right\}$$

is partially ordered by

$$(D, \bar{\mu}) \leq (D', \bar{\mu}') \text{ if and only if } D \subset D' \text{ and } \bar{\mu}'|_D = \bar{\mu}.$$

We will show that a totally ordered subset $\{(D_i, \bar{\mu}_i), i \in I\}$ of $\mathcal{I}$ (that is, for all i, j either $(D_i, \bar{\mu}_i) \leq (D_j, \bar{\mu}_j)$ or $(D_i, \bar{\mu}_i) \geq (D_j, \bar{\mu}_j)$) has an upper bound and then we will apply Zorn's lemma. To this end, observe that D given by

$$C \subset D := \bigcup_{i \in I} D_i \subset E$$

is an L^0–module since $\{(D_i, \bar{\mu}_i), i \in I\}$ is totally ordered. $\bar{\mu} : D \rightarrow L^0$ given by $\bar{\mu}|_{D_i} := \bar{\mu}_i$ is L^0–linear, dominated by p on all of D and $\bar{\mu}|_C = \mu$. Further, in view of Lemma 2.15 below, we can assume that D has the closure property. Hence, $(D, \bar{\mu}) \in \mathcal{I}$ is an upper bound for $\{(D_i, \bar{\mu}_i), i \in I\}$ and Zorn's lemma yields the existence of a maximal element $(D_{\max}, \bar{\mu}_{\max}) \in \mathcal{I}$, i.e.

$$(D_{\max}, \bar{\mu}_{\max}) \leq (D, \bar{\mu}) \in \mathcal{I} \text{ implies } (D_{\max}, \bar{\mu}_{\max}) = (D, \bar{\mu}).$$

Assume that $D_{\max} \neq E$. Then, by the first step of this proof, $\bar{\mu}_{\max}$ extends to

$$\bar{\mu}'_{\max} : \operatorname{span}_{L^0}(D_{\max}, Z) \rightarrow L^0,$$

where $Z \in E \setminus D_{\max}$, which contradicts the maximality of $(D_{\max}, \bar{\mu}_{\max})$. Hence, $D_{\max} = E$ and $\bar{\mu}_{\max}$ is as desired. □

Lemma 2.15 *Let C, μ, p be as in Theorem 2.14. Then μ extends uniquely to an L^0–linear function $\hat{\mu} : \hat{C} \rightarrow L^0$ such that $\hat{\mu}X \leq p(X)$ for all $X \in \hat{C}$.*

Proof. For $Z \in E$, let

$$\hat{\mu}(1_{M(Z|C)}Z) := \lim_{n\to\infty} \mu(1_{M_n}Z), \tag{2.18}$$

where $M(Z \mid C) = \bigcup_{n\in\mathbb{N}} M_n$ as in (2.5). Since for all $n \leq m$

$$\mu(1_{M_n}Z) = \mu(1_{M_m}Z) \text{ on } M_n$$

(2.18) uniquely and unambiguously defines the L^0–linear extension $\hat{\mu} : \hat{C} \to L^0$ of μ to $\hat{C}$. Further, (2.18) guarantees that $\hat{\mu}X \leq p(X)$ for all $X \in \hat{C}$. □

2.3 Locally L^0–convex modules

In this section we establish some facts about locally L^0–convex modules. For more background on general topological spaces we refer to the comprehensive Chapter 2 of [1].

Let us first recall some basic definitions. Let $\mathcal{T}$ be a topology on some set E. Then $K \subset E$ is closed if $K^c \in \mathcal{T}$. The interior, boundary and closure of K are denoted by $\overset{\circ}{K}$, ∂K, $\bar{K}$, respectively. Moreover, $\overset{\circ}{K} \cap \partial K = \emptyset$, K is open if and only if $K = \overset{\circ}{K}$, and K is closed if and only if $K = \bar{K}$. An element $X \in \overset{\circ}{K}, \partial K, \bar{K}$ is an interior, boundary, closure point of K, respectively.

Now let E be an L^0–module and $\mathcal{T}$ the topology induced by some family $\mathcal{P}$ of L^0–semi norms on E, see Definition 2.3 and below. The following result gives one direction in the proof of Theorem 2.4. The converse direction is proved in Corollary 2.24 below.

Lemma 2.16 *$(E, \mathcal{T})$ is a locally L^0–convex module.*

Proof. Let $\mathcal{U}$ denote the neighborhood base given in (2.2). It follows by inspection that each $U \in \mathcal{U}$ is L^0–convex, L^0–absorbent and L^0–balanced as in Definition 2.2. To establish (i) and (ii) of Definition 2.1, let $O \in \mathcal{T}$.

(i) We show that $\widetilde{O} := \{(X, Y) \in E \times E \mid X + Y \in O\}$ is open. Let $(X, Y) \in \widetilde{O}$ and $U = U_{\mathcal{Q},\varepsilon} \in \mathcal{U}$ such that $X + Y + U \subset O$. Then $V = U_{\mathcal{Q},\varepsilon/2}$ satisfies $V + V \subset U$ and hence $(X + V) \times (Y + V) \subset \widetilde{O}$. This means that (X, Y) is an interior point of $\widetilde{O}$ and (i) follows.

(ii) We show that $\widetilde{O} := \{(X, Y) \in E \times L^0 \mid XY \in O\}$ is open. Consider $(X, Y) \in \widetilde{O}$ and $U = U_{\mathcal{Q},\varepsilon} \in \mathcal{U}$ such that $XY + U \subset O$. We find $\varepsilon \in L^0_{++}$ and $W \in \mathcal{U}$ such that

$$W \times \{Z \in L^0 \mid |Z - Y| \leq \varepsilon\} \subset \widetilde{O}$$

as follows. As in the proof of (i) let $V \in \mathcal{U}$ be such that $V + V \subset U$ and let $\varepsilon \in L^0_{++}$ be such that $\varepsilon X \in V$, which is possible since V is L^0–absorbing. Further, since V is L^0–balanced,

$$(Z - Y)X \in V \text{ if } |Z - Y| \leq \varepsilon.$$

V is of the form $V = U_{\mathcal{Q},\delta}$, hence $W := U_{\mathcal{Q},\delta/(\varepsilon+|Y|)}$ satisfies $(\varepsilon + |Y|)W \subset V$ and since W is L^0–balanced

$$ZW \subset V \text{ for all } Z \in L^0 \text{ with } |Z| \leq \varepsilon + |Y|.$$

Finally, for $|Z - Y| \le \varepsilon$ and $X' \in W$ we derive

$$Z(X + X') - YX = (Z - Y)X + ZX' \in V + V \subset U$$

and the assertion is proved. $\square$

Here is a trivial example.

Example 2.17 (Chaos Topology) *The locally L^0–convex topology $\mathcal{T}$ induced by the trivial L^0–semi norm $\|\cdot\| \equiv 0$ on L^0 consists of the sets $\emptyset$ and L^0. $\mathcal{T}$ is called chaos topology and it is an example for a locally L^0–convex topology which is not Hausdorff. Note that $\mathcal{T}$ is locally convex and locally L^0–convex at the same time.*

2.3.1 The countable concatenation property

A technicality we encounter is a certain concatenation property. This concatenation property is crucial in the context of hyperplane separation, cf. Lemma 2.28, Theorem 2.8 and the Examples 2.29 and 2.30 in Section 2.5 below.

The following result motivates the subsequent definition.

Lemma 2.18 *Let $\mathcal{P}$ be a family of L^0–semi norms inducing a locally L^0–convex topology $\mathcal{T}$ on E.*

(i) *For $A \in \mathcal{F}$ and $\|\cdot\| \in \mathcal{P}$, $1_A\|\cdot\|$ is an L^0–semi norm.*

(ii) *For a finite collection $\|\cdot\|_1, \dots, \|\cdot\|_n \in \mathcal{P}$, $\sup_{i=1,\dots,n} \|\cdot\|_i$ is an L^0–semi norm.*

(iii) *Define*

$$\mathcal{P}' := \mathcal{P} \cup \{1_A\|\cdot\| \mid A \in \mathcal{F}, \|\cdot\| \in \mathcal{P}\}$$
$$\mathcal{P}'' := \mathcal{P}' \cup \left\{ \sup_{\|\cdot\| \in \mathcal{Q}} \|\cdot\| \mid \mathcal{Q} \subset \mathcal{P}' \text{ finite} \right\}$$

and denote $\mathcal{T}'$ and $\mathcal{T}''$ the induced locally L^0–convex topologies, respectively. Then $\mathcal{T} = \mathcal{T}' = \mathcal{T}''$; in other words, we may always assume that, with every $\|\cdot\| \in \mathcal{P}$, $\mathcal{P}$ contains $1_A\|\cdot\|$ for all $A \in \mathcal{F}$ and that $\mathcal{P}$ is closed under finite suprema.

Proof. (i) and (ii) follow from the properties of L^0–semi norms.

(iii) Since $\mathcal{P} \subset \mathcal{P}' \subset \mathcal{P}''$ we have $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}''$. The inclusion $\mathcal{T}'' \subset \mathcal{T}$ follows from the fact that for all $\varepsilon \in L^0_{++}$,

$$U_{\{\|\cdot\|\},\varepsilon} \subset U_{\{1_A\|\cdot\|\},\varepsilon} \text{ for all } \|\cdot\| \in \mathcal{P} \text{ and } A \in \mathcal{F} \text{ and}$$
$$U_{\{\|\cdot\|_1,\dots,\|\cdot\|_n\},\varepsilon} = U_{\{\sup_{i=1,\dots,n}\|\cdot\|_i\},\varepsilon} \text{ for all } \|\cdot\|_1, \dots, \|\cdot\|_n \in \mathcal{P}.$$

$\square$

For a finite collection $U_{\mathcal{Q}_1,\varepsilon_1},\dots,U_{\mathcal{Q}_n,\varepsilon_n}$ and a finite collection of pairwise disjoint sets $A_1,\dots,A_n\in\mathcal{F}$ ($A_i\cap A_j=\emptyset$ for $i\neq j$), the preceding lemma shows that $\sum_{i=1}^n 1_{A_i}U_{\mathcal{Q}_i,\varepsilon_i}$ is a neighborhood of $0\in E$. Indeed, let

$$\|\cdot\| := \sum_{i=1}^n 1_{A_i} \sup_{\|\cdot\|\in\mathcal{Q}_i}\|\cdot\| = \sup_{i=1,\dots,n} 1_{A_i} \sup_{\|\cdot\|\in\mathcal{Q}_i}\|\cdot\|$$

and $\varepsilon:=\sum_{i=1}^n 1_{A_i}\varepsilon_i$. Then, $\sum_{i=1}^n 1_{A_i}U_{\mathcal{Q}_i,\varepsilon_i}=U_{\{\|\cdot\|\},\varepsilon}$.

In the case of a countably infinite sequence $(U_{\mathcal{Q}_n,\varepsilon_n})$ and a pairwise disjoint sequence $(A_n)\subset\mathcal{F}$ ($A_i\cap A_j=\emptyset$ for $i\neq j$) the next example illustrates that the above reasoning does not apply, as the L^0–semi norm given by

$$\|\cdot\| := \sum_{n\in\mathbb{N}} 1_{A_n} \sup_{\|\cdot\|\in\mathcal{Q}_n}\|\cdot\| = \sup_{n\in\mathbb{N}} 1_{A_n} \sup_{\|\cdot\|\in\mathcal{Q}_n}\|\cdot\|$$

cannot be assumed to belong to $\mathcal{P}$ in general.

Example 2.19 *Consider the probability space $\Omega=[0,1]$, $\mathcal{F}=\sigma(A_n\mid n\in\mathbb{N})$ the σ-algebra generated by the sets $A_n:=[1-2^{-(n-1)},1-2^{-n}]$, and P the Lebesgue measure. Define $B_n:=\cup_{m\leq n}A_m$, and let $E:=L^0$. For the family $\mathcal{P}$ of L^0–semi norms $|\cdot|_n:=1_{A_n}|\cdot|$, $n\in\mathbb{N}$, we subsequently derive the following:*

(i) $|\cdot|=\sum_{n\in\mathbb{N}}|\cdot|_n\notin\mathcal{P}$.

(ii) *For all $\varepsilon\in L^0_{++}$, $U_{\{|\cdot|\},\varepsilon}=\sum_{n\in\mathbb{N}}1_{A_n}U_{\{|\cdot|_n\},\varepsilon}$ is not a neighborhood of the origin in the locally L^0–convex topology induced by $\mathcal{P}$.*

(iii) *The sequence $(1_{B_n}\frac{1}{n}+1_{B_n^c})_{n\in\mathbb{N}}$ converges to 0 w.r.t. the locally L^0–convex topology induced by $\mathcal{P}$ but it does not converge to 0 in the locally L^0–convex topology induced by $\mathcal{P}\cup\{|\cdot|\}$.*

This leads us to the following definition.

Definition 2.20 *A family $\mathcal{P}$ of L^0–semi norms has the countable concatenation property if*

$$\sum_{n\in\mathbb{N}}1_{A_n}\|\cdot\|_n\in\mathcal{P},$$

for every pairwise disjoint sequence $(A_n)\subset\mathcal{F}$ and for every sequence of L^0–semi norms $(\|\cdot\|_n)$ in $\mathcal{P}$.

If $\mathcal{P}$ is a family of L^0–semi norms which has the countable concatenation property then $(E,\mathcal{T})$ has the countable concatenation property in the sense of Definition 2.7. Conversely, if $(E,\mathcal{T})$ is a topological L^0–module which has the countable concatenation property, where $\mathcal{T}$ is induced by a family $\mathcal{P}$ of L^0–semi norms, we can always assume that $\mathcal{P}$ has the countable concatenation property. Indeed, inspection shows that

$$\left\{\sum_{n\in\mathbb{N}} 1_{A_n}\|\cdot\|_n \mid (A_n)\subset\mathcal{F} \text{ pairwise disjoint}, (\|\cdot\|_n)\subset\mathcal{P}\right\}$$

also induces $\mathcal{T}$.

In view of Lemma 2.18 we can always assume that a finite family of L^0–semi norms has the countable concatenation property.

2.3.2 The index set of nets

The neighborhood base $\mathcal{U}$ of $0\in E$ given in (2.2) is indexed with the collection of all finite subsets of $\mathcal{P}$ and L^0_{++}. We introduce a direction " $\geq$ " on this index set as follows:

$$(\mathcal{R}_2,\alpha_2)\geq(\mathcal{R}_1,\alpha_1) \text{ if and only if } \mathcal{R}_2\subset\mathcal{R}_1 \text{ and } \alpha_1\leq\alpha_2 \tag{2.19}$$

for all finite $\mathcal{R}_1,\mathcal{R}_2\subset\mathcal{P}$ and $\alpha_1,\alpha_2\in L^0_{++}$. We denote nets w.r.t. this index set by $(X_{\mathcal{R},\alpha})$. If E is a topological L^0–module, not necessarily locally L^0–convex, nets are denoted by $(X_\alpha)_{\alpha\in\mathcal{D}}$ or (X_α) for corresponding index set $\mathcal{D}$.

2.4 The gauge function

Let E be an L^0–module.

Definition 2.21 *The gauge function $p_K: E\to\bar{L}^0_+$ of a set $K\subset E$ is defined by*

$$p_K(X):=\text{ess.inf}\{Y\in L^0_+ \mid X\in YK\}. \tag{2.20}$$

The gauge function p_K of an L^0–absorbent set $K\subset E$ maps E into L^0_+. Moreover:

Proposition 2.22 *The gauge function p_K of an L^0–absorbent set $K\subset E$ satisfies:*

(i) $p_K(X)\leq 1$ *for all* $X\in K$.

(ii) $1_A p_K(1_A X)\geq 1_A p_K(X)$ *for all* $X\in E$ *and* $A\in\mathcal{F}$.

(iii) $Yp_K(1_{\{Y>0\}}X)=p_K(YX)$ *for all* $X\in E$ *and* $Y\in L^0_+$*; in particular,* $Yp_K(X)=p_K(YX)$ *if* $Y\in L^0_{++}$.

Proof. (i) This assertion follows immediately from the definition of p_K.

(ii) Let $X\in E$ and $A\in\mathcal{F}$. We have

$$\begin{aligned} 1_A \operatorname*{ess.inf}_{X\in ZK} Z &= 1_A \operatorname*{ess.inf}_{X\in ZK} 1_A Z \\ &\geq 1_A \operatorname*{ess.inf}_{1_A X\in 1_A ZK} 1_A Z \\ &= 1_A \operatorname*{ess.inf}_{1_A X\in ZK} 1_A Z = 1_A \operatorname*{ess.inf}_{1_A X\in ZK} Z, \end{aligned} \tag{2.21}$$

where the inequality in (2.21) follows since $X \in ZK$ implies $1_A X \in 1_A ZK$. Hence, $1_A p_K(X) \geq 1_A p_K(1_A X)$.

(iii) Let $X \in E$, $Y \in L^0_+$ and define $A := \{Y > 0\}$. We have

$$\begin{aligned} Y \operatorname*{ess.inf}_{1_A X \in ZK} Z &= \operatorname*{ess.inf}_{1_A X \in ZK} YZ \\ &\overset{Z' := YZ}{=} \operatorname*{ess.inf}_{1_A XY \in 1_A Z'K} Z' \\ &= \operatorname*{ess.inf}_{1_A XY \in ZK} Z = \operatorname*{ess.inf}_{XY \in ZK} Z, \end{aligned}$$

and hence $Y p_K(1_A X) = p_K(YX)$. ☐

A non empty L^0–absorbent L^0–convex set $K \subset E$ always contains the origin; indeed, let $X \in E$ and $Y_1, Y_2 \in L^0_{++}$ be such that $X/Y_1, -X/Y_2 \in K$. Then, since K is L^0–convex,

$$\frac{Y_1}{Y_1 + Y_2} \frac{X}{Y_1} + \frac{Y_2}{Y_1 + Y_2} \frac{-X}{Y_2} = \frac{X - X}{Y_1 + Y_2} = 0 \in K. \tag{2.22}$$

Depending on the choice of $K \subset E$, the gauge function p_K can be L^0–sublinear or an L^0–semi norm.

Proposition 2.23 *The gauge function p_K of an L^0–absorbent L^0–convex set $K \subset E$ satisfies:*

(i) $p_K(X) = \text{ess.inf}\{Y \in L^0_{++} \mid X \in YK\}$ *for all* $X \in E$.

(ii) $Y p_K(X) = p_K(YX)$ *for all* $Y \in L^0_+$ *and* $X \in E$.

(iii) $p_K(X + Y) \leq p_K(X) + p_K(Y)$ *for all* $X, Y \in E$.

(iv) *For all $X \in E$ there exists a sequence (Z_n) in L^0 such that*

$$Z_n \searrow p_K(X) \text{ a.s.} \tag{2.23}$$

In particular, since $0 \in K$ (cf. (2.22)), p_K is L^0–sublinear.

If in addition to this K is L^0–balanced then p_K satisfies:

(v) $p_K(YX) = |Y| p_K(X)$ *for all* $Y \in L^0$ *and for all* $X \in E$.

In particular, p_K is an L^0–semi norm.

Proof. (i) As "$\leq$" follows from the definition of p_K we only prove the reverse inequality. To this end, let $Y \in L^0_+$ with $X = YZ$ for some $Z \in K$. Then $\{Y = 0\} \subset \{X = 0\}$ and in turn $A := \{Y > 0\} \supset \{X \neq 0\}$. Thus, with $Y_\varepsilon := 1_A Y + 1_{A^c} \varepsilon$ for $\varepsilon \in L^0_{++}$ we have

$$X = 1_A X = Y 1_A Z = Y_\varepsilon 1_A Z \in Y_\varepsilon 1_A K \subset Y_\varepsilon K.$$

The claim now follows since $\text{ess.inf}_{\varepsilon \in L^0_{++}} Y_\varepsilon = Y$.

(ii) To prove this assertion we first show that

$$1_A p_K(1_A X) = 1_A p_K(X) \text{ for all } X \in E \text{ and } A \in \mathcal{F}. \tag{2.24}$$

(ii) then follows from (iii) of Proposition 2.22 together with (2.24).

To establish (2.24), we only have to prove the reverse inequality in (2.21). To this end, let $Y_1, Y_2 \in L^0_+$ with $1_A X = 1_A Y_1 Z_1, X = Y_2 Z_2$ for $Z_1, Z_2 \in K$ and $A \in \mathcal{F}$. In particular, $1_{A^c} X = 1_{A^c} Y_2 Z_2$. We have

$$X = 1_A Y_1 Z_1 + 1_{A^c} Y_2 Z_2 = (1_A Y_1 + 1_{A^c} Y_2)(1_A Z_1 + 1_{A^c} Z_2)$$

and since L^0–convexity of K implies that $1_A Z_1 + 1_{A^c} Z_2 = 1_A Z_1 + (1 - 1_A) Z_2 \in K$ the required inequality follows.

(iii) Let $X_1, X_2 \in E$ and $Y_1, Y_2 \in L^0_{++}$ such that $X_1/Y_1, X_2/Y_2 \in K$. Since K is L^0–convex

$$\frac{Y_1}{Y_1 + Y_2} \frac{X_1}{Y_1} + \frac{Y_2}{Y_1 + Y_2} \frac{X_2}{Y_2} = \frac{X_1 + X_2}{Y_1 + Y_2} \in K.$$

Thus, $p_K(\frac{X_1+X_2}{Y_1+Y_2}) \leq 1$, and hence $p_K(X_1 + X_2) \leq Y_1 + Y_2$. Since Y_1 and Y_2 are arbitrary, we may take the essential infimum over all such pairs Y_1, Y_2 and – in view of (i) – we derive

$$p_K(X_1 + X_2) \leq p_K(X_1) + p_K(X_2).$$

(iv) As in the proof of (2.24), L^0–convexity of K implies that the set

$$\{Y \in L^0_+ \mid X \in YK\}$$

is directed downwards (and upwards) for all $X \in E$.

(v) Let $X \in E, Y \in L^0$ and $A := \{Y \geq 0\}$. Then (2.24) and (ii) imply

$$p_K(YX) = 1_A |Y| p_K(X) + 1_{A^c} |Y| p_K(-X),$$

and hence it remains to prove that $p_K(-X) = p_K(X)$. But since K is L^0–balanced we have $-K = K$ and hence

$$p_K(-X) = p_{-K}(-X) = p_K(X).$$

$\square$

As a consequence of Proposition 2.23, we can now complete the proof of Theorem 2.4:

Corollary 2.24 *Any locally L^0–convex topology $\mathcal{T}$ on E is induced by a family of L^0–semi norms.*

Proof. Let $\mathcal{U}$ be a neighborhood base of $0 \in E$ such that every $U \in \mathcal{U}$ is L^0–absorbent, L^0–convex and L^0–balanced. Then, the family of gauge functions

$$\mathcal{P} := \{p_U \mid U \in \mathcal{U}\},$$

by Proposition 2.23, is a family of L^0–semi norms and the topology induced by $\mathcal{P}$ coincides with $\mathcal{T}$. $\square$

Proposition 2.25 *The gauge function p_K of an L^0–absorbent L^0–convex set $K \subset E$ (recall that $0 \in K$, cf. (2.22)) satisfies:*

(i) *$p_K(X) \geq 1$ for all $X \in E$ with $1_A X \notin 1_A K$ for all $A \in \mathcal{F}$ with $P[A] > 0$.*

If in addition to this, E is a locally L^0–convex module, then p_K satisfies:

(ii) *$p_K(X) < 1$ for all $X \in \overset{\circ}{K}$.*

Proof. To prove (i) let us assume that $\{p_K(X) < 1\}$ has positive P–measure for some $X \in E$ with $X 1_A \notin K$ for all $A \in \mathcal{F}$ with $P[A] > 0$. With (iv) of Proposition 2.23 we know that there is $Y \in L^0_+$ such that $B := \{Y < 1\}$ has positive P–measure and

$$X \in YK.$$

But this is a contradiction as we derive

$$X 1_B \in Y 1_B K \subset 1_B K,$$

where the last inclusion follows from the L^0–convexity of $1_B K$. (Note that $0 \in K$.)

(ii) Let $X \in \overset{\circ}{K}$. Then there exists a neighborhood $U_{\mathcal{Q},\varepsilon}$ ($\mathcal{Q} \subset \mathcal{P}$ finite and $\varepsilon \in L^0_{++}$) of $0 \in E$ such that $X + U_{\mathcal{Q},\varepsilon} \subset K$. In view of Proposition 2.18 we can assume that $\mathcal{P}$ is closed under finite suprema and that $U_{\mathcal{Q},\varepsilon} = U_{\{\|\cdot\|_{\sup}\},\varepsilon}$, where $\|\cdot\|_{\sup} := \sup_{\|\cdot\| \in \mathcal{Q}} \|\cdot\|$. Then, for all $\delta \in L^0_{++}$,

$$\|X - X(1+\delta)\|_{\sup} = \delta \|X\|_{\sup}.$$

Thus, choosing δ such that $\delta \|X\|_{\sup} \leq \varepsilon$, we derive $X(1+\delta) \in K$ and hence $p_K(X) \leq 1/(1+\delta) < 1$. $\square$

2.5 Hyperplane separation

Let E be a locally L^0–convex module.

Let $X \in E$ be such that there is an L^0–linear bijection $\mu : \operatorname{span}_{L^0}(X) \to L^0$. Then, necessarily

$$\mu(YX) = Y \mu X \text{ for all } Y \in L^0 \tag{2.25}$$

and $\mu^{-1} : L^0 \to \operatorname{span}_{L^0}(X)$ is L^0–linear as well. Since μ is a surjection we derive from (2.25) that $P[\mu X \neq 0] = 1$. Further,

$$Y = \mu(\mu^{-1}(Y)) = \mu(\bar{Y} X) = \bar{Y} \mu X$$

for all $Y \in L^0$. Hence, $\bar{Y} = Y/\mu X$ and in turn $\mu^{-1}(Y) = YX/\mu X$. On replacing μ by $\mu/(\mu X)$, we can always assume that $\mu X = 1$. In this case, $\mu(YX) = Y$ and $\mu^{-1} Y = YX$ for all $Y \in L^0$.

Lemma 2.26 *Let $K, M \subset E$ be L^0–convex, K open and non empty. If $1_A M \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$, then there is an L^0–linear function $\mu : E \to L^0$ such that*

$$\mu Y < \mu Z \text{ for all } Y \in K \text{ and } Z \in M. \tag{2.26}$$

Proof. We can assume that M is non empty.

Step 1: Suppose first that $M = \{X\}$ is a singleton.

Without loss of generality, we may assume that $0 \in K$. Indeed, if $0 \notin K$ replace X by $X - Y$ and K by $K - Y$ for some $Y \in K$ which is possible since $K \neq \emptyset$. Note that $\{X - Y\}, K - Y$ remain L^0–convex, that $K - Y$ remains open non empty and that an L^0–linear function $\mu : E \to L^0$ separates $\{X\}$ from K – in the sense of (2.26) – if and only if it separates $\{X - Y\}$ from $K - Y$.

Thus, let K be L^0–convex open non empty and $0 \in K$. (Note that K is L^0–absorbent.) By assumption, $1_A X \notin K$ for all $A \in \mathcal{F}$ with $P[A] > 0$. In particular, $1_A X \neq 0$ for all $A \in \mathcal{F}$ with $P[A] > 0$. Hence, $YX = Y'X$ implies $Y = Y'$ for all $Y, Y' \in L^0$ and $\mu : \mathrm{span}_{L^0}(X) \to L^0$,

$$\mu(YX) := Y \text{ for all } Y \in L^0, \tag{2.27}$$

is a well-defined L^0–linear bijection of $\mathrm{span}_{L^0}(X)$ into L^0. By Proposition 2.23, the gauge function $p_K : E \to L^0$ is L^0–sublinear. We show $p_K(Z) \geq \mu Z$ for all $Z \in \mathrm{span}_{L^0}(X)$. For $Z \in \mathrm{span}_{L^0}(X)$ let $Y \in L^0$ be the unique element with $Z = YX$. From (2.24) in the proof of Proposition 2.23 we derive

$$p_K(YX) = 1_A p_K(1_A YX) + 1_{A^c} p_K(1_{A^c} YX) \tag{2.28}$$

for $A := \{Y \geq 0\}$. Further, with (ii) of Proposition 2.23 and (i) of Proposition 2.25 we know that

$$1_A p_K(1_A YX) = 1_A Y p_K(X) \geq 1_A Y = 1_A \mu(YX) \tag{2.29}$$

and since $p_K \geq 0$

$$1_{A^c} p_K(1_{A^c} YX) \geq 1_{A^c} Y = 1_{A^c} \mu(YX). \tag{2.30}$$

Adding up (2.29) and (2.30), together with (2.28), yield

$$p_K(YX) \geq \mu(YX).$$

Hence, $p_K(Z) \geq \mu Z$ for all $Z \in \mathrm{span}_{L^0}(X)$ and therefore μ extends by the Hahn–Banach Theorem 2.14 to $\mu : E \to L^0$ such that

$$\mu Y \leq p_K(Y) \text{ for all } Y \in E.$$

In particular, for all $Y \in K$

$$\mu Y \leq p_K(Y) < 1 = \mu X,$$

where the strict inequality follows from (ii) of Proposition 2.25 and the equality follows from (2.27).

Step 2: Now let M be as in the lemma. Then, $K - M$ is L^0–convex open non empty and $1_A\{0\} \cap 1_A(K - M) = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. Thus, from the first step of this proof, there is an L^0–linear function $\mu : E \to L^0$ with

$$\mu(Y - Z) < 0 \text{ for all } Y \in K \text{ and } Z \in M$$

and the assertion is proved. □

Lemma 2.27 *Let $K \subset E$ be open L^0–convex with $0 \in K$. If $\mu : E \to L^0$ is L^0–linear such that*

$$\mu(X) \leq p_K(X) \textit{ for all } X \in E$$

then μ is continuous.

Proof. It suffices to show that $\mu^{-1}B_\varepsilon$ is a neighborhood of $0 \in E$ for each ball B_ε centered at $0 \in L^0$. Thus, let $\varepsilon \in L^0_{++}$. The set $U := \varepsilon K \cap -\varepsilon K$ is a neighborhood of $0 \in E$. (Indeed, let $V := U_{\mathcal{Q},\delta} \subset K$, be a neighborhood of $0 \in E$, which exists since K is open and $0 \in K$. Then, $\varepsilon V = U_{\mathcal{Q},\varepsilon\delta}$ is an L^0–balanced neighborhood of $0 \in E$. Further, $\varepsilon V \subset \varepsilon K$, $-\varepsilon V \subset -\varepsilon K$ and since εV is L^0–balanced $\varepsilon V = -\varepsilon V$ and in turn $\varepsilon V \subset \varepsilon K \cap -\varepsilon K$.) Further, for all $X \in U$ we have

$$\begin{aligned} \mu(X) &\leq p_K(X) \leq \varepsilon \text{ and} \\ -\mu(X) = \mu(-X) &\leq p_K(-X) \leq \varepsilon. \end{aligned}$$

Thus, $|\mu(X)| \leq \varepsilon$ and hence $U \subset \mu^{-1}B_\varepsilon$. □

We can now prove Theorem 2.6.

Proof. We can assume that M is non empty. Define $L := K - M$. For $X \in L$, the set $L - X$ is L^0–convex open and $0 \in L - X$. By assumption, $0 \notin 1_A L$ for all $A \in \mathcal{F}$ with $P[A] > 0$ and so $1_A(-X) \notin 1_A(L - X)$. From the first step of the proof of Lemma 2.26 we know that there is an L^0–linear function $\mu : E \to L^0$ such that

$$\mu Y \leq p_{L-X}(Y) \text{ for all } Y \in E.$$

By Lemma 2.27, μ is continuous. Further,

$$\mu Y < \mu(-X) \text{ for all } Y \in L - X.$$

and Theorem 2.6 is proved. □

Lemma 2.28 *Let $\mathcal{P}$ be a family of L^0–semi norms inducing a locally L^0–convex topology on E and let $K \subset E$ be closed with $1_A X + 1_{A^c} X' \in K$ for all $A \in \mathcal{F}$ and $X, X' \in K$. If $\mathcal{P}$ has the countable concatenation property and $X \in E$ satisfies $1_A\{X\} \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$, then there is an L^0–convex, L^0–absorbent and L^0–balanced neighborhood U of $0 \in E$ such that*

$$1_A(X + U) \cap 1_A(K + U) = \emptyset$$

for all $A \in \mathcal{F}$ with $P[A] > 0$.

Proof. We can assume that $K \neq \emptyset$. Via translation by X, it suffices to construct an L^0–convex, L^0–absorbent and L^0–balanced neighborhood U of $0 \in E$ such that

$$1_A U \cap 1_A(K + U) = \emptyset$$

for all $A \in \mathcal{F}$ with $P[A] > 0$.

Step 1: In this step we construct an L^0–convex, L^0–absorbent, L^0–balanced neighborhood U of $0 \in E$ such that $1_A U \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. To this end, define

$$\varepsilon^* := 1 \wedge \operatorname*{ess.sup}_{\mathcal{Q} \subset \mathcal{P} \text{ finite}} \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\mathcal{Q},\varepsilon} \cap K \neq \emptyset\}$$

(Note that for all $\mathcal{Q} \subset \mathcal{P}$ finite there is $\varepsilon \in L^0_{++}$ such that $U_{\mathcal{Q},\varepsilon} \cap K \neq \emptyset$ since all neighborhoods of $0 \in E$ are L^0–absorbent.) Successively we show that ε^* satisfies:

(i) $\varepsilon^* \in L^0_{++}$.

(ii) There is an L^0–semi norm $\|\cdot\|^* \in \mathcal{P}$ such that

$$\frac{\varepsilon^*}{2} < \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\{\|\cdot\|^*\},\varepsilon} \cap K \neq \emptyset\}.$$

(iii) $1_A U_{\{\|\cdot\|^*\},\varepsilon^*/2} \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. (Note that $U_{\{\|\cdot\|^*\},\varepsilon^*/2}$ is L^0–convex, L^0–absorbent, L^0–balanced and closed.)

(i) Suppose $P[A] > 0$, $A := \{\varepsilon^* = 0\}$. Then for all $\mathcal{Q} \subset \mathcal{P}$ finite and for all $\alpha \in L^0_{++}$ there is $X_{\mathcal{Q},\alpha} \in K$ such that

$$1_A X_{\mathcal{Q},\alpha} \in U_{\mathcal{Q},1/\alpha} \cap 1_A K.$$

Hence, for $X \in K$ the net $(1_A X_{\mathcal{Q},\alpha} + 1_{A^c} X)$ converges to $1_{A^c} X$ and $1_A X_{\mathcal{Q},\alpha} + 1_{A^c} X \in K$ for all $\mathcal{Q} \subset \mathcal{P}$ finite and for all $\alpha \in L^0_{++}$. Since K is closed, we derive $1_{A^c} X \in K$, which is impossible as it would imply $0 = 1_A 1_{A^c} X \in 1_A K$.

(ii) For all finite $\mathcal{Q} \subset \mathcal{P}$, let

$$\varepsilon_{\mathcal{Q}} := \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\mathcal{Q},\varepsilon} \cap K \neq \emptyset\}.$$

For finite $\mathcal{Q}, \mathcal{Q}' \subset \mathcal{P}$, $U_{\mathcal{Q} \cup \mathcal{Q}',\varepsilon} \subset U_{\mathcal{Q},\varepsilon}, U_{\mathcal{Q}',\varepsilon}$. Thus, the collection $\{\varepsilon_{\mathcal{Q}} \mid \mathcal{Q} \subset \mathcal{P} \text{ finite}\}$ is directed upwards and hence there is an increasing sequence $(\varepsilon_{\mathcal{Q}_n})$ with $1 \wedge \varepsilon_{\mathcal{Q}_n} \nearrow \varepsilon^*$ a.s. Let

$$\begin{aligned} A_1 &:= \{\varepsilon_{\mathcal{Q}_1} > \varepsilon^*/2\}, \\ A_n &:= \{\varepsilon_{\mathcal{Q}_n} > \varepsilon^*/2\} \setminus A_{n-1} \text{ for all } n \geq 2. \end{aligned}$$

Then, $\bigcup_{n \in \mathbb{N}} A_n \nearrow \Omega$ since $\varepsilon^* > \varepsilon^*/2$. Further, the L^0–semi norm

$$\|\cdot\|^* := \sum_{n \in \mathbb{N}} 1_{A_n} \sup_{\|\cdot\| \in \mathcal{Q}_n} \|\cdot\|$$

is an element of $\mathcal{P}$ since $\mathcal{P}$ has the countable concatenation property and $\|\cdot\|^*$ is as required.

(iii) Finally, assume there is $A \in \mathcal{F}$, $P[A] > 0$, and $X \in K$ such that $1_A X \in 1_A U_{\{\|\cdot\|^*\},\varepsilon^*/2}$. Then

$$1_A \text{ess.inf}\{\varepsilon \in L^0_{++} \mid U_{\{\|\cdot\|^*\},\varepsilon} \cap K \neq \emptyset\} \leq 1_A \frac{\varepsilon^*}{2},$$

in contradiction to the statement in (ii).

Step 2: From the first step we have $\|\cdot\| \in \mathcal{P}$ and $\varepsilon \in L^0_{++}$ such that $1_A U_{\{\|\cdot\|\},\varepsilon} \cap 1_A K = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. This implies $1_A U_{\{\|\cdot\|\},\varepsilon/2} \cap 1_A(K + U_{\{\|\cdot\|\},\varepsilon/2}) = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$ and the assertion follows. $\square$

The next example illustrates, that the countable concatenation property, as an assumption on $\mathcal{P}$ in Lemma 2.28, cannot be omitted.

Example 2.29 *Let $(\Omega, \mathcal{F}, P)$, A_n, and the family $\mathcal{P}$ of L^0–semi norms on $E = L^0$ be as in Example 2.19. From Example 2.19 we know that $\mathcal{P}$ does not have the countable concatenation property. We now further derive the following:*

(i) *The set $K := \{X \in E \mid X \geq 1\}$ is closed with respect to the locally L^0–convex topology on E induced by $\mathcal{P}$.*

Indeed, if $X \notin K$ then there is $n \in \mathbb{N}$ such that $0 < 1 - X =: c \in \mathbb{R}$ on A_n. But then $X + U_{\{1_{A_n}|\cdot|\},c/2}$ defines a neighborhood of X which is disjoint of K. Hence K^c is open.

(ii) *$1_A K \cap \{0\} = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$.*

This follows as $1_{A_n} K \cap \{0\} = \emptyset$, for all atoms $A_n, n \in \mathbb{N}$.

(iii) *For every neighborhood U of $0 \in E$ there exists $A \in \mathcal{F}$ with $P[A] > 0$ such that $1_A K \cap U \neq \emptyset$.*

Indeed, for every neighborhood U of $0 \in E$ there is $n \in \mathbb{N}$ and $\varepsilon \in L^0_{++}$ such that $U_{\{1_{B_n}|\cdot|\},\varepsilon} \subset U$. Note that $P[B_n] < 1$. But now, $1_{B_n^c} K \subset 1_{B_n^c} E = 1_{B_n^c} U_{\{1_{B_n}|\cdot|\},\varepsilon} \subset U$.

We can now prove Theorem 2.8.

Proof. Recall we can assume a family $\mathcal{P}$ of L^0–semi norms induces the locally L^0–convex topology on E and that $\mathcal{P}$ inherits the countable concatenation property from E.

By Lemma 2.28, there is an L^0–convex, L^0–absorbent and L^0–balanced neighborhood U of $0 \in E$ such that

$$1_A(X + U) \cap 1_A(K + U) = \emptyset$$

for all $A \in \mathcal{F}$ with $P[A] > 0$. Since $K + \overset{\circ}{U}, X + \overset{\circ}{U}$ are L^0–convex open and $K + \overset{\circ}{U}$ is non empty Theorem 2.6 yields a continuous L^0–linear function $\mu : E \to L^0$ such that

$$\mu Y < \mu Z \text{ for all } Y \in K + \overset{\circ}{U} \text{ and } Z \in X + \overset{\circ}{U}.$$

Further, from the first step of the proof of Lemma 2.26 we know that there is $X_0 \in E$ such that
$$\mu(YX_0) = Y \text{ for all } Y \in L^0.$$
Since $\overset{\circ}{U}$ is L^0–absorbent and L^0–balanced there is $\varepsilon \in L^0_{++}$ such that $-\varepsilon X_0 \in \overset{\circ}{U}$. Thus,
$$\mu Y < \mu(X - \varepsilon X_0) = \mu X - \varepsilon \text{ for all } Y \in K + \overset{\circ}{U}.$$
In particular,
$$\mu Y + \varepsilon < \mu X \text{ for all } Y \in K,$$
whence Theorem 2.8 is proved. □

We provide an example which illustrates that the countable concatenation property, as an assumption on $\mathcal{P}$ in Theorem 2.8, cannot be omitted.

Example 2.30 *Let $(\Omega, \mathcal{F}, P)$, A_n, and the family $\mathcal{P}$ of L^0–semi norms on $E = L^0$ be as in Example 2.29. Then the closed subset $K := \{X \in E \mid X \geq 1\}$ of E cannot be separated from 0 by a continuous L^0–linear function.*

Indeed, as every L^0–linear function $\mu : E \to L^0$ is of the form
$$\mu X = \sum_{n \in \mathbb{N}} 1_{A_n} a_n X \text{ for all } X \in E,$$
for some sequence $(a_n) \subset \mathbb{R}$, we conclude that $a_n > 0$ for all $n \in \mathbb{N}$ if μ separates 0 from K. Such μ, however, is not continuous at 0. To see this, let $Z := \sum_{n \in \mathbb{N}} 1_{A_n} a_n$, $\varepsilon \in L^0_{++}$ and observe that
$$\mu^{-1}\{Y \in L^0 \mid |Y| \leq \varepsilon\} = \{X \in E \mid |X/Z| \leq \varepsilon\}$$
is not a neighborhood of $0 \in E$.

3 Part II. Duality in locally L^0–convex modules

3.1 Main results

We first recall and introduce some terminology. Let E be an L^0–module. The effective domain of a function $f : E \to \bar{L}^0$ is denoted by $\mathrm{dom} f := \{X \in E \mid f(X) \in L^0\}$. The epigraph of f is denoted by $\mathrm{epi} f := \{(X, Y) \in E \times L^0 \mid f(X) \leq Y\}$. The function f is proper if $f(X) > -\infty$ for all $X \in E$ and $\mathrm{dom} f \neq \emptyset$.

Definition 3.1 *Let E be an L^0–module and $f : E \to \bar{L}^0$ a proper function.*

(i) *f is L^0–convex if $f(YX_1 + (1-Y)X_2) \leq Yf(X_1) + (1-Y)f(X_2)$ for all $X_1, X_2 \in E$ and $Y \in L^0$ with $0 \leq Y \leq 1$.*

(ii) *f has the local property if $1_A f(X) = 1_A f(1_A X)$ for all $X \in E$ and $A \in \mathcal{F}$.*

As a first result in this part, we obtain that L^0–convexity enforces the local property. The proof is given in Section 3.3 below.

Theorem 3.2 *Let E be an L^0–module. A proper function $f : E \to \bar{L}^0$ is L^0–convex if and only if f has the local property and* epi$f$ *is L^0–convex.*

We now address some topological properties of L^0–convex functions.

Definition 3.3 *Let E be a topological L^0–module. A function $f : E \to \bar{L}^0$ is lower semi continuous if for all $Y \in L^0$ the level set $\{X \in E \mid f(X) \leq Y\}$ is closed.*

As one expects from the real case, lower semi continuity of an L^0–convex function can also be characterized in terms of its epigraph. In fact, the following result is proved in Section 3.4.

Proposition 3.4 *Let E be a locally L^0–convex module that has the countable concatenation property. A proper function $f : E \to \bar{L}^0$ that has the local property is lower semi continuous if and only if* epi$f$ *is closed.*

A subset B of a topological L^0–module E is an L^0–barrel if it is L^0–convex, L^0–absorbent, L^0–balanced and closed. A locally L^0–convex module E is an L^0–barreled module if every L^0–barrel is a neighborhood of $0 \in E$. It follows by inspection that L^0–normed modules are L^0–barreled. The following result is proved in Section 3.5.

Proposition 3.5 *Let E be an L^0–barreled module. A proper lower semi continuous L^0–convex function $f : E \to \bar{L}^0$ is continuous on* $\overset{\circ}{\mathrm{dom}} f$.

We now turn to our main, Fenchel–Moreau type, duality results. Let E be a topological L^0–module, and denote by $\mathcal{L}(E, L^0)$ the L^0–module of continuous L^0–linear functions $\mu : E \to L^0$. The conjugate $f^* : \mathcal{L}(E, L^0) \to \bar{L}^0$ of a function $f : E \to \bar{L}^0$ is defined by

$$f^*(\mu) := \underset{X \in E}{\text{ess.sup}}(\mu X - f(X)). \tag{3.31}$$

Further, the conjugate $f^{**} : E \to \bar{L}^0$ of f^* is defined by

$$f^{**}(X) := \underset{\mu \in \mathcal{L}(E, L^0)}{\text{ess.sup}}(\mu X - f^*(\mu)). \tag{3.32}$$

Definition 3.6 *Let E be a topological L^0–module. An element μ of $\mathcal{L}(E, L^0)$ is a subgradient of a function $f : E \to \bar{L}^0$ at $X_0 \in$* dom$f$ *if*

$$\mu(X - X_0) \leq f(X) - f(X_0), \; \textit{for all } X \in E.$$

The set of all subgradients of f at X_0 is denoted by $\partial f(X_0)$.

A pre stage of Theorem 3.7 below, which we will prove in Section 3.6, is given in Kutateladze [16, 17, 18]. However, Kutateladze entirely remains within an algebraic scope as he does not address topological aspects such as continuity. More precisely, he provides necessary and sufficient conditions for the existence of algebraic subgradients of L^0–sublinear functions in terms of the underlying ring. Further, Kutateladze only covers the case of L^0–sublinear functions which take values in L^0 adjoint $+\infty$, that is, $L^0 \cup \{+\infty\}$ rather than functions which take values in $\bar{L}^0$.

Theorem 3.7 *Let E be an L^0–barreled module that has the countable concatenation property. Let $f : E \to \bar{L}^0$ be a proper lower semi continuous L^0–convex function. Then,*

$$\partial f(X) \neq \emptyset \text{ for all } X \in \overset{\circ}{\mathrm{dom}} f.$$

Here is the generalized Fenchel–Moreau duality theorem, the proof of which is given in Section 3.7.

Theorem 3.8 *Let E be a locally L^0–convex module that has the countable concatenation property. Let $f : E \to \bar{L}^0$ be a proper lower semi continuous L^0–convex function. Then,*

$$f = f^{**}.$$

3.2 Financial applications

In this section we illustrate the scope of applications that can be covered by our results. The entropic risk measure $\rho_0 : \bar{L}^0 \to [-\infty, +\infty]$ is defined as

$$\rho_0(X) := \log E[\exp(-X)].$$

Its restriction to the locally convex vector space L^p, $p \in [1, +\infty]$, is proper, convex, lower semi continuous. Classical convex analysis yields the dual representation

$$\rho_0(X) = \sup_{Z \in L^q} (E[ZX] - \rho_0^*(Z))$$

with conjugate function

$$\rho_0^*(Z) = \sup_{X \in L^p} (E[ZX] - \rho_0(X)) \quad (= E[-Z \log(-Z)] \text{ if defined and } = +\infty \text{ otherwise})$$

where $q := p/(p-1)$ ($:= +\infty$ if $p = 1$), cf. [8]. For $p = +\infty$, in particular, ρ_0 is continuous and subdifferentiable on $\overset{\circ}{\mathrm{dom}} \rho_0 = L^\infty$ with unique subgradient $-\exp(-X)/E[\exp(-X)]$ at $X \in L^\infty$.

Market models in stochastic finance involve filtrations which represent the flow of information provided by the market. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}}, P)$ be a filtered probability space. We shall write $L^0(\mathcal{F})$, $L^0(\mathcal{F}_t)$, etc. to express the respective reference σ-algebra. The

$[-\infty,+\infty]$–valued entropic risk measure ρ_0 can be made contingent on the information available at t by modifying it to $\rho_t : \bar{L}^0(\mathcal{F}) \to \bar{L}^0(\mathcal{F}_t)$,

$$\rho_t(X) := \log E[\exp(-X) \mid \mathcal{F}_t].$$

As in the deterministic case, subdifferentiability and dual representation of ρ_t are important aspects in risk management applications. For this reason, ρ_t must be restricted to a space which allows for convex analysis.

The restriction ρ_t to bounded risks, that is $L^\infty(\mathcal{F})$, has been analyzed in [2, 5, 6, 9]. It turns out that ρ_t maps $L^\infty(\mathcal{F})$ into $L^\infty(\mathcal{F}_t)$. Convex analysis of ρ_t can then be carried out by means of scalarization, an idea which goes back to [12, 19, 21].

However, $L^\infty(\mathcal{F})$ is a too narrow model space for financial risks. For instance, it does not contain normal distributed random variables. The space $L^p(\mathcal{F})$, for $p \in [1,+\infty)$, is larger and already sufficient for many applications. But ρ_t restricted to $L^p(\mathcal{F})$ takes values in $\bar{L}^0(\mathcal{F}_t)$ and the scalarization method used in the previous literature can no longer be applied.

Exploiting our results, we thus propose to view ρ_t as a function on the $L^0(\mathcal{F}_t)$–module $L^p_{\mathcal{F}_t}(\mathcal{F})$, defined in Example 2.5, which in fact is much larger than $L^p(\mathcal{F})$ and thus even better apt for applications. The function $\rho_t : L^p_{\mathcal{F}_t}(\mathcal{F}) \to \bar{L}^0(\mathcal{F}_t)$ is proper L^0–convex. Fatou's generalized lemma and lemma 3.10 show that ρ_t is lower semi continuous. Moreover, from Theorem 3.8 we know that the following dual representation applies

$$\begin{aligned}\rho_t(X) &= \operatorname*{ess.sup}_{Z \in L^q_{\mathcal{F}_t}(\mathcal{F})} \left(E[ZX \mid \mathcal{F}_t] - \rho_t^*(Z)\right)\\ &= \operatorname*{ess.sup}_{Y \in L^0(\mathcal{F}_t), Z' \in L^q(\mathcal{F})} \left(Y E[Z'X \mid \mathcal{F}_t] - \rho_t^*(YZ')\right).\end{aligned}$$

For time-consistent dynamic risk assessment, compositions of the form $\rho_t \circ (-\rho_{t+1})$ are another important aspect, cf. [5, 9]. For the entropic risk measure we derive in an ad hoc manner that $\rho_t \circ (-\rho_{t+1}) = \rho_t$ on $\bar{L}^0(\mathcal{F})$. Hence, our results immediately apply to the dynamic risk assessment by means of the entropic risk measure. An extension to more general dynamic risk measures and lower semi continuity as well as subdifferentiability aspects of compositions of lower semi continuous functions is subject to future research.

3.3 Proof of Theorem 3.2

To prove the if statement, let $X_1, X_2 \in E$ and $Y \in L^0$, $0 \le Y \le 1$. The inequality

$$f(YX_1 + (1-Y)X_2) \le Yf(X_1) + (1-Y)f(X_2) \tag{3.33}$$

is trivially valid on $\{f(X_1) = +\infty\} \cup \{f(X_2) = +\infty\}$. Since f is proper there is $X \in \mathrm{dom} f$. Since f has the local property

$$\begin{aligned}X_1' &:= 1_{\{f(X_1) < +\infty\}} X_1 + 1_{\{f(X_1) = +\infty\}} X \in \mathrm{dom} f\\ X_2' &:= 1_{\{f(X_2) < +\infty\}} X_2 + 1_{\{f(X_2) = +\infty\}} X \in \mathrm{dom} f.\end{aligned}$$

From L^0–convexity of epi$f$ we derive

$$f(YX_1' + (1-Y)X_2') \le Yf(X_1') + (1-Y)f(X_2'). \tag{3.34}$$

The local property of f together with (3.33) and (3.34) yields

$$f(YX_1 + (1-Y)X_2) \le Yf(X_1) + (1-Y)f(X_2),$$

that is, f is L^0–convex.

To establish the only if statement, observe that epi$f$ is L^0–convex if f is L^0–convex. Thus, it suffices to prove that f has the local property. This, however, follows from the inequalities

$$\begin{aligned} f(1_A X) &= f(1_A X + 1_{A^c} 0) \le 1_A f(X) + 1_{A^c} f(0) \\ &= 1_A f(1_A(1_A X) + 1_{A^c} X) + 1_{A^c} f(0) \\ &\le 1_A f(1_A X) + 1_{A^c} f(0) \end{aligned}$$

which become equalities if multiplied with 1_A.

3.4 Lower semi continuous functions

Lemma 3.9 *Let E be a topological L^0–module. The essential supremum of a family of lower semi continuous functions $f_i : E \to \bar{L}^0$, $i \in I$, I an arbitrary index set, is lower semi continuous.*

Proof. The assertion follows from the identity

$$\{X \mid X \in E \text{ and } \operatorname*{ess.sup}_{i\in I} f_i(X) \le Y\} = \bigcap_{i\in I} \{X \mid X \in E \text{ and } f_i(X) \le Y\}$$

for all $Y \in L^0$. □

The essential limit inferior ess.liminf$_\alpha X_\alpha$ of a net $(X_\alpha) \subset L^0$ is defined by

$$\operatorname*{ess.liminf}_{\alpha} X_\alpha := \operatorname*{ess.sup}_{\alpha} \operatorname*{ess.inf}_{\beta \ge \alpha} X_\beta$$

Lemma 3.10 *Let E be a locally L^0–convex module that has the countable concatenation property. A proper function $f : E \to \bar{L}^0$ that has the local property is lower semi continuous if and only if*

$$\operatorname*{ess.liminf}_{\alpha} f(X_\alpha) \ge f(X) \tag{3.35}$$

for all nets $(X_\alpha) \subset E$ with $X_\alpha \to X$ for some $X \in E$.

Proof. Assume that f has the local property, is lower semi continuous and let $(X_\alpha) \subset E$ be such that $X_\alpha \to X$ for some $X \in E$. Let $Y \in L^0$ be such that $Y < f(X)$ which is possible since f is proper. By lower semi continuity of f, the set $V := \{Z \in E \mid f(Z) \le Y\}$ is closed and by the local property we have $1_A X' + 1_{A^c} X'' \in V$ for all $A \in \mathcal{F}$ and $X', X'' \in V$. Further,

$$1_A X \notin 1_A V$$

for all $A \in \mathcal{F}$ with $P[A] > 0$. By Lemma 2.28 there is a neighborhood U of $0 \in E$ such that $1_A(X+U) \cap 1_A V = \emptyset$ for all $A \in \mathcal{F}$ with $P[A] > 0$. Since $X_\alpha \to X$ there is α_0 such that $X_\beta \in X + U$ for all $\beta \geq \alpha_0$. Due to the local property, $1_A X_\beta \notin 1_A V$ for all $\beta \geq \alpha_0$ and $A \in \mathcal{F}$ with $P[A] > 0$. Hence, $f(X_\beta) > Y$ for all $\beta \geq \alpha_0$ and in turn

$$\begin{aligned} \operatorname*{ess.liminf}_{\alpha} f(X_\alpha) &= \operatorname*{ess.sup}_{\alpha} \operatorname*{ess.inf}_{\beta \geq \alpha} f(X_\beta) \\ &\geq \operatorname*{ess.inf}_{\beta \geq \alpha_0} f(X_\beta) \geq Y. \end{aligned}$$

Since Y was arbitrary, we deduce (3.35).

Now assume (3.35) and let $Y \in L^0$. We have to show that the set

$$V := \{Z \in E \mid f(Z) \leq Y\}$$

is closed. To this end, let $(X_\alpha) \subset V$ and $X \in E$ with $X_\alpha \to X$ for some $X \in E$. Then, from the inequality $f(X_\alpha) \leq Y$ for each α, we obtain

$$f(X) \leq \operatorname*{ess.liminf}_{\alpha} f(X_\alpha) \leq Y,$$

so $X \in V$. That is, V is closed, and hence f is lower semi continuous. □

Next, we prove Proposition 3.4.

Proof. Define $\phi : E \times L^0 \to \bar{L}^0$ by

$$\phi(X, Y) := f(X) - Y.$$

From Lemma 3.10 and the definition of the product topology we derive that lower semi continuity of f on E is equivalent to lower semi continuity of ϕ on $E \times L^0$. For all $Z \in L^0$ we have

$$\{(X, Y) \in E \times L^0 \mid \phi(X, Y) \leq Z\} = \text{epi} f - (0, Z).$$

Since $E \times L^0$ is a topological L^0–module we derive that $\{(X, Y) \in E \times L^0 \mid \phi(X, Y) \leq Z\}$ is closed if and only if epi$f$ is closed. This proves Proposition 3.4. □

3.5 Lower semi continuous L^0–convex functions

Lemma 3.11 *Let E be a topological L^0–module. If in the neighborhood of $X_0 \in E$ a proper L^0–convex function $f : E \to \bar{L}^0$ is bounded above by $Y_0 \in L^0$ then f is continuous at X_0.*

Proof. On replacing f by $f(\cdot + X_0) - f(X_0)$, we assume that $X_0 = f(X_0) = 0$. Let $\delta \in L^0_{++}$ and $f(X) \leq Y_0$ for all X in a neighborhood V of $0 \in E$. We have to show that there is a neighborhood W_δ of $0 \in E$ such that $|f(X)| \leq \delta$ for all $X \in W_\delta$.

Without loss of generality we can assume that Y_0 is such that $\varepsilon := \delta / Y_0 > 0$ is well defined and $\varepsilon < 1$. Since E is a topological L^0–module $W := V \cap -V$ is a symmetric ($W = -W$) neighborhood of $0 \in E$. We will show that the neighborhood $W_\delta := \varepsilon W$ is as required. Indeed, for all $X \in \varepsilon W$ we have $\pm X/\varepsilon \in V$ and hence L^0–convexity of f implies

$$\begin{aligned} f(X) &\leq (1-\varepsilon) f(0) + \varepsilon f(X/\varepsilon) \leq \varepsilon Y_0 = \delta \\ \text{and } f(X) &\geq (1+\varepsilon) f(0) - \varepsilon f(-X/\varepsilon) \geq -\varepsilon Y_0 = -\delta. \end{aligned}$$

Thus, $|f(X)| \leq \delta$ for all $X \in W_\delta$, whence the required continuity follows. □

Proposition 3.12 *Let E be a topological L^0–module. Let $f : E \to \bar{L}^0$ be a proper L^0–convex function. The following statements are equivalent:*

(i) *There is a non empty open set $O \subset E$ on which f is bounded above by $Y_0 \in L^0$.*

(ii) *f is continuous on $\overset{\circ}{\mathrm{dom}} f$ and $\overset{\circ}{\mathrm{dom}} f \neq \emptyset$.*

Proof. (ii) implies (i) since for every $X_0 \in \overset{\circ}{\mathrm{dom}} f$ and for every $\delta \in L^0_{++}(\mathcal{F})$ there is a neighborhood V of X_0 such that $f(X_0) - \delta \leq f(X) \leq f(X_0) + \delta$ for all $X \in V$. $O := \overset{\circ}{V}$ and $Y_0 := f(X_0) + \delta$ are then as required.

Conversely, let O and Y_0 be as in (i) and take $X_0 \in O$. Then, $X_0 \in \overset{\circ}{\mathrm{dom}} f$, whence $\overset{\circ}{\mathrm{dom}} f \neq \emptyset$. To see that f is continuous on $\overset{\circ}{\mathrm{dom}} f$, let $X_1 \in \overset{\circ}{\mathrm{dom}} f$. Observe that there is $Y_1 \in L^0_{++}$, $Y_1 > 1$, such that $X_2 := X_0 + Y_1(X_1 - X_0) \in \overset{\circ}{\mathrm{dom}} f$. Since E is a topological L^0–module the map $H : E \to E$ given by

$$H(X) := X_2 - \frac{Y_1 - 1}{Y_1}(X_2 - X) \text{ for all } X \in E$$

is continuous and has continuous inverse H^{-1}. As H transforms X_0 into X_1, it transforms O into an open set $H(O)$ containing X_1. By L^0–convexity of f, we have for all $X \in H(O)$

$$\begin{aligned} f(X) &= f\left(\frac{Y_1 - 1}{Y_1} H^{-1}(X) + \frac{1}{Y_1} X_2\right) \\ &\leq \frac{Y_1 - 1}{Y_1} f(H^{-1}(X)) + \frac{1}{Y_1} f(X_2) \\ &\leq \frac{Y_1 - 1}{Y_1} Y_0 + \frac{1}{Y_1} f(X_2). \end{aligned}$$

In other words, for every $X_1 \in \overset{\circ}{\mathrm{dom}} f$ there is a neighborhood of X_1 on which f is bounded above by an element of L^0. By Lemma 3.11, f is continuous at X_1. □

Corollary 3.13 *Let E be a topological L^0–module and $X \in E$. Every proper L^0–convex function $f : \mathrm{span}_{L^0}(X) \to \bar{L}^0$ is continuous (with respect to the trace topology) on* $\overset{\circ}{\mathrm{dom}} f$.

Proof. Without loss of generality we assume that $0 \in \overset{\circ}{\mathrm{dom}} f$, else translate. Then there is a neighborhood U of $0 \in \mathrm{span}_{L^0}(X)$ and $Y \in L^0_{++}$ such that $\widetilde{X} := YX \in U \subset \overset{\circ}{\mathrm{dom}} f$. From L^0–convexity it follows that f is bounded above by $\sup(f(0), f(\widetilde{X}))$ on the open set

$$\{\lambda \widetilde{X} \mid 0 < \lambda < 1, \lambda \in L^0\}$$

and hence, by Proposition 3.12, f is continuous on $\overset{\circ}{\mathrm{dom}} f$. □

We can now prove Proposition 3.5.

Proof. Assume that there is $X_0 \in \overset{\circ}{\mathrm{dom}} f$. By translation, we may assume $X_0 = 0$. Take $Y_0 \in L^0$ such that $f(0) < Y_0$. By assumption, the level set $C := \{X \in E \mid f(X) \leq Y_0\}$ is closed. Further, for all $X \in E$ the net $(X/Y)_{Y \in L^0_{++}}$ converges to $0 \in E$. By Corollary 3.13, the restriction of f to $\mathrm{span}_{L^0}(X)$ is continuous at 0, hence $f(X/Y) < Y_0$ for large Y which implies that C is L^0–absorbent. Hence, $C \cap -C$ is an L^0–barrel and in turn a neighborhood of $0 \in E$. Thus, C is a neighborhood of $0 \in E$ and since f is bounded above by Y_0 on all of C it is continuous at 0. This proves Proposition 3.5. □

3.6 Subdifferentiability

Let E be a topological L^0–module. Recall the Definitions (3.31) and (3.32) of the conjugates f^* and f^{**} of a function $f : E \to \bar{L}^0$ and f^*, respectively. The effective domain of f^* is given by the set

$$\left\{ \mu \in \mathcal{L}(E, L^0) \mid \exists Y \in L^0 : \operatorname*{ess.sup}_{X \in E}(\mu X - f(X)) \leq Y \right\}.$$

If f is proper, then f^* maps its effective domain into L^0 and f^* is L^0–convex if f is so. The effective domain of f^{**} is given by the set

$$\left\{ X \in E \mid \exists Y \in L^0 : \operatorname*{ess.sup}_{\mu \in \mathcal{L}(E, L^0)} (\mu X - f^*(\mu)) \leq Y \right\}.$$

Again, if f^* is proper f^{**} maps its effective domain into L^0 and f^{**} is L^0–convex if f^* is so. Since for all $X \in E$ and $\mu \in \mathcal{L}(E, L^0)$,

$$f^*(\mu) \geq \mu X - f(X) \tag{3.36}$$

we have for all $X \in E$

$$f(X) \geq f^{**}(X). \tag{3.37}$$

For $\mu \in \mathcal{L}(E, L^0)$ and $X_0 \in \mathrm{dom} f$ we have

$$\mu \in \partial f(X_0) \text{ if and only if } f(X_0) = \mu X_0 - f^*(\mu). \tag{3.38}$$

Indeed, $\mu \in \partial f(X_0)$ by definition means

$$f(X_0) \leq \mu X_0 - (\mu X - f(X)) \text{ for all } X \in E.$$

This is equivalent to

$$f(X_0) \leq \mu X_0 - \underset{X \in E}{\text{ess.sup}}(\mu X - f(X)) = \mu X_0 - f^*(\mu)$$

which, by (3.36), is equivalent to $f(X_0) = \mu X_0 - f^*(\mu)$.

With (3.37) and (3.38) we know that $\mu \in \partial f(X_0)$ maximizes (3.32) at X_0, i.e.

$$f^{**}(X_0) = \mu X_0 - f^*(\mu).$$

Lemma 3.14 *Let E be an L^0–barreled module that has the countable concatenation property. Let $f : E \to \bar{L}^0$ be a proper lower semi continuous function that has the local property. Equivalent are:*

(i) $\overset{\circ}{\mathrm{dom}} f \neq \emptyset$.

(ii) $\overset{\circ}{\mathrm{epi}} f \neq \emptyset$.

Further, for all $X \in \mathrm{dom} f$, $(X, f(X)) \in \partial \mathrm{epi} f$ and $1_A(X, f(X)) \notin 1_A \overset{\circ}{\mathrm{epi}} f$ for all $A \in \mathcal{F}$ with $P[A] > 0$.

Proof. To prove that (i) implies (ii), let $\varepsilon \in L^0_{++}$ and $X \in \overset{\circ}{\mathrm{dom}} f$. We claim

$$(X, f(X) + \varepsilon) \in \overset{\circ}{\mathrm{epi}} f. \tag{3.39}$$

To verify this, we show that there is a neighborhood U of $(X, f(X)+\varepsilon)$ such that $U \subset \mathrm{epi} f$. By Proposition 3.5, f is continuous at X. Hence, there is a neighborhood U_E of X such that

$$f(X) + \varepsilon/3 \geq f(X') \text{ for all } X' \in U_E.$$

This implies

$$(X, f(X) + \varepsilon) \in U_E \times U_{L^0} \subset \mathrm{epi} f,$$

where

$$U_{L^0} := \{Y \in L^0 \mid |f(X) + \varepsilon - Y| \leq \varepsilon/3\}.$$

$U := U_E \times U_{L^0}$ is as required and (3.39) is proved.

Conversely, to prove that (ii) implies (i), let $(X, Y) \in \overset{\circ}{\text{epi}} f$. Then there are neighborhoods U_E and U_{L^0} of X and Y respectively such that $U := U_E \times U_{L^0} \subset \text{epi} f$. In particular, $f(X') < +\infty$ for all $X' \in U_E$ and hence $X \in \overset{\circ}{\text{dom}} f$.

Next, let $X \in \text{dom} f$. To prove $(X, f(X)) \in \partial \text{epi} f$ we show that every $U \subset E \times L^0$ of the form

$$U := U_E \times \{Y \in L^0 \mid |f(X) - Y| \leq \varepsilon\},$$

$U_E \subset E$ a neighborhood of X, satisfies

$$U \cap \text{epi} f \neq \emptyset \neq U \cap \text{epi} f^c.$$

Observe $(X, f(X) - \varepsilon/2), (X, f(X) + \varepsilon/2) \in U$ and $(X, f(X) - \varepsilon/2) \notin \text{epi} f$ and $(X, f(X) + \varepsilon/2) \in \text{epi} f$, which proves $(X, f(X)) \in \partial \text{epi} f$. For fixed $A \in \mathcal{F}$ with $P[A] > 0$, we show in a similar way that $1_A(X, f(X)) \notin 1_A \overset{\circ}{\text{epi}} f$. Observe that every $U \subset E \times L^0$ of the form

$$U := U_E \times \{Y \in L^0 \mid |1_A f(X) - Y| \leq \varepsilon\},$$

$U_E \subset E$ a neighborhood of $1_A X$, satisfies

$$U \cap \text{epi} f^c \neq \emptyset.$$

Indeed, $1_A(X, f(X) - \varepsilon/2) \in U$ and yet $1_A(X, f(X) - \varepsilon/2) \notin 1_A \text{epi} f$ by the local property of f. This proves $1_A(X, f(X)) \notin 1_A \overset{\circ}{\text{epi}} f$. □

Next, we prove Theorem 3.7.

Proof. Let $X_0 \in \overset{\circ}{\text{dom}} f$. We separate $(X_0, f(X_0))$ from $\overset{\circ}{\text{epi}} f$ by means of Theorem 2.6. By Lemma 3.14, $\overset{\circ}{\text{epi}} f$ is non empty, $(X_0, f(X_0)) \in \partial \text{epi} f$ and

$$1_A\{(X_0, f(X_0))\} \cap 1_A \overset{\circ}{\text{epi}} f = \emptyset \text{ for all } A \in \mathcal{F} \text{ with } P[A] > 0.$$

Hence, there are continuous L^0–linear functions $\mu_1 : E \to L^0$ and $\mu_2 : L^0 \to L^0$ such that

$$\mu_1 X + \mu_2 Y < \mu_1 X_0 + \mu_2 f(X_0) \text{ for all } (X, Y) \in \overset{\circ}{\text{epi}} f. \tag{3.40}$$

From (3.40) together with the fact that $\mu_2 Y = Y \mu_2 1$ for all $Y \in L^0$ we derive that $\mu_2 1 < 0$. We will show that $-\mu_1/\mu_2 1 \in \partial f(X_0)$. To this end, let $X \in E$, $A := \{f(X) = +\infty\}$ and $\widetilde{X} := 1_A X_0 + 1_{A^c} X$. Then, $\widetilde{X} \in \text{dom} f$ and in turn $(\widetilde{X}, f(\widetilde{X})) \in \partial \text{epi} f$. Thus, there is a net $(X_{\mathcal{R},\alpha}, Y_{\mathcal{R},\alpha}) \subset \overset{\circ}{\text{epi}} f$ which converges to $(\widetilde{X}, f(\widetilde{X}))$ and for which

$$\mu_1 X_{\mathcal{R},\alpha} + Y_{\mathcal{R},\alpha} \mu_2 1 < \mu_1 X_0 + \mu_2 f(X_0) \text{ for all } \mathcal{R}, \alpha. \tag{3.41}$$

Since μ_1 is continuous we may pass to limits in (3.41) yielding

$$\frac{-\mu_1(\widetilde{X} - X_0)}{\mu_2 1} \leq f(\widetilde{X}) - f(X_0).$$

Finally, from the local property of f and μ_1 we derive

$$\frac{-\mu_1(X - X_0)}{\mu_2 1} \leq f(X) - f(X_0)$$

and since $X \in E$ was arbitrary we conclude that $-\mu_1/\mu_2 1$ indeed is a subgradient of f at X_0. This proves Theorem 3.7. □

3.7 Proof of the Fenchel–Moreau duality theorem 3.8

In this section, we prove Theorem 3.8. The proof follows a known pattern, cf. Proposition A.6 in [10]; however, it contains certain subtleties due to our L^0–convex framework.

We fix $X_0 \in E$, and proceed in two steps.

Step 1: Let $\beta \in L^0$ with $\beta < f(X_0)$. In this step, we show there is a continuous function $h : E \to L^0$ of the form

$$h(X) = \mu X + Z, \tag{3.42}$$

where $\mu : E \to L^0$ is continuous L^0–linear and $Z \in L^0$, such that $h(X_0) = \beta$ and $h(X) \leq f(X)$ for all $X \in E$. To this end, we separate (X_0, β) from epi$f$ by means of Theorem 2.8. It applies since $\beta < f(X_0)$ and the local property of f imply

$$1_A\{(X_0, \beta)\} \cap 1_A \mathrm{epi} f = \emptyset \text{ for all } A \in \mathcal{F} \text{ with } P[A] > 0.$$

(Note, epi$f$ is closed by Proposition 3.4.) Hence, there are continuous L^0–linear functions $\mu_1 : E \to L^0$ and $\mu_2 : L^0 \to L^0$ such that

$$\delta := \operatorname*{ess.sup}_{(X,Y)\in \mathrm{epi} f} \mu_1 X + \mu_2 Y < \mu_1 X_0 + \mu_2 \beta. \tag{3.43}$$

This has two consequences:

(i) $\mu_2 1 \leq 0$.

Indeed, $\mu_2 Y = Y \mu_2 1$ for all $Y \in L^0$. Further, $(X, Y) \in \mathrm{epi} f$ for arbitrarily large Y as long as $f(X) \leq Y$. Hence, for large $Y \in L^0$, $\mu_1 X + \mu_2 Y$ is large on $\{\mu_2 1 > 0\}$ and yet bounded above by $\mu_1 X_0 + \mu_2 \beta$. This implies $P[\mu_2 1 > 0] = 0$.

(ii) $\{f(X_0) < +\infty\} \subset \{\mu_2 1 < 0\}$.

Indeed, define $\widetilde{X}_0 := 1_{\{f(X_0)<+\infty\}} X_0 + 1_{\{f(X_0)=+\infty\}} X$ for some $X \in \mathrm{dom} f$. (f is proper by assumption.) By L^0–convexity of f, $\widetilde{X}_0 \in \mathrm{dom} f$. Local property of f and (3.43) imply on $\{f(X_0) < +\infty\}$

$$\mu_1 X_0 + \mu_2 f(X_0) = \mu_1 \widetilde{X}_0 + \mu_2 f(\widetilde{X}_0) < \mu_1 X_0 + \mu_2 \beta.$$

Hence, $f(X_0)\mu_2 1 = \mu_2 f(X_0) < \mu_2 \beta = \beta \mu_2 1$ on $\{f(X_0) < +\infty\}$ and so $\mu_2 1 < 0$ on $\{f(X_0) < +\infty\}$.

We distinguish the cases $X_0 \in \mathrm{dom} f$ and $X_0 \notin \mathrm{dom} f$.

*C*ase 1. Assume $X_0 \in \mathrm{dom} f$. By (ii), $\mu_2 1 < 0$. Thus, define h by

$$h(X) := -\frac{\mu_1(X - X_0)}{\mu_2 1} + \beta \text{ for all } X \in E$$

which is as required. Indeed, $h(X) \leq f(X)$ for all $X \in \mathrm{dom} f$ as a consequence of (3.43). If $X \notin \mathrm{dom} f$ we have

$$1_B h(X) = 1_B h(X') \leq 1_B f(X') = 1_B f(X), \tag{3.44}$$

where $X' = 1_B X + 1_{B^c} X''$ for some $X'' \in \mathrm{dom} f$ and $B = \{f(X) < +\infty\}$. Hence, $h(X) \leq f(X)$ for all $X \in E$.

*C*ase 2. Assume $X_0 \notin \mathrm{dom} f$. Then chose any $X_0' \in \mathrm{dom} f$ and let h' be the corresponding L^0–affine minorant as constructed in case 1 above. Define $A_1 := \{\mu_2 1 < 0\}$, $A_2 := A_1^c$ and $h_1, h_2 : E \to L^0$,

$$\begin{aligned} h_1(X) :&= 1_{A_1}\left(-\frac{\mu_1(X - X_0)}{\mu_2 1} + \beta\right), \\ h_2(X) :&= \begin{cases} 1_{A_2}\left(h'(X) + \beta - h'(X_0)\right) & \text{on } \{h'(X_0) \geq \beta\} \\ 1_{A_2}\left(h'(X) + \frac{\beta - h'(X_0)}{\widetilde{h}(X_0)}\widetilde{h}(X)\right) & \text{on } \{h'(X_0) < \beta\} \end{cases}, \end{aligned}$$

with the convention $0/0 := 0$, where $\widetilde{h} : E \to L^0$,

$$\widetilde{h}(X) := \delta - \mu_1 X.$$

Note that on $\{\mu_2 1 = 0\}$, $\widetilde{h}(X_0) < 0$ and $\widetilde{h}(X) \geq 0$ for all $X \in \mathrm{dom} f$. It follows that

$$h := h_1 + h_2$$

is as required. (As in (3.44) we see $h(X) \leq f(X)$ for all $X \in E$.)

Step 2: Recall $f \geq f^{**}$, cf. (3.37). By way of contradiction, assume $f(X_0) > f^{**}(X_0)$ on a set of positive measure. Then there is $\beta \in L^0$ with $\beta > f^{**}(X_0)$ on a set of positive measure and $\beta < f(X_0)$. The first step of this proof yields $h : E \to L^0$,

$$h(X) = \mu X + Z \text{ for all } X \in E,$$

for continuous L^0–linear $\mu : E \to L^0$ and $Z \in L^0$, such that $h(X_0) = \beta$ and $h(X) \leq f(X)$ for all $X \in E$. We derive a contradiction as

$$\begin{aligned} f^{**}(X_0) &\geq \mu X_0 - f^*(\mu) \\ &= \mu X_0 - \operatorname*{ess.sup}_{X \in E}(\mu X - f(X)) \\ &\geq \mu X_0 - \operatorname*{ess.sup}_{X \in E}(\mu X - h(X)) = \beta \end{aligned}$$

negates $\beta > f^{**}(X_0)$ on a set of positive measure. This finishes the proof of Theorem 3.8.

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