

Monotone and Cash-Invariant Convex Functions and Hulls*

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Abstract

This paper provides some useful results for convex risk measures. In fact, we consider convex functions on a locally convex vector space E which are monotone with respect to the preference relation implied by some convex cone and invariant with respect to some numeraire (“cash”). As a main result, for any function f , we find the greatest closed convex monotone and cash-invariant function majorized by f . We then apply our results to some well-known risk measures and problems arising in connection with insurance regulation.

Key words: constrained risk measures, convex duality, infimal convolution, insurance regulation, monotone and cash-invariant functions and hulls.

1 Introduction

It has become a standard in modern risk management to assess the riskiness of a portfolio by means of convex risk measures (see for instance [2, 15, 16, 17] and the references therein). Formally, a *convex risk measure* is a convex function $\rho : L^p \rightarrow (-\infty, \infty]$ which is

- *monotone*: $\rho(X) \leq \rho(Y)$ for $X \geq Y$, and

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- *cash-invariant*: $\rho(X + c) = \rho(X) - c$ for all $c \in \mathbb{R}$.

Here, and in what follows, we write $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ with corresponding norm $\|\cdot\|_p$ for some reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $p \in [1, \infty]$. By convention, $X \geq Y$ means $X \geq Y$ a.s.

Now suppose an insurance company uses ρ to assess the riskiness of its portfolio, where the risk measure ρ is specified by the regulator. The objective of the company is then to minimize $\rho(X)$ by admissible modifications of its risk profile X . The regulator may enforce the insurance company to assume positions $X \in M$ only, where $M \subset L^p$ is some closed convex and cash-invariant ($M + c = M$ for $c \in \mathbb{R}$) set. The optimization task of the company's risk management is thus to minimize $\rho^M(X)$, where $\rho^M := \rho$ on M and $\rho^M := \infty$ outside of M . This modified risk measure ρ^M is still convex and cash-invariant, but it lacks monotonicity in general.

This paper provides an extended analysis of monotone and cash-invariant functions, which will be useful for tackling the afore mentioned constrained optimization problem. Often, M is some affine space $\{X_0 + \sum_{j \geq 0} x^j Z_j \mid x^j \in \mathbb{R}\}$ with spanning elements $X_0, Z_j \in L^p$, such that changing the coordinates may be appropriate. That is, considering the convex function $f(x) := \rho(X_0 + \sum_{j \geq 0} x^j Z_j)$ on $\mathbb{R}^{\mathbb{N}}$. But then monotonicity and cash-invariance have to be translated accordingly for f .

Throughout the vast financial and insurance literature there is a variety of risk measures – or premium principles – in use, see e.g. [8, 9, 16, 18, 22]. However, many of these traditional risk measures f fail either to be monotone or cash-invariant. In our paper we provide the monotone and cash-invariant hulls which are the greatest convex functions with these properties majorized by f . We then argue below that, from a regulatory point of view, the insurance company is allowed to replace the original risk measure by its monotone hull, which usually yields lower capital requirements.

In [12], optimal diversification between business units under constraints is studied. The business units are only allowed to take positions in closed convex and cash-invariant sets. Mathematically, the problem is treated by considering a constrained version of the infimal convolution of the risk measures of the business units. We show here that this constrained convolution is a monotone hull of the classical infimal convolution.

As for the mathematical analysis, it turns out that the underlying structures become most clear when we replace L^p by some locally convex vector space E . Indeed, the theory of convex functions has been well established on such spaces and often one wants to extend the risk model framework beyond L^p anyway.

In the context of optimal risk exchange, Pareto optimality and optimal allocation problems, the infimal convolution has been developed and explored in [3, 4, 7, 8, 9, 12, 19, 20, 21]. Fair prices for optimal risk exchange based on economic equilibrium theory have been studied in [5, 6, 9, 11, 12]. Most of these articles originally deal with an unconstrained setup and can be extended to admit constraints. We expect that our results can then be applied.

The remainder of the paper is as follows. In Section 2 we provide the ingredients and formal setup for the study. Section 3 introduces and discusses the extended monotonicity (“ \mathcal{P} -monotonicity”) and cash-invariance (“ Π -invariance”) properties for convex functions on E . In particular, we find the most conservative \mathcal{P} -monotone (and Π -invariant) closed convex risk measure on E . A subsection deals with the case where E is a normed vector space, and we derive a well-known continuity and representation results for convex risk measures on L^∞ . We finish this section with a first example of a constrained risk measure, mentioned above. In Section 4 we define the \mathcal{P} -monotone (and Π -invariant) hull of a convex function f , and we show that its closure is the greatest closed convex \mathcal{P} -monotone (and Π -invariant) function majorized by f . This is then applied to the infimal convolution of convex functions. Section 5 contains a selection of important examples. We find the monotone and cash-invariant hulls of some traditional risk measures. In Section 6 we finally provide an economic interpretation of our results with regard to insurance regulation, which ties up with the discussion started in this introduction. It turns out that any risk measure can be replaced by its \mathcal{P} -monotone hull to determine the regulatory capital charge. In Section A we recall some basic definitions and results from convex analysis. For the sake of readability, we postponed most of the proofs to the appendix.

2 Ingredients

In accordance with the introduction, we fix the following ingredients:

- A Hausdorff locally convex topological vector space E with topological dual E^* .

Facts: The weak topologies $\sigma(E, E^*)$ and $\sigma(E^*, E)$ on E and E^* , respectively, are both Hausdorff and locally convex (see e.g. Section 5.14 in [1]).

Example 1: $E = L^p$, $p \in [1, \infty]$, with norm topology. For $p < \infty$, we have $(L^p)^* = L^{p/(p-1)}$, but notice that the inclusion $L^1 \subset (L^\infty)^*$ is

strict in general.

Example 2: $E = L^\infty$ with topology $\sigma(L^\infty, L^1)$. Now the dual pairing is $(L^\infty, \sigma(L^\infty, L^1))^* = (L^1, \sigma(L^1, L^\infty))$.

- A $\sigma(E, E^*)$ -closed convex cone $\mathcal{P} \subset E$ inducing a *preference relation* on E :

$$X \geq_{\mathcal{P}} Y \quad :\Leftrightarrow \quad X - Y \in \mathcal{P}.$$

Facts: The *Bipolar Theorem* (e.g. Theorem 5.91 in [1]) states that

$$X \geq_{\mathcal{P}} Y \quad \Leftrightarrow \quad \langle \mu, X - Y \rangle \leq 0 \quad \forall \mu \in \mathcal{P}^\circ, \quad (1)$$

where

$$\mathcal{P}^\circ := \{\mu \in E^* \mid \langle \mu, Z \rangle \leq 0 \quad \forall Z \in \mathcal{P}\}$$

is the $\sigma(E^*, E)$ -closed convex *polar cone* of \mathcal{P} (see Lemma 5.90 in [1]). Hence $\mu \notin \mathcal{P}^\circ$ if and only if there exists $X \geq_{\mathcal{P}} 0$ with $\langle \mu, X \rangle > 0$.

It follows by inspection that the relation $\geq_{\mathcal{P}}$ is *reflexive* ($X \geq_{\mathcal{P}} X$) and *transitive* ($X \geq_{\mathcal{P}} Y$ and $Y \geq_{\mathcal{P}} Z$ imply $X \geq_{\mathcal{P}} Z$). But notice that $\geq_{\mathcal{P}}$ is *antisymmetric* ($X \geq_{\mathcal{P}} Y$ and $Y \geq_{\mathcal{P}} X$ imply $X = Y$) if and only if \mathcal{P}° separates points in E .

Example 1: $E = L^p$, for $p \in [1, \infty]$, and $X \geq_{\mathcal{P}} Y$ if $X \geq Y$. Then $\mathcal{P} = L^p_+$ and $\mathcal{P}^\circ = (L^p)_-^* := \{\mu \in (L^p)^* \mid \langle \mu, X \rangle \leq 0 \quad \forall X \geq 0\}$.

Example 2: $E = L^\infty$ with topology $\sigma(L^\infty, L^1)$. Then $\mathcal{P} = L^\infty_+$, as above, and $\mathcal{P}^\circ = L^1_-$.

- A *numeraire* $\Pi \in E \setminus \{0\}$.

Example: $E = L^p$ and $\Pi = 1$.

For the convenience of the reader, we have collected some basic definitions and facts from convex analysis that will be used throughout the text in Section A below. The standard reference is Rockafellar [25] and Ekeland and Témam [10].

3 \mathcal{P} -Monotone and Π -Invariant Functions

The following concepts extend the monotonicity and cash-invariance property of convex risk measures.

Definition 3.1. A function $f : E \rightarrow [-\infty, +\infty]$ is called

- (i) \mathcal{P} -monotone if $f(X) \leq f(Y)$ for all $X \geq_{\mathcal{P}} Y$;
- (ii) Π -invariant if $f(X + c\Pi) = f(X) - c$ for all $c \in \mathbb{R}$ and $X \in E$.

If $E = L^p$ with the usual order $\mathcal{P} = L^p_+$ and $\Pi = 1$, then we say monotone and cash-invariant, respectively. This convention is in line with the notion of a convex risk measure defined in Section 1 above.

For brevity and further use we define the closed convex set

$$\mathcal{D} := \{\mu \in E^* \mid \langle \mu, \Pi \rangle = -1\}. \quad (2)$$

\mathcal{P} -monotonicity and Π -invariance can be characterized in terms of the effective domain of the conjugate function (see also [17]). The proof is postponed to Section B.

Lemma 3.2. *A function $f : E \rightarrow [-\infty, +\infty]$ with $f \not\equiv +\infty$ is*

- (i) \mathcal{P} -monotone only if $\text{dom}(f^*) \subset \mathcal{P}^\circ$ and $\partial f(X) \subset \mathcal{P}^\circ \forall X \in \text{dom}(-f)$;
- (ii) Π -invariant only if $\text{dom}(f^*) \subset \mathcal{D}$ and $\partial f(X) \subset \mathcal{D} \forall X \in \text{dom}(-f)$.

Conversely, a closed convex function $f : E \rightarrow (-\infty, +\infty]$ is

- (iii) \mathcal{P} -monotone if $\text{dom}(f^*) \subset \mathcal{P}^\circ$;
- (iv) Π -invariant if $\text{dom}(f^*) \subset \mathcal{D}$.

As a consequence, we can find the most conservative closed convex \mathcal{P} -monotone and/or Π -invariant functions. The proof is given in Section C.

Lemma 3.3. *The following equalities hold*

$$\delta(\cdot \mid \mathcal{P}) = \delta^*(\cdot \mid \mathcal{P}^\circ) \quad \text{and} \quad \delta^*(\cdot \mid \mathcal{P}) = \delta(\cdot \mid \mathcal{P}^\circ); \quad (3)$$

$$\delta^*(X \mid \mathcal{D}) = \begin{cases} -\lambda, & \text{if } X = \lambda\Pi, \\ +\infty, & \text{else.} \end{cases} \quad (4)$$

Moreover,

- (i) $\delta(\cdot \mid \mathcal{P})$ is the greatest closed convex \mathcal{P} -monotone, and
- (ii) $\delta^*(\cdot \mid \mathcal{D})$ is the greatest closed convex Π -invariant, and
- (iii) $\delta^*(\cdot \mid \mathcal{P}^\circ \cap \mathcal{D})$ is the greatest closed convex \mathcal{P} -monotone Π -invariant function on E that is zero at $X = 0$, respectively.

Example 3.4. For $E = L^p$ with the usual order $\mathcal{P} = L^p_+$ and $\Pi = 1$, the greatest monotone cash-invariant function is

$$\delta^*(X \mid \mathcal{P}^\circ \cap \mathcal{D}) = -\text{ess inf } X.$$

3.1 The Case of a Normed Vector Space

In this section we assume that E is a normed vector space. Then we can give sufficient conditions under which every \mathcal{P} -monotone and Π -invariant function is continuous.

Lemma 3.5. *Assume that*

$$Z \geq_{\mathcal{P}} -\kappa \|Z\|_E \Pi \quad \forall Z \in E. \quad (5)$$

for some finite constant $\kappa > 0$ (in particular, $\Pi \geq_{\mathcal{P}} 0$).

Then every \mathcal{P} -monotone and Π -invariant function $f : E \rightarrow [-\infty, +\infty]$ satisfies: either $f \equiv -\infty$, $f \equiv +\infty$ or f is \mathbb{R} -valued and Lipschitz continuous:

$$|f(X) - f(Y)| \leq \kappa \|X - Y\|_E \quad \forall X, Y \in E. \quad (6)$$

Proof. From (5) we have that

$$f(X) - \kappa \|X - Y\|_E = f(Y + (X - Y) + \kappa \|X - Y\|_E \Pi) \leq f(Y)$$

for all $X, Y \in E$. Hence either $f \equiv -\infty$, $f \equiv +\infty$ or f is \mathbb{R} -valued, and (6) follows. \square

We can improve the statement of Lemma 3.5 and give sufficient conditions in dual terms for (5) to hold. The proof is postponed to Section D

Lemma 3.6. *Assume that*

$$\|\mu\|_{E^*} := \sup_{\|X\|_E=1} \langle \mu, X \rangle \leq -\kappa \langle \mu, \Pi \rangle \quad \forall \mu \in \mathcal{P}^\circ \quad (7)$$

for some $\kappa > 0$. Then $\mathcal{P}^\circ \cap \mathcal{D}$ is $\sigma(E^*, E)$ -compact and (5) holds.

Moreover, every proper convex \mathcal{P} -monotone and Π -invariant function f satisfies

$$f(X) = \max_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} (\langle \mu, X \rangle - f^*(\mu)) \quad \forall X \in E. \quad (8)$$

As a corollary we derive the well-known continuity and representation property of convex risk measures on L^∞ :

Corollary 3.7. *A convex risk measure $\rho : L^\infty \rightarrow (-\infty, \infty]$ is \mathbb{R} -valued and 1-Lipschitz continuous. Moreover*

$$\rho(X) = \max\{\langle \mu, X \rangle - \rho^*(\mu) \mid \mu \in (L^\infty)_-^*, \langle \mu, 1 \rangle = -1\} \quad \forall X \in L^\infty.$$

Proof. This follows from Lemma 3.6 since $\langle \mu, 1 \rangle = -1$ and $|\langle \mu, X \rangle| \leq \|\mu\|_{E^*} \|X\|_\infty$, for all $\mu \in \mathcal{P}^\circ \cap \mathcal{D}$ and $X \in L^\infty$. \square

3.2 Illustration

For a convex risk measure $\rho : L^\infty \rightarrow (-\infty, \infty]$ and some linearly independent random variables X_0 and $Z_0 \equiv 1, Z_1, \dots, Z_n \in L^\infty$, we consider the finite dimensional restriction

$$f(x) := \rho \left(X_0 + \sum_{j=0}^n x^j Z_j \right), \quad x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1},$$

of ρ to the affine space $M = \{X_0 + \sum_{j=0}^n x^j Z_j \mid x^j \in \mathbb{R}\}$. Then $f : E := \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is continuous convex and

(i) \mathcal{P} -monotone, for the closed convex cone

$$\mathcal{P} := \left\{ x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_{j=0}^n x^j Z_j \geq 0 \right\}, \quad \text{and}$$

(ii) Π -invariant, for $\Pi := (1, 0, \dots, 0)$.

Such functions f are studied in [12] in connection with an optimal capital and risk transfer problem for insurance groups.

4 \mathcal{P} -Monotone and Π -Invariant Hulls

This section contains our main results. In what follows, we let $f : E \rightarrow (-\infty, +\infty]$ be some proper function.

Definition 4.1. *The \mathcal{P} -monotone hull of f is defined as*

$$f_{\mathcal{P}}(X) := \inf_{X \geq_{\mathcal{P}} Y} f(Y) = f \square \delta(\cdot \mid \mathcal{P})(X). \quad (9)$$

If f is convex then $f_{\mathcal{P}}$ is convex, but not closed in general, and part (iii) of Lemma 3.2 does not apply. Nevertheless, the terminology for $f_{\mathcal{P}}$ is justified, as our first main result shows. The proof is postponed to Section E.

Theorem 4.2. *$f_{\mathcal{P}}$ is \mathcal{P} -monotone with $f_{\mathcal{P}} \leq f$, and $f_{\mathcal{P}} = f$ if and only if f is \mathcal{P} -monotone. Moreover,*

$$f_{\mathcal{P}}^* = f^* + \delta(\cdot \mid \mathcal{P}^\circ).$$

In particular,

$$f_{\mathcal{P}}^* = f^* \quad \text{on } \mathcal{P}^\circ,$$

and $\text{cl}(f_{\mathcal{P}}) = f_{\mathcal{P}}^{**}$ is the greatest closed convex \mathcal{P} -monotone function majorized by f .

The monotone hull of utility functions has been derived and studied independently in [23] and [24]. We interpret their result for mean-variance preferences in our context in Section 5.3 below.

We can say more if the infimum in (9) is attained, in terms of the subgradients of $f_{\mathcal{P}}$. The proof is postponed to Section F.

Theorem 4.3. *Let $X \geq_{\mathcal{P}} Y$ and $\mu \in \partial f_{\mathcal{P}}(X)$. Then the following are equivalent:*

- (i) $f_{\mathcal{P}}(X) = f(Y)$
- (ii) $\langle \mu, X \rangle = \langle \mu, Y \rangle$ and $f(Y) = \langle \mu, Y \rangle - f^*(\mu)$
- (iii) $\langle \mu, X \rangle = \langle \mu, Y \rangle$ and $\mu \in \partial f(Y)$
- (iv) $\mu \in \partial \delta(X - Y \mid \mathcal{P}) \cap \partial f(Y)$.

As a first useful application of Theorem 4.2 we define the \mathcal{P} -convolution of two proper functions f, g as

$$f \square_{\mathcal{P}} g(X) := f \square g \square \delta(\cdot \mid \mathcal{P})(X) = \inf_{X \geq_{\mathcal{P}} X_1 + X_2} (f(X_1) + g(X_2)).$$

Corollary 4.4. *$\text{cl}(f \square_{\mathcal{P}} g) = (f^* + g^* + \delta(\cdot \mid \mathcal{P}^\circ))^*$ is the greatest closed convex \mathcal{P} -monotone function majorized by $f \square g$.*

Proof. This follows from Theorem 4.2 and (29). □

We now extend Definition 4.1.

Definition 4.5. *The \mathcal{P} -monotone Π -invariant hull of f is defined as*

$$f_{\mathcal{P}, \Pi} := f \square_{\mathcal{P}} \delta^*(\cdot \mid \mathcal{D}) = f \square \delta^*(\cdot \mid \mathcal{D}) \square \delta(\cdot \mid \mathcal{P}) \quad (10)$$

From (4) we immediately derive the equality

$$f_{\mathcal{P}, \Pi}(X) = \inf \{ f(Y) - \lambda \mid X \geq_{\mathcal{P}} Y + \lambda \Pi \}. \quad (11)$$

If f is convex then $f_{\mathcal{P}, \Pi}$ is convex, but in general not closed, and part (iv) of Lemma 3.2 does not apply. Nevertheless, the terminology in Definition 4.5 is justified even if f is not closed. The proof is given in Section G.

Theorem 4.6. *$f_{\mathcal{P}, \Pi}$ is \mathcal{P} -monotone Π -invariant with $f_{\mathcal{P}, \Pi} \leq f$, and $f_{\mathcal{P}, \Pi} = f$ if and only if f is \mathcal{P} -monotone Π -invariant. Moreover,*

$$f_{\mathcal{P}, \Pi}^* = f^* + \delta(\cdot \mid \mathcal{P}^\circ \cap \mathcal{D}).$$

In particular,

$$f_{\mathcal{P},\Pi}^* = f^* \quad \text{on } \mathcal{P}^\circ \cap \mathcal{D},$$

and $\text{cl}(f_{\mathcal{P},\Pi}) = f_{\mathcal{P},\Pi}^{**}$ is the greatest closed convex \mathcal{P} -monotone Π -invariant function majorized by f .

Remark 4.7. For $\mathcal{P} = \{0\}$ notice that $\mathcal{P}^\circ = E^*$ and $X \geq_{\{0\}} Y$ is equivalent to $X = Y$, so that $f_{\{0\}} = f$ and $\square_{\{0\}} = \square$. We thus can define the Π -invariant hull of f as

$$f_{\Pi} := f_{\{0\},\Pi} = f \square \delta^*(\cdot | \mathcal{D}) = \inf_{a \in \mathbb{R}} (f(\cdot - a\Pi) - a).$$

The statements of Theorem 4.6 carry over to f_{Π} with \mathcal{P}° replaced by E^* .

Combining Theorems 4.2 and 4.6 and Remark 4.7, we obtain

Corollary 4.8. If f is \mathcal{P} -monotone then $f_{\Pi} = f_{\mathcal{P},\Pi}$ is \mathcal{P} -monotone. If f is Π -invariant then $f_{\mathcal{P}} = f_{\mathcal{P},\Pi}$ is Π -invariant.

We finally give an alternative representation for the \mathcal{P} -monotone Π -invariant hull of f . Notice that equation (12) below is not trivial, as the infimal convolution of two closed convex functions need not be closed in general, see Example 4.11 below and Section 9 in [25] for more on this. We have the following Theorem, the proof of which is given in Section H.

Theorem 4.9. Suppose $\mathcal{P}^\circ \cap \mathcal{D} \neq \emptyset$. Then

$$\delta^*(\cdot | \mathcal{D}) \square \delta^*(\cdot | \mathcal{P}^\circ) = \delta^*(\cdot | \mathcal{P}^\circ \cap \mathcal{D}). \quad (12)$$

Hence

$$f_{\mathcal{P},\Pi} = f \square \delta^*(\cdot | \mathcal{P}^\circ \cap \mathcal{D}). \quad (13)$$

Example 4.10. In the setup of Example 3.4, equation (13) reads as

$$f_{\mathcal{P},\Pi}(X) = \inf_{Y \in L^p} (f(X - Y) - \text{ess inf } Y).$$

The assumption in Theorem 4.9 cannot be omitted, as the following example shows.

Example 4.11. Let \mathcal{P} be the linear span of Π . Then $\mathcal{P}^\circ = \{\mu \in E^* \mid \langle \mu, \Pi \rangle = 0\}$, and therefore $\mathcal{P}^\circ \cap \mathcal{D} = \emptyset$. It follows that $\delta^*(\cdot | \mathcal{P}^\circ \cap \mathcal{D}) \equiv -\infty$. On the other hand, simple calculations show that

$$\delta^*(\cdot | \mathcal{D}) \square \delta^*(\cdot | \mathcal{P}^\circ)(X) = \begin{cases} -\infty, & X \in \mathcal{P}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Hence (12), and thus (13), does not hold.

5 Examples

Theorem 4.6 provides the recipe for constructing the greatest closed convex \mathcal{P} -monotone Π -invariant function majorized by f , for any proper function $f : E \rightarrow (-\infty, +\infty]$. In this section we illustrate the usefulness and practicability of the above results. In all examples below we have $E = L^p$ with the usual order $\mathcal{P} = L^p_+$, for some $p \in [1, \infty]$, and $\Pi = 1$. Recall the convention in Definition 3.1 for this case, and notice that $\mathcal{D} = \{Z \in L^q \mid \mathbb{E}[Z] = -1\}$ if $E^* = L^q$.

We consider various well-known risk measures which are widely used in practice (see e.g. [9], Chapter 5 in [18] or Section 5.4 in [22]), but fail to be either monotone or cash-invariant.

5.1 L^p -Deviation Risk Measures

Let $p \in [1, \infty)$ and $E = L^p$, and set $q = p/(p-1)$ ($= +\infty$ if $p = 1$). Fix $\alpha > 0$ and define the L^p -deviation risk measure as

$$f(X) = \mathbb{E}[-X] + \alpha \|X - \mathbb{E}[X]\|_p.$$

The L^p -deviation risk measure is continuous convex and cash-invariant, but fails to be monotone in general.

Proposition 5.1. *For $\mathcal{Q} = \{\alpha(\mathbb{E}[g] - g) - 1 \mid g \in L^q, \|g\|_q \leq 1\}$ the mapping*

$$\rho(X) = \sup_{Z \in \mathcal{Q} \cap L^q_-} \mathbb{E}[ZX]$$

is the greatest closed convex monotone cash-invariant function majorized by the L^p -deviation risk measure f .

Proof. We modify the calculations of Example 7 in [8]. For every $Z = \alpha(\mathbb{E}[g] - g) - 1 \in \mathcal{Q}$, we deduce by Hölder's inequality,

$$\begin{aligned} \mathbb{E}[ZX] &= \mathbb{E}[-X] + \mathbb{E}[Z(X - \mathbb{E}[X])] \\ &= \mathbb{E}[-X] + \mathbb{E}[(Z + 1 - \alpha\mathbb{E}[g])(X - \mathbb{E}[X])] \\ &\leq \mathbb{E}[-X] + \alpha \|g\|_q \|X - \mathbb{E}[X]\|_p \leq f(X). \end{aligned} \tag{14}$$

The element $g = -\text{sign}(X - \mathbb{E}[X]) \frac{|X - \mathbb{E}[X]|^{p-1}}{\|X - \mathbb{E}[X]\|_p^{p-1}}$ is in L^q and satisfies $\|g\|_q \leq 1$. Plugging g in (14), we obtain $\mathbb{E}[ZX] = \mathbb{E}[-X] + \alpha \|X - \mathbb{E}[X]\|_p$. This shows that $f(X) = \max_{Z \in \mathcal{Q}} \mathbb{E}[ZX]$, and the claim now follows from Theorem 4.6. \square

Notice that $\mathcal{Q} \cap L_-^q = \{\alpha(\mathbb{E}[g] - g) - 1 \mid g \in L^q, \|g\|_q \leq 1, \mathbb{E}[g] - g \leq 1/\alpha\}$. For $p = 2$ it is thus particularly simple to find elements X where the L^2 -deviation risk measure coincides with its monotone hull.

Corollary 5.2. *Let $p = 2$. Then $\rho(X) = f(X)$ for every $X \in L^2$ with*

$$X \leq \mathbb{E}[X] + \frac{\|X - \mathbb{E}[X]\|_2}{\alpha}. \quad (15)$$

Proof. From the proof of Proposition 5.1 we know that $f(X) = \mathbb{E}[ZX]$ for $Z = \alpha \frac{X - \mathbb{E}[X]}{\|X - \mathbb{E}[X]\|_2} - 1$. Since $Z \in \mathcal{Q} \cap L_-^q$ whenever (15) holds, the claim follows. \square

5.2 L^p -Semi-Deviation Risk Measures

Consider the setup of Section 5.1. Closely related to the L^p -deviation risk measure is the L^p -semi-deviation risk measure

$$f(X) = \mathbb{E}[-X] + \alpha \|(X - \mathbb{E}[X])_-\|_p,$$

see [13] and Example 7 in [8]. It can be shown with similar arguments as in Proposition 5.1 (see Example 7 in [8]) that

$$f(X) = \sup_{Z \in \mathcal{R}} \mathbb{E}[ZX] \quad (16)$$

for $\mathcal{R} = \{\alpha(\mathbb{E}[g] - g) - 1 \mid g \in L^q, g \geq 0, \|g\|_q \leq 1\}$. Moreover the supremum in (16) is attained at $Z = \alpha(\mathbb{E}[g] - g) - 1$ for

$$g = \frac{(X - \mathbb{E}[X])_-^{p-1}}{\|(X - \mathbb{E}[X])_-\|_p^{p-1}} \quad (= 1_{\{X < \mathbb{E}[X]\}} \text{ if } p = 1). \quad (17)$$

Hence f is a continuous convex and cash-invariant function, but fails to be monotone for $\alpha > 1$ in general. We can now extend these results as follows.

Proposition 5.3. *The greatest closed convex monotone cash-invariant function majorized by the L^p -semi-deviation risk measure f is*

$$\rho(X) = \sup_{Z \in \mathcal{R} \cap L_-^q} \mathbb{E}[ZX].$$

We have $\rho(X) = f(X)$ for all $X \in L^p$ with

$$\begin{cases} \mathbb{E}[(X - \mathbb{E}[X])_-^{p-1}] \leq \frac{\|(X - \mathbb{E}[X])_-\|_p^{p-1}}{\alpha}, & \text{if } p > 1 \\ \mathbb{P}[X < \mathbb{E}[X]] \leq \frac{1}{\alpha}, & \text{if } p = 1. \end{cases} \quad (18)$$

Moreover, the L^p -semi-deviation risk measure f is monotone (that is, $f = \rho$) if $\alpha \leq 1$.

We remark that the last statement of this proposition has already been proved in [13] and [8].

Proof. The first statement follows from the preceding remarks and Theorem 4.6. As for the second statement we note that

$$\mathcal{R} \cap L_-^q = \{\alpha(\mathbb{E}[g] - g) - 1 \mid g \in L^q, g \geq 0, \|g\|_q \leq 1, \mathbb{E}[g] - g \leq 1/\alpha\}. \quad (19)$$

Hence $Z = \alpha(\mathbb{E}[g] - g) - 1$ with g given in (17) lies in $\mathcal{R} \cap L_-^q$ if (18) holds.

Finally, it follows from $\mathbb{E}[g] - g \leq \mathbb{E}[g] \leq \|g\|_q \leq 1$ and (19) that $\mathcal{R} \subset L_-^q$ if $\alpha \leq 1$, which proves the last statement. \square

5.3 Mean- L^p Risk Measure

Consider the setup of Section 5.1. We obtain a variation of the L^p -(semi)-deviation risk measure as follows. Let

$$f(X) = \mathbb{E}[-X] + \frac{\alpha}{p} \mathbb{E}[|X|^p]$$

(this can be further generalized by replacing $\frac{\alpha}{p} \mathbb{E}[|\cdot|^p]$ by any convex function $g : L^p \rightarrow \mathbb{R}$). We define the *mean- L^p risk measure* f_1 as cash-invariant hull of f (see Remark 4.7)

$$f_1(X) = \inf_{\lambda \in \mathbb{R}} \left(\mathbb{E}[-X] + \frac{\alpha}{p} \mathbb{E}[|X - \lambda|^p] \right). \quad (20)$$

The infimum in (20) is attained for $\hat{\lambda} \in \mathbb{R}$ satisfying

$$\mathbb{E} \left[\text{sign}(X - \hat{\lambda}) |X - \hat{\lambda}|^{p-1} \right] = 0.$$

In the case $p = 2$, we obtain $\hat{\lambda} = \mathbb{E}[X]$, and

$$-f_1(X) = \mathbb{E}[X] - \frac{\alpha}{2} \mathbb{E}[|X - \mathbb{E}[X]|^2]$$

reduces to the classical mean-variance utility function on L^2 . Its monotone hull, $-f_{L^2_+,1}$, has been derived and studied in [23]. In what follows, we derive their result as an application of Theorem 4.6.

Since

$$f^*(Z) = \sup_{X \in L^p} \{\mathbb{E}[(Z + 1)X] - g(X)\} = g^*(Z + 1), \quad (21)$$

for $g = \frac{\alpha}{p} \mathbb{E}[|\cdot|^p]$, the calculation of the monotone cash-invariant hull of f reduces to the specification of g^* . Using similar arguments as for (24) below, we deduce for $Z \in L^q$ that

$$\begin{aligned} g^*(Z) &= \sup_{X \in L^p} \mathbb{E} \left[ZX - \frac{\alpha}{p} |X|^p \right] \\ &= \mathbb{E} \left[|Z| \left(\frac{|Z|}{\alpha} \right)^{\frac{1}{p-1}} - \frac{\alpha}{p} \left(\frac{|Z|}{\alpha} \right)^{\frac{p}{p-1}} \right] = \frac{1}{\alpha^{\frac{1}{p-1}q}} \mathbb{E}[|Z|^q], \end{aligned} \quad (22)$$

where we have used the optimizer $\hat{X} = \text{sign}(Z) \left(\frac{|Z|}{\alpha} \right)^{\frac{1}{p-1}}$. Combining Theorem 4.6, (21) and (22) implies that

$$\rho(X) = \sup \left\{ \mathbb{E}[ZX] - \frac{1}{\alpha^{\frac{1}{p-1}q}} \mathbb{E}[|Z+1|^q] \mid Z \in L^q_-, \mathbb{E}[Z] = -1 \right\}$$

is the greatest closed convex monotone cash-invariant function majorized by f . For the case $p = 2$, we obtain the utility function

$$-\rho(X) = \inf \left\{ \mathbb{E}[ZX] + \frac{1}{2\alpha} (\mathbb{E}[Z^2] - 1) \mid Z \in L^2_+, \mathbb{E}[Z] = 1 \right\}$$

which induces the *monotone mean-variance preference order* over L^2 . For more details and applications of the monotone mean-variance preference order we refer to [23].

5.4 L^p -Semi-Moment Risk Measure

Consider the setup of Section 5.1. The L^p -semi-moment risk measure $f : E \rightarrow \mathbb{R}$ is defined as

$$f(X) = \frac{1}{\alpha} \mathbb{E}[X_-^p],$$

which is continuous, convex and monotone, but not cash-invariant. In view of Corollary 4.8, the monotone cash-invariant hull of f is

$$f_1(X) = \inf_{\lambda \in \mathbb{R}} \left(\frac{1}{\alpha} \mathbb{E}[(X - \lambda)_-^p] - \lambda \right) = \frac{1}{\alpha} \mathbb{E}[(X - \hat{\lambda})_-^p] - \hat{\lambda}, \quad (23)$$

where the optimizer $\hat{\lambda}$ satisfies $\mathbb{E}[(X - \hat{\lambda})_-^{p-1}] = \frac{\alpha}{p}$ for $p > 1$. For $p = 1$, it can be shown that $\hat{\lambda}$ satisfies $\mathbb{P}[X < \hat{\lambda}] \leq \alpha \leq \mathbb{P}[X \leq \hat{\lambda}]$. That is, $\hat{\lambda}$ is an α -quantile of X . In this case, the right hand side of (23) is well known

as *expected shortfall* or *conditional value-at-risk* or *tail value-at-risk* of X (see e.g. [16]). We remark that the minimization representation (23) of the expected shortfall has already been proved in [26].

Hence the expected shortfall is the greatest closed convex monotone cash-invariant function majorized by the L^1 -semi-moment risk measure.

5.5 Exponential Risk Measure

Here we let $E = (L^\infty, \sigma(L^\infty, L^1))$. The *exponential risk measure* $f : E \rightarrow \mathbb{R}$ is defined as

$$f(X) = \mathbb{E}[\exp(-X)] - 1.$$

It can be checked that f is continuous convex and monotone, but obviously fails to be cash-invariant. For the monotone cash-invariant hull we calculate, according to Corollary 4.8,

$$f_1(X) = \inf_{\lambda \in \mathbb{R}} (\mathbb{E}[\exp(-X + \lambda)] - 1 - \lambda) = \inf_{\lambda \in \mathbb{R}} (\exp(\lambda) \mathbb{E}[\exp(-X)] - 1 - \lambda).$$

Plugging in the optimizer $\hat{\lambda} = -\log \mathbb{E}[\exp(-X)]$, we obtain

$$f_1(X) = \log \mathbb{E}[\exp(-X)],$$

which is closed convex and known as *entropic risk measure* (see e.g. [16]).

Hence the entropic risk measure is the greatest closed convex monotone cash-invariant function majorized by the exponential risk measure.

Remark 5.4. *It should be obvious how to generalize the examples in Sections 5.4 and 5.5 by setting $f(X) = \mathbb{E}[g(X)]$ for some (smooth) convex function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g(0) = 0$.*

5.6 Logarithmic Risk Measure

We consider the setup of Section 5.5, and define the *logarithmic risk measure* as

$$f(X) = \begin{cases} \mathbb{E}[-\log(X)] - 1 & \text{if } X > 0 \\ +\infty & \text{else.} \end{cases}$$

This function is monotone but fails to be cash-invariant.

For $Z \in L^1$ such that $Z < 0$, it follows that

$$f^*(Z) = \sup_{X > 0, X \in L^\infty} \mathbb{E}[XZ + \log(X)] + 1 = \mathbb{E}[\log(-1/Z)]. \quad (24)$$

Indeed, for every $\omega \in \Omega$, $x \mapsto xZ(\omega) + \log(x)$ is maximal at $x = -\frac{1}{Z(\omega)} > 0$. Hence,

$$\sup_{X>0, X \in L^\infty} \mathbb{E}[XZ + \log(X)] + 1 \leq \mathbb{E}[\log(-1/Z)].$$

Notice that by Jensen's inequality $(\log(-1/Z))^- \in L^1$, hence the expectation on the right-hand side is well defined (may be $+\infty$). On the other hand, monotone convergence yields

$$f^*(Z) \geq \lim_{n \rightarrow \infty} \mathbb{E}[(-1/Z \wedge n)Z + \log(-1/Z \wedge n)] + 1 = \mathbb{E}[\log(-1/Z)],$$

which shows (24). This also implies that $f^*(Z) = +\infty$ if $Z \in L_-^1$ with $\mathbb{P}[Z = 0] > 0$.

Together with Theorem 4.6 we now deduce that

$$f_{L_+^1, 1}^{**}(X) = \sup\{\mathbb{E}[XZ] - \mathbb{E}[\log(-1/Z)] \mid Z \in L^1, Z < 0, \mathbb{E}[Z] = -1\}$$

is the greatest closed convex monotone cash-invariant function majorized by the logarithmic risk measure.

5.7 Logarithmic Certainty Equivalent Risk Measure

We consider the setup of Section 5.5 and define the *logarithmic certainty equivalent risk measure* as

$$f(X) = \begin{cases} -\exp \mathbb{E}[\log(X)] & \text{if } X > 0, \log(X) \in L^1 \\ +\infty & \text{else.} \end{cases}$$

Notice that $-f(X)$ is the certainty equivalent of the logarithmic utility function. That is,

$$\mathbb{E}[\log(X)] = \log(-f(X)), \quad (25)$$

whenever the left hand side is defined and finite.

It follows from (25) that f is monotone. However, f is neither cash-invariant nor convex in general.

By convex duality for $\exp : \mathbb{R} \rightarrow \mathbb{R}$, we obtain the representation

$$\exp(x) = \sup_{y \geq 0} (xy - y \log(y) + y).$$

For $Z \in L^1$ with $Z < 0$, we thus calculate

$$\begin{aligned} f^*(Z) &= \sup_{X>0, X \in L^\infty} (\mathbb{E}[XZ] + \sup_{y \geq 0} (y\mathbb{E}[\log(X)] - y \log(y) + y)) \\ &= \sup_{y \geq 0} \sup_{X>0, X \in L^\infty} \mathbb{E}[XZ + y \log(X) - y \log(y) + y] \\ &= \sup_{y \geq 0} y \mathbb{E}[\log(-1/Z)], \end{aligned}$$

where the last equality follows by adapting the argumentation for (24). Furthermore, if $Z \in L_-^1$ such that $\mathbb{P}[Z = 0] > 0$, then $f^*(Z) = +\infty$. Hence, for $Z \in L_-^1 \cap \mathcal{D} = \{Y \in L_-^1 \mid \mathbb{E}[Y] = -1\}$, it follows that $f^*(Z) = 0$ if $Z \equiv -1$ and $f^*(Z) = +\infty$ else.

Consequently, the negative expectation, $-\mathbb{E}[\cdot]$, is the greatest closed convex monotone cash-invariant function majorized by the logarithmic certainty equivalent risk measure.

6 Economic Interpretation

We resume the view point of the risk management of an insurance company as discussed in the introduction.

We interpret the relation $\geq_{\mathcal{P}}$ in the sense that $X \geq_{\mathcal{P}} Y$ makes the position X objectively preferable to Y . In particular, a position $X \geq_{\mathcal{P}} 0$ is understood not to bear any downside risk.

Hence a $\geq_{\mathcal{P}}$ -compatible risk measure f on E has to be proper convex and \mathcal{P} -monotone. (We do not insist that f has to be cash- or Π -invariant.) It can always be normalized so that $f(0) = 0$.

Lemma 3.3 then states that the most conservative closed \mathcal{P} -monotone risk measure vanishing at $X = 0$ is the indicator function $\delta(\cdot \mid \mathcal{P})$.

Thus, an institution equipped with the risk measure $\delta(\cdot \mid \mathcal{P})$ cannot assume any downside risk at all (unless charged an infinite amount of capital which is impossible). Such an institution could be, for instance, the non-risk taking holding company of an insurance group.

Now suppose that the regulator specifies a proper convex function f which is to be used by the insurance company to assess the riskiness of its portfolio $X \in E$. The greater $f(X)$, the more capital is required to carry the risk X .

Due to legal constraints, the function f may fail to be \mathcal{P} -monotone. Indeed, in reality there are legal constraints on the mobility of capital between business units of the insurance company. Risk transfers have to be defined via contingent capital notes, such as retrocession or surplus participation between the business units. Formally, we let the initial risk profile of business unit i be X_i , which sum up to the total portfolio, $X = \sum_i X_i$, and suppose there are legally enforceable contingent capital notes Z_j . In a first step, the risk management will thus find the optimal risk structure (λ_{ij}) which leads to the company's capital charge

$$\inf_{\sum_i \lambda_{ij} = 0 \forall j} \sum_i f_i(X_i + \sum_j \lambda_{ij} Z_j) =: f(X), \quad (26)$$

where f_i denotes the regulator-specified risk measure for business unit i . The so obtained convex function f on E is not \mathcal{P} -monotone in general, as it is demonstrated in Example 4.1 in [11].

However, we may assume that the company is allowed to share any “positive” portion $Z \geq_{\mathcal{P}} 0$ of its portfolio with the holding company as long as the capital charge for the company is readjusted to $f(X - Z)$. The total amount of capital charged to the insurance company and the holding company is thus the sum

$$f(X - Z) + \delta(Z \mid \mathcal{P}) = f(X - Z).$$

Accepting the preceding arguments there is nothing that forbids the management to optimize the risk profile by sharing “positive” portions $Z \geq_{\mathcal{P}} 0$ of the portfolio with the holding company (or the shareholders), so that the resulting capital charge becomes (in the limit)

$$\inf_{Z \geq_{\mathcal{P}} 0} f(X - Z).$$

But this value is just $f_{\mathcal{P}}(X)$, the \mathcal{P} -monotone hull of f evaluated at the initial risk profile X .

This argumentation shows that a regulator-specified risk measure f can always be replaced by (the closure of) its \mathcal{P} -monotone hull $f_{\mathcal{P}}$. The regulator will still accept the resulting capital charge.

Another way to see how this works for (26) is to rewrite f as infimal convolution

$$f = f_1^{M_1} \square \dots \square f_m^{M_m}$$

where $f_i^{M_i} := f_i + \delta(\cdot \mid M_i)$ for $M_i := \{X_i + \sum_j \lambda_j Z_j \mid \lambda_j \in \mathbb{R}\}$, assuming m business units. The \mathcal{P} -monotone hull of f is then just the \mathcal{P} -convolution $f_{\mathcal{P}} = f_1^{M_1} \square_{\mathcal{P}} \dots \square_{\mathcal{P}} f_m^{M_m}$, as shown in Corollary 4.4.

A Some Facts from Convex Analysis

For the convenience of the reader we collect here some standard definitions and results in convex analysis. For more background we refer to Rockafellar [25] and Ekeland and Témam [10].

A function $f : E \rightarrow [-\infty, +\infty]$ is *convex* if

$$f(\lambda X + (1 - \lambda)Y) \leq \lambda f(X) + (1 - \lambda)f(Y) \quad \forall X, Y \in E, \quad \forall \lambda \in [0, 1],$$

whenever the right-hand side is defined. We write $\text{dom}(f) = \{f < \infty\}$ for the *effective domain*, and

$$\partial f(X) = \{\mu \in E^* \mid f(Y) \geq f(X) + \langle \mu, Y - X \rangle \quad \forall Y \in E\}$$

for the convex set of *subgradients* of f . We call f *proper* if $f > -\infty$ and $\text{dom}(f) \neq \emptyset$.

The *closure* of f is denoted by $\text{cl}(f)$ and defined as $\text{cl}(f) \equiv -\infty$, if $f(X) = -\infty$ for some X , and as greatest convex lower semicontinuous function majorized by f , else. A convex function f is called *closed* if $f = \text{cl}(f)$. In [10], the closure of f is called the Γ -regularization of f .

A convex set $A \subset E$ is closed if and only if it is $\sigma(E, E^*)$ -closed. As a consequence, a convex function f is lower semicontinuous if and only if f is lower semicontinuous with respect to $\sigma(E, E^*)$.

The *conjugate function* of a function $f : E \rightarrow [-\infty, +\infty]$,

$$f^*(\mu) = \sup_{X \in E} (\langle \mu, X \rangle - f(X)),$$

is a closed convex function on E^* . Moreover, $(\text{cl}(f))^* = f^*$, and the following convex duality relation holds (Proposition 4.1 in Chapter I of [10])

$$f^{**} = \text{cl}(f). \quad (27)$$

As for the subgradients, we have (Proposition 5.1 in Chapter I of [10])

$$\mu \in \partial f(X) \quad \Leftrightarrow \quad f(X) + f^*(\mu) = \langle \mu, X \rangle. \quad (28)$$

The *infimal convolution* of two proper functions $f_1, f_2 : E \rightarrow (-\infty, \infty]$ is defined as

$$f_1 \square f_2(X) := \inf_{X_1 + X_2 = X} (f_1(X_1) + f_2(X_2)).$$

From this definition we have

$$(f_1 \square f_2)^* = f_1^* + f_2^* \quad (29)$$

(the first part of Theorem 16.4 in [25] carries over to E). Furthermore, if f_1, f_2 are convex then $f_1 \square f_2$ is convex (Theorem 5.4 in [25] carries over to E).

The *indicator function* of a set $\mathcal{C} \subset E$ is defined as

$$\delta(X \mid \mathcal{C}) := \begin{cases} 0, & X \in \mathcal{C} \\ +\infty, & X \notin \mathcal{C}. \end{cases}$$

Then $\delta(\cdot | \mathcal{C})$ is convex lower semicontinuous if and only if \mathcal{C} is convex closed. The conjugate is the *support function* of \mathcal{C} ,

$$\delta^*(\mu | \mathcal{C}) = \sup_{X \in \mathcal{C}} \langle \mu, X \rangle.$$

Notice that E and E^* can be interchanged in the definition of δ and δ^* .

B Proof of Lemma 3.2

Let $f : E \rightarrow [-\infty, +\infty]$ be a function with $f \not\equiv +\infty$. If $f \equiv -\infty$ then $\text{dom}(f^*) = \emptyset$, and there is nothing to prove. Hence we may assume that there exists $X_0 \in E$ with $f(X_0) \in \mathbb{R}$. Notice that

$$\partial f(X) = \emptyset \quad \forall X \in \{f = +\infty\}. \quad (30)$$

Suppose f is \mathcal{P} -monotone. Let $\mu \in E^* \setminus \mathcal{P}^\circ$. Then there exists $X \geq_{\mathcal{P}} 0$ with $\langle \mu, X \rangle > 0$. Hence $f(X_0 + kX) \leq f(X_0)$ and we conclude

$$f^*(\mu) \geq \langle \mu, X_0 + kX \rangle - f(X_0 + kX) \geq k\langle \mu, X \rangle + \langle \mu, X_0 \rangle - f(X_0) \quad \forall k \geq 1.$$

Hence $f^*(\mu) = +\infty$ and therefore $\text{dom}(f^*) \subset \mathcal{P}^\circ$. Now let $X \in \text{dom}(-f) \cap \text{dom}(f)$ (see (30)) and $\mu \in \partial f(X)$. Then $f(X) \in \mathbb{R}$ and thus

$$\langle \mu, Z \rangle \leq f(X + Z) - f(X) \leq 0 \quad \forall Z \geq_{\mathcal{P}} 0.$$

Therefore $\mu \in \mathcal{P}^\circ$, and assertion (i) follows.

Suppose f is Π -invariant. Let $\mu \in E^*$. Then

$$f^*(\mu) \geq \langle \mu, X_0 + c\Pi \rangle - f(X_0 + c\Pi) = c(\langle \mu, \Pi \rangle + 1) + \langle \mu, X_0 \rangle - f(X_0) \quad \forall c \in \mathbb{R}.$$

We conclude that $f^*(\mu) < \infty$ only if $\langle \mu, \Pi \rangle = -1$, whence $\text{dom}(f^*) \subset \mathcal{D}$. Now let $X \in \text{dom}(-f) \cap \text{dom}(f)$ (see (30)) and $\mu \in \partial f(X)$. Then $f(X) \in \mathbb{R}$ and thus

$$-c = f(X + c\Pi) - f(X) \geq c\langle \mu, \Pi \rangle \quad \forall c \in \mathbb{R}.$$

Hence $\mu \in \mathcal{D}$, and assertion (ii) follows.

Now let $f : E \rightarrow (-\infty, +\infty]$ be closed convex. Suppose first that $\text{dom}(f^*) \subset \mathcal{P}^\circ$. Then

$$f(X) = f^{**}(X) = \sup_{\mu \in \mathcal{P}^\circ} (\langle \mu, X \rangle - f^*(\mu)).$$

In view of the Bipolar Theorem (1), f is thus \mathcal{P} -monotone, and assertion (iii) follows.

Suppose now that $\text{dom}(f^*) \subset \mathcal{D}$. Then

$$f(X + c\Pi) = \sup_{\mu \in \mathcal{D}} (\langle \mu, X + c\Pi \rangle - f^*(\mu)) = f(X) - c,$$

and the lemma is proved.

C Proof of Lemma 3.3

Since \mathcal{P}° is a cone, $\delta^*(X | \mathcal{P}^\circ) = \sup_{\mu \in \mathcal{P}^\circ} \langle \mu, X \rangle$ can only take values 0 or ∞ . In view of the Bipolar Theorem (1), $\delta^*(X | \mathcal{P}^\circ) = 0$ if and only if $X \in \mathcal{P}$, and the first equality in (3) follows. The second follows by a dual argument.

Straightforward inspection shows that the conjugate of the closed convex function on the right hand side of (4) is the indicator function $\delta(\cdot | \mathcal{D})$ of \mathcal{D} , which proves (4).

Now let f be a closed convex function with $f(0) = 0$. Then $f^*(\mu) \geq \langle \mu, 0 \rangle - f(0) = 0$. From part (i) of Lemma 3.2 we thus derive

$$f(X) = f^{**}(X) = \sup_{\mu \in \mathcal{P}^\circ} (\langle \mu, X \rangle - f^*(\mu)) \leq \sup_{\mu \in \mathcal{P}^\circ} \langle \mu, X \rangle = \delta^*(X | \mathcal{P}^\circ)$$

if f is \mathcal{P} -monotone, which together with (3) proves (i). Parts (ii) and (iii) follows similarly from part (ii) of Lemma 3.2.

D Proof of Lemma 3.6

From (7) we have, for any $Z \in E$,

$$|\langle \mu, Z \rangle| \leq \|\mu\|_{E^*} \|Z\|_E \leq -\kappa \|Z\|_E \langle \mu, \Pi \rangle \quad \forall \mu \in \mathcal{P}^\circ.$$

Hence (5) follows. Moreover, we have $\|\mu\|_{E^*} \leq -\kappa \langle \mu, \Pi \rangle = \kappa$ for all $\mu \in \mathcal{P}^\circ \cap \mathcal{D}$. Since $\mathcal{P}^\circ \cap \mathcal{D}$ is also $\sigma(E^*, E)$ -closed, it is thus $\sigma(E^*, E)$ -compact by Alaoglu's Theorem (e.g. Theorem 6.25 in [1]).

Let f be proper convex \mathcal{P} -monotone and Π -invariant function. Then f satisfies (6), by Lemma 3.5. In particular, $\langle \mu, X \rangle - f^*(\mu)$ is $\sigma(E^*, E)$ -upper semicontinuous in μ . Hence (8) follows since, by parts (i) and (ii) Lemma 3.2,

$$f(X) = f^{**}(X) = \sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} (\langle \mu, X \rangle - f^*(\mu)) \quad (31)$$

and $\mathcal{P}^\circ \cap \mathcal{D}$ is $\sigma(E^*, E)$ -compact.

Remark D.1. Note that, in view of (28) and (31), property (8) is in fact equivalent to

$$\partial f(X) \cap \mathcal{P}^\circ \cap \mathcal{D} \neq \emptyset \quad \forall X \in E.$$

E Proof of Theorem 4.2

\mathcal{P} -monotonicity of $f_{\mathcal{P}}$ and $f_{\mathcal{P}} \leq f$ follow from the transitivity and reflexivity of $\geq_{\mathcal{P}}$, respectively. If f is \mathcal{P} -monotone then $f_{\mathcal{P}}(X) = \inf_{X \geq_{\mathcal{P}} Y} f(Y) \geq f(X)$, whence the first part of the theorem is proved.

By Lemma 3.3, we have $f_{\mathcal{P}} = f \square \delta^*(\cdot | \mathcal{P}^\circ)$, and thus $f_{\mathcal{P}}^* = f^* + \delta(\cdot | \mathcal{P}^\circ)$, see (29). Theorem 4.2 now follows from Lemma E.1 below.

Lemma E.1. *Let $\mathcal{C} \subset E^*$ be a $\sigma(E^*, E)$ -closed convex set. Then $\hat{f} := (f^* + \delta(\cdot | \mathcal{C}))^* = \text{cl}(f \square \delta^*(\cdot | \mathcal{C}))$ is the greatest closed convex function majorized by f and whose conjugate function has its effective domain contained in \mathcal{C} .*

Proof of Lemma E.1. The equality for \hat{f} follows from (27) and (29).

From the convex duality (27) we have that

$$f_1 \leq f_2 \quad \Rightarrow \quad f_1^* \geq f_2^* \quad \Rightarrow \quad \text{cl}(f_1) \leq \text{cl}(f_2) \quad (32)$$

for all functions $f_1, f_2 : E \rightarrow [-\infty, \infty]$.

In view of (32) we thus have $\hat{f} \leq \text{cl}(f) \leq f$. Now let $g \leq f$ be a closed convex function with $\text{dom}(g^*) \subset \mathcal{C}$. Then $g^* = g^* + \delta(\cdot | \mathcal{C}) \geq f^* + \delta(\cdot | \mathcal{C}) = \hat{f}^*$, whence $g = g^{**} \leq \hat{f}$. \square

F Proof of Theorem 4.3

Notice that, in view of Lemma 3.2 and Theorem 4.2, we have $\mu \in \mathcal{P}^\circ$ and thus $f^*(\mu) = f_{\mathcal{P}}^*(\mu)$.

(i) \Rightarrow (ii): in view of the preceding remark, we deduce from (28)

$$f_{\mathcal{P}}(X) = \langle \mu, X \rangle - f_{\mathcal{P}}^*(\mu) \leq \langle \mu, Y \rangle - f^*(\mu) \leq f^{**}(Y) \leq f(Y) = f_{\mathcal{P}}(X).$$

(ii) \Rightarrow (i): we deduce from (28)

$$f_{\mathcal{P}}(X) \leq f(Y) = \langle \mu, Y \rangle - f^*(\mu) = \langle \mu, X \rangle - f_{\mathcal{P}}^*(\mu) \leq f_{\mathcal{P}}(X).$$

(ii) \Leftrightarrow (iii): this follows from (28).

(iii) \Leftrightarrow (iv): in view of (28) and (3) we have

$$\mu \in \partial \delta(X - Y | \mathcal{P}) \Leftrightarrow \langle \mu, X - Y \rangle = \delta(X - Y | \mathcal{P}) + \delta(\mu | \mathcal{P}^\circ) = 0.$$

G Proof of Theorem 4.6

In view of (10) and Theorem 4.2 we conclude that $f_{\mathcal{P},\Pi}$ is \mathcal{P} -monotone. Let $c \in \mathbb{R}$. Using (11), we calculate

$$\begin{aligned}
f_{\mathcal{P},\Pi}(X + c\Pi) &= \inf \{f(Y) - \lambda \mid X + c\Pi \geq_{\mathcal{P}} Y + \lambda\Pi\} \\
&= \inf \{f(Y) - \lambda \mid X \geq_{\mathcal{P}} Y + (\lambda - c)\Pi\} \\
&= \inf \{f(Y) - (\lambda' + c) \mid X \geq_{\mathcal{P}} Y + \lambda'\Pi\} \\
&= \inf \{f(Y) - \lambda' \mid X \geq_{\mathcal{P}} Y + \lambda'\Pi\} - c \\
&= \inf \{f(Y) + \delta^*(Z \mid \mathcal{D}) \mid X \geq_{\mathcal{P}} Y + Z\} - c \\
&= f_{\mathcal{P},\Pi}(X) - c.
\end{aligned}$$

Hence $f_{\mathcal{P},\Pi}$ is Π -invariant. For $c = 0$, we deduce from the above equalities in particular that $f_{\mathcal{P},\Pi} \leq f$. If f is \mathcal{P} -monotone Π -invariant, Theorem 4.2 and (4) yield

$$f_{\mathcal{P},\Pi}(X) = f \square \delta^*(\cdot \mid \mathcal{D})(X) = \inf_{\lambda \in \mathbb{R}} (f(X - \lambda\Pi) - \lambda) = f(X),$$

which proves the first part of the theorem.

Moreover, in view of (29), (10) and (3),

$$f_{\mathcal{P},\Pi}^* = f^* + \delta(\cdot \mid \mathcal{D}) + \delta(\cdot \mid \mathcal{P}^\circ) = f^* + \delta(\cdot \mid \mathcal{P}^\circ \cap \mathcal{D}).$$

The theorem now follows from Lemma E.1.

H Proof of Theorem 4.9

Note that (13) is a consequence of (12) and Theorem 4.6. It thus remains to prove (12).

We first claim that

$$\inf_{\lambda \in \mathbb{R}} \sup_{\mu \in \mathcal{P}^\circ} (\langle \mu, X \rangle + \lambda(\langle \mu, \Pi \rangle + 1)) = \sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle. \quad (33)$$

Indeed, we can assume $\sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle < \infty$, otherwise (33) is obvious. Following the arguments in the proof of Lemma 1 in [14], we define the nonempty convex set

$$\Xi := \left\{ (y_0, y_1) \in \mathbb{R}^2 \mid \begin{array}{l} y_0 < \langle \mu, X \rangle \\ y_1 = \langle \mu, \Pi \rangle + 1, \text{ for some } \mu \in \mathcal{P}^\circ \end{array} \right\}.$$

Since $\mathcal{P}^\circ \cap \mathcal{D} \neq \emptyset$, it follows that $\sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle \in \mathbb{R}$ and by construction $(\sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle, 0) \notin \Xi$. Hence, by the separating hyperplane theorem, there exists a nonzero vector $(c_0, c_1) \in \mathbb{R}^2$ such that

$$c_0 y_0 + c_1 y_1 \leq c_0 \sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle \quad (34)$$

for all (y_0, y_1) in the closure of Ξ . Since y_0 can become arbitrarily small, it follows that $c_0 \geq 0$. If $c_0 = 0$ then $c_1(\langle \mu, \Pi \rangle + 1) \leq 0$ for all $\mu \in \mathcal{P}^\circ$. But for $\mu \in \mathcal{P}^\circ \cap \mathcal{D}$ and $\lambda > 0$ we have $\langle \mu, \Pi \rangle = -1$ and $\lambda \mu \in \mathcal{P}^\circ$ and thus

$$\langle \lambda \mu, \Pi \rangle + 1 \begin{cases} > 0, & \lambda < 1 \\ < 0, & \lambda > 1 \end{cases}$$

and therefore $c_1 = 0$. This shows that $c_0 > 0$. For $c^* := \frac{c_1}{c_0}$, inequality (34) implies

$$\sup_{\mu \in \mathcal{P}^\circ} \{ \langle \mu, X \rangle + c^* (\langle \mu, \Pi \rangle + 1) \} \leq \sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle. \quad (35)$$

In view of (35) we have

$$\begin{aligned} \inf_{\lambda \in \mathbb{R}} \sup_{\mu \in \mathcal{P}^\circ} (\langle \mu, X \rangle + \lambda (\langle \mu, \Pi \rangle + 1)) &\leq \sup_{\mu \in \mathcal{P}^\circ} \{ \langle \mu, X \rangle + c^* (\langle \mu, \Pi \rangle + 1) \} \\ &\leq \sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle \leq \inf_{\lambda \in \mathbb{R}} \sup_{\mu \in \mathcal{P}^\circ} (\langle \mu, X \rangle + \lambda (\langle \mu, \Pi \rangle + 1)), \end{aligned}$$

whence the claim (33) is proved.

Combining (4) and (33) we conclude

$$\begin{aligned} \delta^*(\cdot | \mathcal{D}) \square \delta^*(\cdot | \mathcal{P}^\circ)(X) &= \inf_{Y \in E} (\delta^*(Y | \mathcal{D}) + \sup_{\mu \in \mathcal{P}^\circ} \langle \mu, X - Y \rangle) \\ &= \inf_{\lambda \in \mathbb{R}} \sup_{\mu \in \mathcal{P}^\circ} (-\lambda + \langle \mu, X - \lambda \Pi \rangle) \\ &= \inf_{\lambda \in \mathbb{R}} \sup_{\mu \in \mathcal{P}^\circ} (\langle \mu, X \rangle + \lambda (\langle \mu, \Pi \rangle + 1)) \\ &= \sup_{\mu \in \mathcal{P}^\circ \cap \mathcal{D}} \langle \mu, X \rangle = \delta^*(X | \mathcal{P}^\circ \cap \mathcal{D}) \end{aligned}$$

and (12) is proved.

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