Conditional L_p-spaces and the duality of modules over f-algebras

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Abstract. Motivated by dynamic asset pricing, we extend the dual pairs' theory of Dieudonné (1942) and Mackey (1945) to pairs of modules over a Dedekind complete f-algebra with multiplicative unit. The main tools are:

- a Hahn-Banach Theorem for modules of this kind;
- a topology on the f-algebra that has the special feature of coinciding with the norm topology when the algebra is a Banach algebra and with the strong order topology of Filipovic, Kupper, and Vogelpoth (2009), when the algebra of all random variables on a probability space $(\Omega, \mathcal{G}, P)$ is considered.

As a leading example, we study in some detail the duality of conditional L_p-spaces.

Keywords: Dual pairs, Hahn-Banach Theorem for modules, complete L_0-normed modules, automatic continuity

1. Introduction

In order to study the testable implications of the Fundamental Theorem of Asset Pricing in a dynamic setting, Hansen and Richard [**HaRi**] consider pricing functions π that map time T payoffs, modeled as $\mathcal{G}_T$-measurable random variables, into prices that are also random variables, but are constrained to be in the information set of traders at the time t when the portfolio decisions are made, that is, are $\mathcal{G}_t$-measurable.[1]

Denoting by $L_0(\mathcal{G}_T) = L_0(\Omega, \mathcal{G}_T, P)$ the space of all $\mathcal{G}_T$-measurable random variables, [**HaRi**] replace the classical Hilbert space $L_2(\mathcal{G}_T) = L_2(\Omega, \mathcal{G}_T, P)$ with its conditional version

$$L_2^{\mathcal{G}_t}(\mathcal{G}_T) = \left\{x \in L_0(\mathcal{G}_T) : E[x^2 \mid \mathcal{G}_t] \text{ is a.s. finite}\right\}$$

and they consider pricing functions $\pi : L_2^{\mathcal{G}_t}(\mathcal{G}_T) \to L_0(\mathcal{G}_t)$ that are linear in the following sense:

$$\pi(ax+by) = a\pi(x) + b\pi(y) \quad \text{for all } a, b \in L_0(\mathcal{G}_t) \text{ and all } x, y \in L_2^{\mathcal{G}_t}(\mathcal{G}_T)$$

and bounded in the following sense: there exists $c \in L_0(\mathcal{G}_t)$ such that

$$|\pi(x)| \leq c\sqrt{E[x^2 \mid \mathcal{G}_t]} \quad \text{for all } x \in L_2^{\mathcal{G}_t}(\mathcal{G}_T).$$

Then, by means of a conditional counterpart to the Riesz Representation Theorem, they show that pricing functions that embody conditioning information can be represented as conditional expectations, thus extending both the unconditional and the conditional results of Harrison and Kreps [**HaKr**]. More in general, they show how the main insights and results of unconditional asset pricing find a natural extension in this fundamentally more powerful setting.

The key intuition of [**HaRi**] is replacing the unconditional duality $\langle x, y\rangle = E[xy] \in \mathbb{R}$ of the Hilbert space $L_2(\mathcal{G}_T)$ with the conditional duality $\langle x, y\rangle^{\mathcal{G}_t} = E[xy \mid \mathcal{G}_t] \in L_0(\mathcal{G}_t)$ of the Hilbert module $L_2^{\mathcal{G}_t}(\mathcal{G}_T)$.

This paper subsumes a previous manuscript by the second and fifth author titled "Complete L^0-normed modules and automatic continuity of monotone convex functions" and first circulated in 2008.

[1] With the natural convention $t < T$ and $\mathcal{G}_t \subset \mathcal{G}_T$.

This paper stems from the observation that a general theory of conditional L_p-spaces can be developed along the lines suggested by [**HaRi**]. Then it expands with the objective of understanding what kind of duality obtains when the real field $\mathbb{R}$ is replaced by a Dedekind complete f-algebra A with multiplicative unit (see Aliprantis and Burkinshaw [**AlBu**]). The specific choice of algebras of this kind is motivated by the possibility of encompassing, together with $L_0(\mathcal{G})$ also $L_\infty(\mathcal{G})$, as well as $\mathbb{R}^K$, for any nonempty set K, ℓ^∞ and some other important Banach algebras.[2]

The novel ingredient is a *strong order topology* on the f-algebra A that have the special feature of coinciding with the norm topology when A is a unitary algebra (see de Jonge and van Rooij [**dJvR**]), and with the topology introduced by Filipovic, Kupper, and Vogelpoth [**FKV**] when $A = L_0(\mathcal{G})$. The strong order topology on A allows a natural definition of weak topologies on A-modules. A version of the Hahn-Banach Extension Theorem then allows to generalize the Dual Pairs' Theory of Dieudonné [**Di**] and Mackey [**Ma**] to pairs of A-modules.

The paper is concluded by returning to modules of random variables in order to exemplify the implications of our findings.

For technical reasons the order of sections is different from the one presented above. Specifically: Section 2 introduces the theory of conditional L_p-spaces, thus providing a concrete example of modules over an f-algebra. Section 3 presents an Hahn-Banach Extension Theorem, the corresponding Kantorovich Extension Theorem appears in Section 4, and the related Hahn Extension Theorem follows in Section 7. The strong order topology appears in Section 5 and it is used to define weak topologies on A-modules in the subsequent Section 6, where the Dual Pairs' Theory is faithfully extended to A-modules. As anticipated, additional results on modules of random variables are presented in the final Section 8.

2. Conditional L_p-spaces

In this section we present an important class of modules over an f-algebra: the conditional L_p-spaces. The treatment here is elementary and the study of these spaces will be continued in the final section where the tools developed in the main part of the paper will be available.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

We denote by $X(\mathcal{F}) = X(\Omega, \mathcal{F}, P)$ the family of all equivalence classes of extended-real-valued functions, almost surely defined on Ω and almost surely equal to some $\mathcal{F}$-measurable function from Ω to $[-\infty, \infty]$, see Fremlin [**Fr**, Section 241]. The vector space $L_0(\mathcal{F})$ of $\mathcal{F}$-*measurable random variables*, consists of all elements of $X(\mathcal{F})$ which admit a real-valued representative, and, as usual, $L_p(\mathcal{F})$ is the subspace of $L_0(\mathcal{F})$ consisting of all random variables with finite absolute p-th moment.

Letting $X(\mathcal{G})$ be the subset of $X(\mathcal{F})$ consisting of all equivalence classes which admit a $\mathcal{G}$-measurable representative, for every $x \in X(\mathcal{F})$ such that either $\int_\Omega x^+ dP < \infty$ or $\int_\Omega x^- dP < \infty$ (for example because $x \geq 0$), there exists a unique $x^\star \in X(\mathcal{G})$ such that

$$\int_G x^\star dP = \int_G x dP \quad \forall G \in \mathcal{G}$$

such $x^\star$ is called *conditional expectation* of x (given $\mathcal{G}$ with respect to P) and denoted $E^{\mathcal{G}} x$. We refer to Loéve [**Lo**, Section 27] for the general properties of conditional expectations.

For every $p \in [1, \infty)$, the set

$$L_p^{\mathcal{G}}(\mathcal{F}) = \left\{ x \in L_0(\mathcal{F}) : E^{\mathcal{G}} |x|^p \in L_0(\mathcal{G}) \right\}$$

[2]Another natural extension of the approach of [**HaRi**] is considering general Hilbert modules and their self-duality, this analysis is carried on by Cerreia-Vioglio, Maccheroni, and Marinacci [**CMM**].

is the subspace of $L_0(\mathcal{F})$ consisting of all random variables with *a.s.* finite *conditional* absolute p-th moment. Analogously,

$$L_\infty^{\mathcal{G}}(\mathcal{F}) = \{x \in L_0(\mathcal{F}) : |x| \le a \text{ for some } a \in L_0(\mathcal{G})\}.$$

We call these spaces *conditional L_p-spaces*. Then we define

$$\|x\|_p^{\mathcal{G}} = \begin{cases} \sqrt[p]{E^{\mathcal{G}}|x|^p} & \text{if } p \in [1,\infty) \\ \inf_{L_0(\mathcal{G})}\{a \in L_0(\mathcal{G}) : |x| \le a\} & \text{if } p = \infty \end{cases}$$

for all $x \in L_p^{\mathcal{G}}(\mathcal{F})$.[3]

We are ready for the first proposition. In reading it remember that the definitions of $L_0(\mathcal{G})$-module, $L_0(\mathcal{G})$-norm, and submodule are formally identical to those of vector space, norm, and linear subspace, where the real field $\mathbb{R}$ is replaced by $L_0(\mathcal{G})$.

Proposition 1. *$L_0(\mathcal{F})$ is an $L_0(\mathcal{G})$-module and, for every $p \in [1,\infty]$,*

- *$L_p^{\mathcal{G}}(\mathcal{F})$ is a submodule of $L_0(\mathcal{F})$;*
- *$\|\cdot\|_p^{\mathcal{G}} : L_p^{\mathcal{G}}(\mathcal{F}) \to L_0(\mathcal{G})^+$ is an $L_0(\mathcal{G})$-norm;*
- *$L_p^{\mathcal{G}}(\mathcal{F}) = L_0(\mathcal{G})L_p(\mathcal{F})$ is the submodule of $L_0(\mathcal{F})$ generated by $L_p(\mathcal{F})$.*

Proof. The first part is routine. We only check the last point. If $x \in L_p^{\mathcal{G}}(\mathcal{F})$, then

$$x = \left(1 + \|x\|_p^{\mathcal{G}}\right)\left[\left(1 + \|x\|_p^{\mathcal{G}}\right)^{-1} x\right]$$

but $\left(1 + \|x\|_p^{\mathcal{G}}\right), \left(1 + \|x\|_p^{\mathcal{G}}\right)^{-1} \in L_0(\mathcal{G})^+$ and $\left\|\left(1 + \|x\|_p^{\mathcal{G}}\right)^{-1} x\right\|_p^{\mathcal{G}} \le 1$.

- For $p \in [1,\infty)$ this implies $E^{\mathcal{G}}\left|\left(1 + \|x\|_p^{\mathcal{G}}\right)^{-1} x\right|^p \le 1$ and integrating both sides of the inequality delivers $\left(1 + \|x\|_p^{\mathcal{G}}\right)^{-1} x \in L_p(\mathcal{F})$.
- For $p = \infty$ this implies $\left|\left(1 + \|x\|_\infty^{\mathcal{G}}\right)^{-1} x\right| \le 1$ and $\left(1 + \|x\|_\infty^{\mathcal{G}}\right)^{-1} x \in L_\infty(\mathcal{F})$.

The generic choice of x implies $L_p^{\mathcal{G}}(\mathcal{F}) \subseteq L_0(\mathcal{G})L_p(\mathcal{F})$.

Conversely, let $x = a_1y_1 + \ldots + a_ny_n$ with $a_1, \ldots, a_n \in L_0(\mathcal{G})$ and $y_1, \ldots, y_n \in L_p(\mathcal{F})$, that is, assume x belongs to the submodule of $L_0(\mathcal{F})$ generated by $L_p(\mathcal{F})$. For every $i = 1, \ldots, n$,

- if $p \in [1,\infty)$, $E^{\mathcal{G}}(|a_iy_i|^p) = E^{\mathcal{G}}(|a_i|^p|y_i|^p) = |a_i|^p E^{\mathcal{G}}(|y_i|^p)$, but $|a_i|^p \in L_0(\mathcal{G})$ and $E^{\mathcal{G}}(|y_i|^p) \in L_1(\mathcal{G}) \subseteq L_0(\mathcal{G})$, then $|a_i|^p E^{\mathcal{G}}(|y_i|^p) \in L_0(\mathcal{G})$, and a_iy_i belongs to $L_p^{\mathcal{G}}(\mathcal{F})$;
- if $p = \infty$, $|a_iy_i| \le |a_i||y_i| \le |a_i|\alpha_i$ where $\alpha_i \in \mathbb{R}$ is such $|y_i| \le \alpha_i$, thus $a_iy_i \in L_\infty^{\mathcal{G}}(\mathcal{F})$.

Since $L_p^{\mathcal{G}}(\mathcal{F})$ is a module, $a_1y_1 + \ldots + a_ny_n \in L_p^{\mathcal{G}}(\mathcal{F})$, and this implies $L_0(\mathcal{G})L_p(\mathcal{F}) \subseteq L_p^{\mathcal{G}}(\mathcal{F})$.■

Remark 1. *Notice that, in general, given a subset S of an $L_0(\mathcal{G})$-module, the submodule generated by S, denoted by $L_0(\mathcal{G})S$ consists of all elements $a_1s_1 + \ldots + a_ns_n$ with $n \in \mathbb{N}$, $a_1, \ldots, a_n \in L_0(\mathcal{G})$ and $s_1, \ldots, s_n \in S$. The proof above shows more, that $x \in L_p^{\mathcal{G}}(\mathcal{F})$ if and only if $x = ay$ for some $a \in L_0(\mathcal{G})^+$ and $y \in L_p(\mathcal{F})$.*

Given a (normed) $L_0(\mathcal{G})$-module, the definition of (bounded) $L_0(\mathcal{G})$-linear form $\pi : L \to L_0(\mathcal{G})$ is formally identical to the one of (bounded) linear functional, where the real field $\mathbb{R}$ is replaced by $L_0(\mathcal{G})$ (see the introduction).

For the present analysis, the fundamental example of bounded $L_0(\mathcal{G})$-linear form is

$$\tilde{E}^{\mathcal{G}}x = E^{\mathcal{G}}x^+ - E^{\mathcal{G}}x^- \quad \forall x \in L_1^{\mathcal{G}}(\mathcal{F}).$$

First notice that $E^{\mathcal{G}}(x^+), E^{\mathcal{G}}(x^-) \le E^{\mathcal{G}}|x|$ and so $\tilde{E}^{\mathcal{G}}x$ is a well defined element of $L_0(\mathcal{G})$ for all $x \in L_1^{\mathcal{G}}(\mathcal{F})$. Moreover, if $x \in L_1^{\mathcal{G}}(\mathcal{F})$, and either $\int_\Omega x^+dP < \infty$ or $\int_\Omega x^-dP < \infty$,

[3]Notice that $\|x\|_\infty^{\mathcal{G}}$ is well defined and $|x| \le \|x\|_\infty^{\mathcal{G}}$ because $L_0(\mathcal{G})$ is super Dedekind complete.

then $\tilde{E}^{\mathcal{G}}x = E^{\mathcal{G}}x^+ - E^{\mathcal{G}}x^- = E^{\mathcal{G}}x$. For this reason we can write $E^{\mathcal{G}}x$ instead of $\tilde{E}^{\mathcal{G}}x$ for all $x \in L_1^{\mathcal{G}}(\mathcal{F})$. The verification that $E^{\mathcal{G}} : L_1^{\mathcal{G}}(\mathcal{F}) \to L_0(\mathcal{G})$ is a bounded $L_0(\mathcal{G})$-linear form relies on the basic properties of conditional expectations.

THEOREM 1. *Let $p \in [1, \infty)$ and q be the conjugate exponent of p. If*

$$\pi : L_p^{\mathcal{G}}(\mathcal{F}) \to L_0(\mathcal{G})$$

is a bounded $L_0(\mathcal{G})$-linear form, there exists $y \in L_q^{\mathcal{G}}(\mathcal{F})$ such that

$$\pi(x) = E^{\mathcal{G}}(xy) \quad \forall x \in L_p^{\mathcal{G}}(\mathcal{F}). \tag{2.1}$$

Conversely, for every $y \in L_q^{\mathcal{G}}(\mathcal{F})$, (2.1) defines a bounded $L_0(\mathcal{G})$-linear form on $L_p^{\mathcal{G}}(\mathcal{F})$.

PROOF. Let $\pi : L_p^{\mathcal{G}}(\mathcal{F}) \to L_0(\mathcal{G})$ be a bounded $L_0(\mathcal{G})$-linear form. There exists $c \in L_0(\mathcal{G})$ such that

$$|\pi(x)| \le c\|x\|_p^{\mathcal{G}} \le (1 + |c|)\|x\|_p^{\mathcal{G}} \quad \forall x \in L_p^{\mathcal{G}}(\mathcal{F})$$

and hence we can define an auxiliary $L_0(\mathcal{G})$-linear form

$$\tilde{\pi}(x) = (1 + |c|)^{-1}\pi(x) \quad \forall x \in L_p^{\mathcal{G}}(\mathcal{F}).$$

Then, for all $x \in L_p^{\mathcal{G}}(\mathcal{F})$, $|\tilde{\pi}(x)| \le \|x\|_p^{\mathcal{G}}$ and $E|\tilde{\pi}(x)|^p \le E\left(\|x\|_p^{\mathcal{G}}\right)^p = E\left(E^{\mathcal{G}}|x|^p\right) = E|x|^p$. Therefore, if $x \in L_p(\mathcal{F})$ then $\tilde{\pi}(x) \in L_p(\mathcal{G})$, and the Jensen's inequality implies

$$|E\tilde{\pi}(x)|^p \le E|\tilde{\pi}(x)|^p \le E|x|^p \text{ that is } |E\tilde{\pi}(x)| \le \sqrt[p]{E|x|^p} = \|x\|_p.$$

But then $E \circ \tilde{\pi} : L_p(\mathcal{F}) \to \mathbb{R}$ is a bounded linear functional and the classical Riesz Representation Theorem delivers the existence of $z \in L_q(\mathcal{F})$ (for future reference, notice that z is positive if π is positive) such that

$$E\tilde{\pi}(x) = \int_\Omega \tilde{\pi}(x)\,dP = \int_\Omega xz dP \quad \forall x \in L_p(\mathcal{F}).$$

By $L_0(\mathcal{G})$-linearity $\int_G \tilde{\pi}(x)\,dP = \int_\Omega \tilde{\pi}(1_G x)\,dP = \int_\Omega (1_G x)\,z dP = \int_G (xz)\,dP$ for all $G \in \mathcal{G}$ and $x \in L_p(\mathcal{F})$, that is,

$$\tilde{\pi}(x) = E^{\mathcal{G}}(xz) \quad \forall x \in L_p(\mathcal{F}).$$

But, by Remark 1, for all $u \in L_p^{\mathcal{G}}(\mathcal{F})$ there are $a \in L^0(\mathcal{G})$ and $x \in L_p(\mathcal{F})$ such that $u = ax$ and $\tilde{\pi}(u) = \tilde{\pi}(ax) = a\tilde{\pi}(x) = aE^{\mathcal{G}}(xz) = E^{\mathcal{G}}(axz) = E^{\mathcal{G}}(uz)$ and $\pi(u) = (1 + |c|)\tilde{\pi}(u) = (1 + |c|)E^{\mathcal{G}}(uz) = E^{\mathcal{G}}(uy)$, where $y = (1 + |c|)z \in L_q^{\mathcal{G}}(\mathcal{F})$ because of Remark 1 again (for future reference, notice that y is positive if π is positive). The rest is routine. ■

3. The extension theorem

In this section we present a perfect analogue of the Hahn-Banach Theorem for modules over f-algebras. Our result jointly extends the pioneer theorem of Vincent-Smith [**VS**], which corresponds to the special case in which the algebra is unitary, and the recent result of [**FKV**] for modules over $L_0(\mathcal{G})$ (which is not unitary). At the same time, the importance of this theorem is not the greater generality, but the fact that (as it happens for the classical Hahn-Banach Theorem for vector spaces) it is the backbone of all duality theory on modules.

We refer the reader to [**dJvR**] and [**AlBu**] for an introductory treatment of Riesz spaces and Riesz algebras.

DEFINITION 1. *A* Riesz algebra *is a Riesz space A endowed with an associative multiplication such that for every $a \in A$ the maps $b \mapsto ab$ and $b \mapsto ba$ are linear, and $ab \ge 0$ for all $a, b \ge 0$.*

*A Riesz algebra A is an f-*algebra *if $b \wedge c = 0$ implies $ab \wedge c = ba \wedge c = 0$ for all $a \ge 0$.*[4]

Also recall that:

[4]The definition of f-algebra dates back to Birkhoff and Pierce [**BiPi**].

- A Riesz space is called *Dedekind complete* whenever every nonempty bounded above subset has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum).
- A *weak order unit* for a Dedekind complete Riesz space A is an element $e \geq 0$ such that $e \wedge a \neq 0$ for all $a > 0$.
- A *multiplicative unit* for a Riesz algebra A is an element $e \neq 0$ such that $ea = ae = a$ for all $a \in A$.

The following proposition provides a geometrically intuitive characterization of Dedekind complete f-algebras with multiplicative unit.[5]

Proposition 2. *A Dedekind complete Riesz algebra A with multiplicative unit e is an f-algebra if and only if e is a weak order unit. In this case, A is commutative.*

Since we shall consider only this kind of algebras we give them a name.

Definition 2. *A* Stonean algebra *is a Dedekind complete f-algebra with multiplicative unit.*

The multiplicative unit will always be denoted by e.[6]

Theorem 2. *Let A be Stonean algebra, E be an A-module, and $p : E \to A$ be a function satisfying*

$$\begin{cases} p(ax) = ap(x) & \forall x \in E \text{ and } \forall a > 0 \\ p(x+y) \leq p(x) + p(y) & \forall x, y \in E. \end{cases} \tag{3.1}$$

Let $L \subseteq E$ be a submodule and $f : L \to A$ be an A-linear form such that

$$f(z) \leq p(z) \qquad \forall z \in L.$$

Then there exists an A-linear form $g : E \to A$ such that $g_{|L} = f$ and $g(x) \leq p(x)$ for all $x \in E$.

3.1. Proof of Theorem 2. Notice that $p(0) = p(2e0) = 2ep(0) = 2p(0)$ so that $p(0) = 0$. We prove the theorem under slightly weaker assumptions, that will be used later. Condition (3.1) implies that $p(ax) = ap(x)$ for all $x \in E$ and all $a \in A^+$, we will instead *only assume*

$$\begin{cases} p(ax) = ap(x) & \forall x \in E \text{ and } \forall a \in A^{++} \cup C_e \\ p(x+y) \leq p(x) + p(y) & \forall x, y \in E \end{cases} \tag{3.2}$$

where

$$A^{++} = \left\{a \in A^+ : a^{-1} \text{ exists in } A\right\}$$

is the set of all *positive invertibles*, and

$$C_e = \left\{v \in A^+ : v \wedge (e - v) = 0\right\}$$

is the set of all *components of* e.[7]

Consider the set

$$P = \left\{ h : D(h) \to A \;\middle|\; \begin{array}{l} D(h) \text{ is a submodule of } E \text{ that contains } L \\ h \text{ is an } A\text{-linear form} \\ h_{|L} = f \text{ and } h(x) \leq p(x) \text{ for all } x \in D(h) \end{array} \right\}.$$

Clearly $P \ni f$ so P is nonempty, and the relation defined by

$$h_2 \succeq h_1 \iff D(h_2) \supseteq D(h_1) \text{ and } h_{2|D(h_1)} = h_1$$

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[5]It can be easily obtained by combining [**dJvR**, Theorem 15.9] and [**AlBu**, Theorem 2.64], and it actually holds for Archimedean Riesz algebras with multiplicative unit.

[6]The definitions of A-module E, A-submodule $L \subseteq E$, and A-linear form $f : L \to A$ are formally identical to those of vector space, linear subspace, linear form, where the real field $\mathbb{R}$ is replaced by A.

[7]These sets are quite important in the theory of f-algebras, see, e.g., Zaanen [**Za**, Theorem 142.2] and [**AlBu**, Theorem 1.49]. Also $0 \in C_e$ implies $p(0) = 0$.

is a partial order on P. We claim that every totally ordered subset Q in P has an upper bound. Indeed, let $Q=\{h_i\}_{i\in I}\subseteq P$ be a totally ordered subset. If Q is empty, then it is bounded by f, else set

$$D(h)=\bigcup_{i\in I}D(h_i)\text{ and }h(x)=h_i(x)\text{ if }x\in D(h_i).$$

It is easy to check that h is well defined, belongs to P, and is an upper bound for Q. We can therefore apply Zorn's lemma and obtain a maximal element g of P.

If $D(g)=E$, then the proof is finished. Suppose $D(g)\subset E$, set $M=D(g)$, and choose $z\in E\setminus M$. It is easy to check that

$$N=\{x+az:(x,a)\in M\times A\}$$

is the smallest submodule of E containing M and z. A contradiction to the maximality of g will be obtained by finding an A-linear form $h:N\to A$ such that

$$h_{|M}=g\text{ and }h(x+az)\le p(x+az)\quad\forall(x,a)\in M\times A. \tag{3.3}$$

Remark 2. *In looking for such an h, notice that, if it exists,*

$$g(x)+ah(z)\le p(x+az)\quad\forall(x,a)\in M\times A$$

in particular, for $(x,a)=(u,-e)$ and $(x,a)=(w,e)$, this implies $-p(u-z)+g(u)\le h(z)\le p(w+z)-g(w)$ for all $u,w\in M$, and by Dedekind completeness

$$\sup\{-p(u-z)+g(u):u\in M\}\le h(z)\le\inf\{p(w+z)-g(w):w\in M\}.$$

Claim 1. *$a(z,M)=\sup\{-p(u-z)+g(u):u\in M\}$ and $b(z,M)=\inf\{p(w+z)-g(w):w\in M\}$ exist in A, they are unique, and $a(z,M)\le b(z,M)$.*

Proof of Claim 1. Since $g\in P$, for every $u,w\in M$

$$g(u)+g(w)=g(u+w)\le p(u+w)=p(u-z+w+z)\le p(u-z)+p(w+z)$$

that is, $-p(u-z)+g(u)\le p(w+z)-g(w)$. Then $A(z,M)=\{-p(u-z)+g(u):u\in M\}$ is bounded above and every element $p(w+z)-g(w)$ of A is an upper bound. By Dedekind completeness, the least upper bound $a(z,M)$ of $A(z,M)$ exists in A, it is unique, and $a(z,M)\le p(w+z)-g(w)$ for all $w\in M$. Analogously, $B(z,M)=\{p(w+z)-g(w):w\in M\}$ admits a unique greatest lower bound $b(z,M)$, so that

$$\inf\{p(w+z)-g(w):w\in M\}=b(z,M)\ge a(z,M)=\sup\{-p(u-z)+g(u):u\in M\}$$

as wanted. □

Claim 2. *For each $c\in[a(z,M),b(z,M)]$ the function*

$$h_c(x+az)=g(x)+ac\quad\forall(x,a)\in M\times A$$

is a well defined A-linear form $h_c:N\to A$ such that $h_{c|M}=g$ and $h_c(y)\le p(y)$ for all $y\in N$.

This claim concludes the proof which, so far, has been identical to the one of the Hahn-Banach Theorem.[8] When $A=\mathbb{R}$, the proof of Claim 2 is very simple: h_c is obviously well defined, linear, and it extends g; while the invertibility of all $a\in\mathbb{R}^+\setminus\{0\}$, together with the positive homogeneity of g and p, guarantees that

$$-p(u-z)+g(u)\le c=h_c(z)=c\le p(w+z)-g(w)\quad\forall u,w\in M\implies h_c(y)\le p(y)\quad\forall y\in N.$$

In our case, the proof Claim 2 is more delicate and requires some lemmata.

Lemma 1. *Let A be Stonean algebra, then $C_e=\left\{v\in[0,e]:v^2=v\right\}$.*

[8] See, e.g., Brezis [**Br**, pages 1–3].

PROOF. If $v \in C_e$, then $e - v \in C_e$, and $e - v \geq 0$ implies $v \leq e$. But A is an Archimedean f-algebra with multiplicative unit, then $v \perp u$ if and only if $vu = 0$ (see [**AlBu**, page 131]), thus

$$v \wedge (e - v) = 0 \implies v \perp (e - v) \implies v (e - v) = 0 \implies v = v^2.$$

Conversely, if $v \in [0, e]$ and $v^2 = v$, then $v \geq 0$ and $v \leq e$, that is, $e - v \geq 0$. Since A is an Archimedean f-algebra with multiplicative unit, then

$$v^2 = v \implies v (e - v) = 0 \implies v \perp (e - v) \implies v \wedge (e - v) = 0.$$

As wanted. ■

Let S be an extremally disconnected compact Hausdorff space and $C^\infty (S)$ the collection of all continuous functions $\varphi : S \to [-\infty, \infty]$ for which the open set $\operatorname{dom} (\varphi) = \{s \in S : -\infty < \varphi (s) < \infty\}$ is dense in S. Given $\varphi, \psi \in C^\infty (S)$, define $\varphi + \psi$ and $\varphi\psi$ as the unique elements of $C^\infty (S)$ such that for every $s \in \operatorname{dom} (\varphi) \cap \operatorname{dom} (\psi)$

$$(\varphi + \psi) (s) = \varphi (s) + \psi (s) \text{ and } (\varphi\psi) (s) = \varphi (s) \psi (s) .$$

Endowed with these operations and the pointwise order, $C^\infty (S)$ is a Dedekind complete f-algebra with unit 1_S (see [**dJvR**, page 122] and Luxemburg and Zaanen [**LuZa**, pages 295 and 323]) that contains $C (S)$ as a subalgebra, notice that the operations in $C (S)$ coincide with the usual pointwise ones.

LEMMA 2. *If A is a Stonean algebra, then there exist an extremally disconnected compact Hausdorff space S and a multiplicative Riesz isomorphism*

$$\begin{array}{cccc} T: & A & \to & \hat{A} \\ & a & \mapsto & \hat{a} \end{array}$$

of A onto a solid f-subalgebra $\hat{A}$ of $C^\infty (S)$ such that $\hat{e} = 1_S$.

The proof is omitted, since it readily follows by the versions of the Ogasawara-Maeda Theorem of [**dJvR**, Theorem 15.9], or [**AlBu**, Theorem 2.64], and the fact that since $\hat{A}$ is an order dense Riesz subspace of the Archimedean Riesz space $C^\infty (S)$ and $\hat{A}$ is Dedekind complete in its own right, then $\hat{A}$ is an ideal of $C^\infty (S)$ (see [**dJvR**, Lemma 13.21] or [**AlBu**, Theorem 2.31]). In what follows, Lemmata 1 and 2 will be repeatedly used without reference.

LEMMA 3. *Let A be a Stonean algebra.*

(i) *If $a \geq \varepsilon e$ for some $\varepsilon \in \mathbb{R}^{++}$, then $a \in A^{++}$.*
(ii) *For every $a \in A$ there exists $v \in C_e$ such that $va = a^+$.*

PROOF. (i) Notice that $\varepsilon^{-1} a \geq e \geq 0$, then by [**dJvR**, Corollary 15.10] there exists $b \in A$ such that $\left(\varepsilon^{-1} a\right) b = e$, but then $a \left(\varepsilon^{-1} b\right) = \left(\varepsilon^{-1} a\right) b = e$ and $\varepsilon^{-1} b = a^{-1}$.

(ii) Let $a \in A$ and set $V = \overline{\{s \in S : \hat{a} (s) > 0\}}$. Since $\hat{a}$ is continuous and S is extremally disconnected, then V is a clopen set in S and

$$\{s \in S : \hat{a} (s) > 0\} \subseteq V \subseteq \{s \in S : \hat{a} (s) \geq 0\} \tag{3.4}$$

next we show that $1_V \in \hat{A}$ and $1_V \hat{a} = \hat{a}^+$.

Since V is clopen, then $1_V \in C (S) \subseteq C^\infty (S)$. Since $\hat{A}$ is an ideal of $C^\infty (S)$, then $|1_V| \leq |1_S| = 1_S \in \hat{A}$ implies $1_V \in \hat{A}$. By definition, $1_V \hat{a}$ is the only $\psi \in C^\infty (S)$ such that

$$\psi (s) = 1_V (s) \hat{a} (s) \quad \forall s \in \operatorname{dom} (1_V) \cap \operatorname{dom} (\hat{a}) = \operatorname{dom} (\hat{a}) . \tag{3.5}$$

But the function $\varphi : S \to [-\infty, \infty]$ defined by $\varphi (s) = 1_V (s) \hat{a} (s)$ for all $s \in S$ is continuous. In fact, given any net $s_\eta \to s$ in S,

- if $s \in V$, then $\varphi (s) = 1_V (s) \hat{a} (s) = \hat{a} (s)$ and, since V is open, there exists η_V such that $s_\eta \in V$ for all $\eta \gtrsim \eta_V$, so that $\varphi (s_\eta) = 1_V (s_\eta) \hat{a} (s_\eta) = \hat{a} (s_\eta)$ for all $\eta \gtrsim \eta_V$, but $\hat{a} (s_\eta) \to \hat{a} (s)$ and hence $\varphi (s_\eta) \to \varphi (s)$;

- else $s \in V^c$, and $\varphi(s) = 1_V(s)\hat{a}(s) = 0$ and, since V^c is open, there exists η_V such that $s_\eta \in V^c$ for all $\eta \succsim \eta_V$, so that $\varphi(s_\eta) = 1_V(s_\eta)\hat{a}(s_\eta) = 0$ for all $\eta \succsim \eta_V$, and hence $\varphi(s_\eta) \to \varphi(s)$.

Now, the continuous functions $\varphi, \psi : S \to [-\infty, \infty]$ coincide on $\mathrm{dom}(\hat{a})$ which is dense in S, thus

$$(1_V\hat{a})(s) = \psi(s) = \varphi(s) = 1_V(s)\hat{a}(s) \quad \forall s \in S.$$

In turn, by (3.4),

- if $\hat{a}(s) > 0$ then $s \in V$ and $(1_V\hat{a})(s) = 1_V(s)\hat{a}(s) = \hat{a}(s) = \sup\{\hat{a}(s), 0\} = (\sup\{\hat{a}, 0\})(s) = \hat{a}^+(s)$;
- if $\hat{a}(s) = 0$ then $(1_V\hat{a})(s) = 1_V(s)\hat{a}(s) = 0 = \sup\{\hat{a}(s), 0\} = (\sup\{\hat{a}, 0\})(s) = \hat{a}^+(s)$;
- if $\hat{a}(s) < 0$ then $s \in V^c$ and $(1_V\hat{a})(s) = 1_V(s)\hat{a}(s) = 0 = \sup\{\hat{a}(s), 0\} = (\sup\{\hat{a}, 0\})(s) = \hat{a}^+(s)$;

that is, $1_V\hat{a} = \hat{a}^+$.

Since $T^{-1} : \hat{A} \to A$ is a multiplicative Riesz isomorphism too, by setting $v = T^{-1}(1_V)$, we have that

- $0 \leq 1_V \leq 1_S$ implies $T^{-1}(0) \leq T^{-1}(1_V) \leq T^{-1}(1_S)$, that is, $0 \leq v \leq e$,
- $(1_V)^2 = 1_V$ implies $vv = T^{-1}(1_V)T^{-1}(1_V) = T^{-1}(1_V 1_V) = T^{-1}(1_V) = v$,
- $1_V\hat{a} = \hat{a}^+$ implies $va = T^{-1}(1_V)T^{-1}(\hat{a}) = T^{-1}(1_V\hat{a}) = T^{-1}(\hat{a}^+) = \left(T^{-1}(\hat{a})\right)^+ = a^+$,

that is, $v \in C_e$ and $va = a^+$. ■

Lemma 4. *Let g, M, z, $a(z, M)$, $b(z, M)$ be defined as above, then, for each $c \in [a(z, M), b(z, M)]$,*

$$g(x) + ac \leq p(x + az) \quad \forall (x, a) \in M \times A. \tag{3.6}$$

Proof. Arbitrarily choose $x \in M$. Let $a \geq 0$, then $a + n^{-1}e \geq n^{-1}e$ belongs to A^{++} for all $n \in \mathbb{N}$ by Lemma 3.

Since $c \leq \inf\{p(w + z) - g(w) : w \in M\}$, then

$$c \leq p\left(\left[a + n^{-1}e\right]^{-1} x + z\right) - g\left(\left[a + n^{-1}e\right]^{-1} x\right) \quad \forall n \in \mathbb{N}$$

by (3.2), $\left[a + n^{-1}e\right] c \leq p\left(x + \left[a + n^{-1}e\right] z\right) - g(x) \leq p(x + az) - g(x) + n^{-1}p(z)$ and

$$g(x) + ac - p(x + az) \leq n^{-1}|p(z) - c| \quad \forall n \in \mathbb{N}$$

since A is Archimedean, then $g(x) + ac - p(x + az) \leq 0$.

Since, $c \geq \sup\{-p(u - z) + g(u) : u \in M\}$, then

$$c \geq -p\left(\left[a + n^{-1}e\right]^{-1} x - z\right) + g\left(\left[a + n^{-1}e\right]^{-1} x\right) \quad \forall n \in \mathbb{N}$$

by (3.2), $-\left[a + n^{-1}e\right] c \leq p\left(x - \left[a + n^{-1}e\right] z\right) - g(x)$ and

$$g(x) - ac - p(x - az) \leq n^{-1}|p(-z) + c| \quad \forall n \in \mathbb{N}$$

so that $g(x) - ac - p(x - az) \leq 0$. Summing up, we have

$$g(x) \pm ac \leq p(x \pm az) \quad \forall a \geq 0. \tag{3.7}$$

Now take any $a \in A$. By Lemma 3, there exists $v \in C_e$ such that $va = a^+$, also $e - v \in C_e$ and $(e - v)a = a - a^+ = -a^-$, thus

$$va^+ = vva = va = a^+ \tag{3.8}$$

$$-(e - v)a^- = (e - v)(e - v)a = (e - v)a = -a^-. \tag{3.9}$$

But then (3.2), (3.7), and (3.8) imply

$$vg(x) + a^+c = v\left(g(x) + a^+c\right) \leq vp\left(x + a^+z\right) = p\left(vx + va^+z\right) = p(vx + vaz) = vp(x + az)$$

while setting $\bar{v} = e - v \geq 0$, (3.2), (3.7), and (3.9) imply

$$\bar{v}g(x) - a^-c = \bar{v}\left(g(x) - a^-c\right) \leq \bar{v}p\left(x - a^-z\right) = p\left(\bar{v}x - \bar{v}a^-z\right) = p(\bar{v}x + \bar{v}az) = \bar{v}p(x + az)$$

and addition of the two above inequalities delivers (3.6). ■

Proof of Claim 2. Arbitrarily choose $c \in [a(z,M), b(z,M)]$. Assume $u + az = w + bz$ for some $u, w \in M$ and $a, b \in A$, then $(u-w) + (a-b)z = 0$ and $((u-w),(a-b)) \in M \times A$, by (3.6)

$$g(u-w) + (a-b)c \le p((u-w)+(a-b)z) = p(0) = 0$$

whence $g(u) + ac \le g(w) + bc$, and analogously, $g(w) + bc \le g(u) + ac$. Therefore

$$\begin{array}{rccc} h_c: & N & \to & A \\ & x + az & \mapsto & g(x) + ac \end{array}$$

is well defined, and (3.6) guarantees $h_c(y) \le p(y)$ for all $y \in N$. Obviously, h_c is A-linear and extends g.

4. Positive extensions

In this section, as a simple, but important, corollary of Theorem 2 we obtain a version of the Kantorovich Extension Theorem.[9]

Theorem 3. *Let A be a Stonean algebra, E be an ordered A-module, M be a majorizing submodule, and $f: M \to A$ be a positive A-linear form. Then there exists a positive A-linear form $g: E \to A$ such that $g_{|M} = f$.*

4.1. Proof of Theorem 3. We will repeatedly use the fact that for any two nonempty index sets I and J, if $a = \inf_{i\in I} a_i$ and $b = \inf_{j \in J} b_j$ exist in A, then by [**LuZa**, Theorem 13.1]

$$a + b = \inf\{a_i + b_j : i \in I \text{ and } j \in J\}. \tag{4.1}$$

For each $x \in E$, set

$$M_x = \{u \in M : u \ge x\}$$
$$p(x) = \inf_{u \in M_x} f(u)$$

that is, $p(x) = \inf\{f(u) : M \ni u \ge x\}$. It is routine to check that p is well defined, monotone,[10] $p(z) = f(z)$ for all $z \in M$, and p is subadditive.[11]

Before applying Theorem 2, we need to check that

$$p(ax) = ap(x) \quad \forall x \in E \text{ and } \forall a \in A^{++} \cup C_e. \tag{4.2}$$

First notice that

$$p(x+z) = p(x) + f(z) = p(x) + p(z) \quad \forall (x,z) \in E \times M.$$

In fact, $u \in M_x$ implies $M \ni u + z \ge x + z$ and $u + z \in M_{x+z}$ and, conversely, $w \in M_{x+z}$ implies $w = (w-z) + z$ with $w - z = u \in M_x$, that is, $M_{x+z} = M_x + z$; therefore, by (4.1),

$$p(x+z) = \inf f(M_{x+z}) = \inf f(M_x + z) = \inf_{u \in M_x}\{f(u) + f(z)\} = p(x) + f(z).$$

In turn, this allows to show that, given $a \in A$,

$$\text{if } p(ax) = ap(x) \text{ for all } x \in E^+, \text{ then } p(ay) = ap(y) \quad \forall y \in E. \tag{4.3}$$

In fact, for each $y \in E$ there exists $z \in M$ such that $-y \le z$, then $y + z \ge 0$, therefore

$$p(ay) + af(z) = p(ay + az) = p(a(y+z)) = ap(y+z) = a(p(y) + f(z)) = ap(y) + af(z).$$

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[9] The definitions of ordered A-module E and positive A-linear form $f: M \to A$ are formally identical to those of ordered vector space and positive linear form, where the real field $\mathbb{R}$ is replaced by A. Recall that a linear subspace $M \subseteq E$ (and in particular an A-submodule) is *majorizing* if and only if for each $x \in E$ there exists $z \in M$ such that $x \le z$.

[10] If $x \ge y$, then $M_x \subseteq M_y$, $f(M_x) \subseteq f(M_y)$, and $p(x) = \inf f(M_x) \ge \inf f(M_y) = p(y)$.

[11] By (4.1), $p(x) + p(y) = \inf\{f(u) + f(w) : u \in M_x \text{ and } w \in M_y\} = \inf\{f(u+w) : u \in M_x \text{ and } w \in M_y\}$, but $u + w \in M_{x+y}$ for every $u \in M_x$ and $w \in M_y$, thus $f(u+w) \in f(M_{x+y})$; hence $f(M_{x+y})$ contains $\{f(u+w) : u \in M_x \text{ and } w \in M_y\}$ and its infimum, $\inf f(M_{x+y}) = p(x+y)$, is smaller than $p(x) + p(y)$.

Now let $x \in E^+$, $v \in C_e$, and set $\bar{v} = e - v \in C_e$, clearly

$$\begin{aligned} vp(x) + \bar{v}p(x) = p(x) &= \inf\{vf(z) + \bar{v}f(z) : M \ni z \geq x\} \\ &= \inf\{f(vz) + f(\bar{v}z) : M \ni z \text{ and } vz \geq vx \text{ and } \bar{v}z \geq \bar{v}x\}. \end{aligned}$$

But for every $z \in M$ such that $vz \geq vx$ and $\bar{v}z \geq \bar{v}x$, $vz \in M_{vx}$ and $\bar{v}z \in M_{\bar{v}x}$; then

$$f(vz) + f(\bar{v}z) \in \{f(u) + f(w) : u \in M_{vx} \text{ and } w \in M_{\bar{v}x}\}$$

that is, $\{f(vz) + f(\bar{v}z) : M \ni z \text{ and } vz \geq vx \text{ and } \bar{v}z \geq \bar{v}x\} \subseteq \{f(u) + f(w) : u \in M_{vx} \text{ and } w \in M_{\bar{v}x}\}$ and

$$\begin{aligned} p(x) &= \inf\{f(vz) + f(\bar{v}z) : M \ni z \text{ and } vz \geq vx \text{ and } \bar{v}z \geq \bar{v}x\} \\ &\geq \inf\{f(u) + f(w) : u \in M_{vx} \text{ and } w \in M_{\bar{v}x}\} = p(vx) + p(\bar{v}x) \end{aligned}$$

by (4.1) again. The converse inequality follows by subadditivity of p, in fact, $p(x) = p(vx + \bar{v}x) \leq p(vx) + p(\bar{v}x)$, summing up,

$$vp(x) + \bar{v}p(x) = p(x) = p(vx) + p(\bar{v}x).$$

Now, $vM_x \subseteq M_{vx}$, and so $vp(x) = v\inf f(M_x) \leq \inf vf(M_x) = \inf f(vM_x) = \inf\{f(vz) : z \in M_x\}$, and

$$\inf\{f(vz) : z \in M_x\} \leq \inf\{f(vu) : M \ni u \text{ and } vu \geq vx\} \tag{4.4}$$

in fact, if $u \in M$ and $vu \geq vx$, then taking any $w \in M_x$ we have $\bar{v}w \geq \bar{v}x$, and setting $z_{uv} = vu + \bar{v}w \in M$, it follows

$$M \ni z_{uv} \geq vx + \bar{v}x = x \text{ and } f(vz_{uv}) = f(vu)$$

hence $\inf\{f(vz) : M \ni z \geq x\} \leq f(vu)$, but this is true for every $u \in M$ such that $vu \geq vx$. We have shown

$$vp(x) \leq \inf\{f(vu) : M \ni u \text{ and } vu \geq vx\}.$$

Moreover, for each $w \in M_{vx}$, since $w \geq vx \geq 0$, we have $w = ew \geq vw \geq v^2x = vx$, so that

$$f(w) \geq f(vw) \geq \inf\{f(vu) : M \ni u \text{ and } vu \geq vx\}$$

and $vp(x) \leq \inf\{f(vu) : M \ni u \text{ and } vu \geq vx\} \leq \inf_{w \in M_{vx}} f(w) = p(vx)$. Since this is true for a generic $v \in C_e$, then it also holds for $\bar{v}$, we conclude

$$vp(x) + \bar{v}p(x) = p(vx) + p(\bar{v}x) \text{ and } vp(x) \leq p(vx) \text{ and } \bar{v}p(x) \leq p(\bar{v}x)$$

but this implies $vp(x) = p(vx)$ and $\bar{v}p(x) = p(\bar{v}x)$.[12] Together with (4.3) this shows (4.2) for all $a \in C_e$.

If $a \in A^{++}$, then $a^{-1} \in A^{++}$.[13] Moreover, $aM_x = M_{ax}$ for all $x \in E$.[14] As a consequence

$$ap(x) = a\inf f(M_x) \leq \inf af(M_x) = \inf f(aM_x) = \inf f(M_{ax}) = p(ax)$$

and hence $ap(x) \leq p(ax)$ for all $x \in E$ and all $a \in A^{++}$. Therefore,

$$a^{-1}p(ax) \leq p\left(a^{-1}ax\right) = p(x) \text{ i.e. } p(ax) \leq ap(x) \quad \forall a \in A^{++} \text{ and } x \in E$$

this shows (4.2) for all $a \in A^{++}$. By Theorem 2, we have that there exists an A-linear form $g : E \to A$ such that $g_{|M} = f$ and $g(x) \leq p(x)$ for all $x \in E$. In particular for every $x \geq 0$, $g(-x) \leq p(-x) \leq p(0) = 0$ and $g(x) \geq 0$.

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[12] $a_1 + a_2 = b_1 + b_2$ and $a_1 \leq b_1$ and $a_2 \leq b_2$, imply $a_1 = b_1 + (b_2 - a_2)$ and $b_2 - a_2 \geq 0$ so $a_1 \geq b_1$, analogously $a_2 = (b_1 - a_1) + b_2$ and $b_1 - a_1 \geq 0$ so $a_2 \geq b_2$.

[13] Set $b = a^{-1}$, then $e = a\left(b^+ - b^-\right)$. But $b^+ \wedge b^- = 0$, A is an f-algebra, and $a \geq 0$, then $ab^+ \wedge b^- = 0$ and $ab^+ \wedge ab^- = 0$. Therefore $e = ab^+ - ab^-$ and $ab^+ \wedge ab^- = 0$, so that $ab^+ = e^+ = e = ab$ and $b = b^+$.

[14] E.g. $aM_x \subseteq M_{ax}$, $a^{-1}M_{ax} \subseteq M_x$, $a\left(a^{-1}M_{ax}\right) \subseteq aM_x$, $M_{ax} \subseteq aM_x$.

5. The strong order topology

In this section a novel topology on Stonean algebras is defined. This topology coincides with the norm topology when A is a unitary algebra (see [**dJvR**]), and with the topology introduced by [**FKV**] when $A = L_0(\mathcal{G})$. Its definition and properties are analogous to those of the usual topology of the real line. More importantly, it will allow us to develop a natural duality theory for modules over f-algebras in the next section.

The set of positive invertibles A^{++} of a Stonean algebra has remerkable properties.[15]

- A^{++} is a sublattice of A;
- A^{++} is a commutative l-group with unit e, see Birkhoff [**Birk**];
- A^{++} is contained in the set of all weak order units;
- A^{++} contains the set of all strong order units.[16]

Example 1. *Let S be an extremally disconnected compact Hausdorff space. If $A = C^{\infty}(S)$,*

$$A^{++} = \{\varphi \in C^{\infty}(S) : [\varphi > 0] \text{ is dense in } S\}$$

coincides with the set of all weak order units of A. While, when the principal ideal $A_e = C(S)$ of A is considered,

$$(A_e)^{++} = \{\varphi \in C(S) : [\varphi > 0] = S\}$$

coincides with the set of all strong order units of A_e which is strictly included in $A_e \cap A^{++}$.

Needless to say that, if $A = \mathbb{R}$, then $A^{++} = \mathbb{R}^{++}$. The above properties suggest that this analogy is rather strong. The next proposition seems to be conclusive in this respect. We will often denote by r a generic element of A^{++}.

Proposition 3. *Let A be a Stonean algebra, and set*

$$a \gg b \iff a - b \in A^{++}$$

then:

(i) $r \gg 0 \iff r \in A^{++}$;
(ii) *if $a \in A$ and $a \geq r \gg 0$, then $a \gg 0$;*
(iii) *$\gg$ is a strict partial order (that is, an antireflexive and transitive binary relation);*
(iv) *if $a \gg b$, then $a + c \gg b + c$ for all $c \in A$ and $ra \gg rb$ for all $r \in A^{++} \cup \mathbb{R}^{++}$.*

Proof. The only non-routine point is (ii), which in turn implies that A^{++} is closed under addition, and since A^{++} is also closed under multiplication, the other properties follow.

(ii) By Lemma 2, A can be considered as a f-subalgebra and an ideal of $C^{\infty}(S)$, with unit 1_S, for some extremally disconnected, compact, and Hausdorff space S. Since r is positive and invertible in A, it is positive and invertible in $C^{\infty}(S)$, then $[r > 0]$ is dense in S. But $a \geq r \geq 0$ implies $[r > 0] \subseteq [a > 0]$, then $[a > 0]$ is dense in S and a is positive and invertible in $C^{\infty}(S)$. For each s in the open and dense $[r > 0] \cap \operatorname{dom}(r) \cap [a > 0] \cap \operatorname{dom}(a)$

$$r^{-1}(s) = \frac{1}{r(s)} \geq \frac{1}{a(s)} = a^{-1}(s)$$

then $0 \leq a^{-1} \leq r^{-1}$. Since A^{++} is a group, $r^{-1} \in A^{++} \subseteq A$, but A is an ideal of $C^{\infty}(S)$ and therefore $a^{-1} \in A$ (because $\left|a^{-1}\right| \leq \left|r^{-1}\right|$). ■

We call $\gg$ the *strong order* on A (in analogy with the strong order on $\mathbb{R}^n$). Define, for every $a \in A$ and $r \in A^{++}$,

$$B(a, r) = \{b \in A : |b - a| \ll r\} \text{ and } \bar{B}(a, r) = \{b \in A : |b - a| \leq r\}$$

and notice that:

[15] See again [**Za**, Theorem 142.2].

[16] If a is a strong order unit, there exists $n \in \mathbb{N}$ such that $na \geq e \geq 0$, that is, $a \geq n^{-1}e \geq 0$, so $a \in A^{++}$ by Lemma 3.

- $a \in B(a,r)$,
- for each $b \in B(a,r)$ we have $B(b, r - |b-a|) \subseteq B(a,r)$,[17]
- if $r_1, r_2 \gg 0$ and $r_1 \leq r_2$, then $B(a,r_1) \subseteq B(a,r_2)$,[18]
- for each $b \in B(a_1,r_1) \cap B(a_2,r_2)$ we have
$$b \in B(b, (r_1 - |b-a_1|) \wedge (r_2 - |b-a_2|)) \subseteq B(a_1,r_1) \cap B(a_2,r_2).^{19}$$

That is $\{B(a,r) : (a,r) \in A \times A^{++}\}$ is a basis for a topology on A that we call *strong order topology*, $a_\eta \overset{so}{\to} a$ means that the net a_η converges to a in this topology.

PROPOSITION 4. *Let A be a Stonean algebra, then:*

(i) *the strong order topology is the order topology generated by the strong order;*
(ii) *for each $a \in A$, $\{B(a,r)\}_{r\gg 0}$ and $\{\bar{B}(a,r)\}_{r\gg 0}$ are neighborhood bases for the strong order topology at a;*
(iii) *$a_\eta \overset{so}{\to} a$ implies $a_\eta \overset{o}{\to} a$;*
(iv) *if e is a strong unit, the strong order topology coincides with the supnorm topology.*

PROOF. (i) Notice that $|a| \ll b$ implies $-a \leq |a| \ll b$ and $a \leq |a| \ll b$, then $-b \ll a \ll b$, and conversely $-b \ll a \ll b$ implies $b-a, b+a \in A^{++}$, which is a lattice, so that $(b-a) \wedge (b+a) \gg 0$, that is, $b - (a \vee -a) \gg 0$ and $|a| \ll b$. Therefore
$$B(a,r) = \{b \in A : -r \ll b-a \ll r\} = \{b \in A : a-r \ll b \ll a+r\}$$
$$\{b \in A : c \ll b \ll d\} = \left\{b \in A : \frac{d+c}{2} - \frac{d-c}{2} \ll b \ll \frac{d+c}{2} + \frac{d-c}{2}\right\} = B\left(\frac{d+c}{2}, \frac{d-c}{2}\right)$$
for all $a, c, d \in A$ with $c \ll d$ and all $r \in A^{++}$.[20]

(ii) Obviously $\{B(a,r)\}_{r\gg 0}$ is a neighborhood basis at a. Moreover notice that for every $\alpha \in (0,1)$, $B(a,\alpha r) \subseteq \bar{B}(a,\alpha r)$ and
$$b \in \bar{B}(a,\alpha r) \iff -\alpha r \leq b-a \leq \alpha r$$
but $r - \alpha r = (1-\alpha) r \gg 0$, then $-r \ll -\alpha r \ll \alpha r \ll r$, thus $B(a,\alpha r) \subseteq \bar{B}(a,\alpha r) \subseteq B(a,r)$.

(iii) By (ii), $a_\eta \overset{so}{\to} a$ if and only if for every $r \gg 0$ there exists η_r such that $|a_\eta - a| \leq r$ for all $\eta \succsim \eta_r$. But when A^{++} is directed by the inverse order ($r_1 \succsim r_2$ if and only if $r_1 \leq r_2$), then the net defined by $b_r = r$ for all $r \in A^{++}$ decreases to 0, denoted $b_r \downarrow 0$. Thus $a_\eta \overset{so}{\to} a$ implies that there exists $b_r \downarrow 0$ such that for every r there exists η_r such that $|a_\eta - a| \leq b_r$ for all $\eta \succsim \eta_r$, that is, $a_\eta \overset{o}{\to} a$.[21]

(iv) For each $a \in A$ and every $\rho \in \mathbb{R}^{++}$, the closed unit ball $\bar{B}_\infty(a,\rho)$ induced by the supnorm $\|\cdot\|_\infty$ coincides with $\bar{B}(a,\rho e)$. Thus $a_\eta \overset{so}{\to} a$ implies $a_\eta \overset{\|\cdot\|_\infty}{\to} a$. Conversely, each $r \in A^{++}$ is a strong unit, then there exists $n = n(r) \in \mathbb{N}$ such that $e \leq nr$ and
$$\bar{B}_\infty\left(a, \frac{1}{n}\right) = \bar{B}\left(a, \frac{1}{n}e\right) = \left\{s \in S : |b-a| \leq \frac{1}{n}e\right\} \subseteq \bar{B}(a,r).$$
Thus $a_\eta \overset{\|\cdot\|_\infty}{\to} a$ implies $a_\eta \overset{so}{\to} a$. ■

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[17] In fact, $r - |b-a| \gg 0$ and letting $r_1 = r - |b-a|$, for each $c \in B(b,r_1)$ we have
$$|c-a| \leq |c-b| + |b-a| \ll r_1 + |b-a| = r$$
but $a_1 \leq a_2 \ll a_3$ implies $a_3 - a_1 \geq a_3 - a_2 \gg 0$ that is $a_1 \ll a_3$, whence $|c-a| \ll r$.

[18] In fact, $b \in B(a,r_1) \iff r_1 \gg |b-a|$, then $r_2 \geq r_1 \gg |b-a|$, but $a_3 \geq a_2 \gg a_1$ implies $a_3 - a_1 \geq a_2 - a_1 \gg 0$ that is $a_3 \gg a_1$, whence $r_2 \gg |b-a|$.

[19] In fact, $r_1 - |b-a_1|, r_2 - |b-a_2| \in A^{++}$ and so
$$A^{++} \ni (r_1 - |b-a_1|) \wedge (r_2 - |b-a_2|) \leq r_i - |b-a_i|$$
thus $B(b, (r_1 - |b-a_1|) \wedge (r_2 - |b-a_2|)) \subseteq B(b, r_i - |b-a_i|) \subseteq B(a_i, r_i)$ for $i = 1,2$.

[20] Notice that $c \ll d$ implies $2^{-1}(d-c) \in A^{++}$ and that A has no largest and no smallest element since $a + e \gg a \gg a - e$ for all $a \in A$.

[21] See Abramovich and Aliprantis [**AbAl**, Exercise 1.2.4].

THEOREM 4. *Let A be a Stonean algebra, then the strong order topology is Hausdorff and all operations are continuous.*

PROOF. We have seen that $a_\eta \overset{so}{\to} a$ implies that $a_\eta \overset{o}{\to} a$, and order limits are unique.

Addition and lattice operations. If $a_\eta \overset{so}{\to} a$ and $b_\eta \overset{so}{\to} b$, then for each $r \gg 0$ there exists η_1 such that $|a_\eta - a| \leq 2^{-1} r$ for all $\eta \succsim \eta_1$, there exists η_2 such that $|b_\eta - b| \leq 2^{-1} r$ for all $\eta \succsim \eta_2$. Taking $\eta_3 \succsim \eta_1, \eta_2$, for all $\eta \succsim \eta_3$,

$$|(a_\eta + b_\eta) - (a + b)| \leq |a_\eta - a| + |b_\eta - b| \leq r$$
$$||a_\eta| - |a|| \leq |a_\eta - a| \leq 2^{-1} r \leq r$$

then $a_\eta + b_\eta \overset{so}{\to} a + b$, $|a_\eta| \overset{so}{\to} |a|$. Moreover, if $\alpha \in \mathbb{R}$, then for each $r \gg 0$, since $a_\eta \overset{so}{\to} a$, there exists η_4 such that $|a_\eta - a| \leq (1 + |\alpha|)^{-1} r$ for all $\eta \succsim \eta_4$, thus

$$|\alpha a_\eta - \alpha a| \leq |\alpha| \, |a_\eta - a| \leq \frac{|\alpha|}{1 + |\alpha|} r \leq r$$

and $\alpha a_\eta \overset{so}{\to} \alpha a$. Continuity of the other lattice operations follows, in fact

$$a_\eta \overset{\vee}{\wedge} b_\eta = \frac{1}{2} \left[(a_\eta + b_\eta) \pm |a_\eta - b_\eta| \right] \overset{so}{\to} \frac{1}{2} \left[(a + b) \pm |a - b| \right] = a \overset{\vee}{\wedge} b.$$

Multiplication. Notice that

- $|c_\eta - c| \overset{so}{\to} 0$ if and only if for each $r \gg 0$ eventually $|c_\eta - c| \leq r$, that is, $c_\eta \overset{so}{\to} c$;
- if $d_\eta \overset{so}{\to} 0$ and eventually $|c_\eta| \leq |d_\eta|$, then $c_\eta \overset{so}{\to} 0$;
- if $c_\eta \overset{so}{\to} c$ and $d \gg 0$, for each $r \gg 0$ eventually $|c_\eta - c| \leq d^{-1} r$, then
$$|d c_\eta - dc| \leq |d| \, |c_\eta - c| \leq r$$
so that $d c_\eta \overset{so}{\to} dc$.

Finally, if $a_\eta \overset{so}{\to} a$ and $b_\eta \overset{so}{\to} b$, then $|a_\eta b_\eta - ab| \leq |a_\eta b_\eta - a_\eta b| + |a_\eta b - ab| \leq |a_\eta| \, |b_\eta - b| + |a_\eta - a| \, |b|$, but $|a_\eta| \overset{so}{\to} |a|$ implies that eventually $|a_\eta| \in B(|a|, e + |b|)$, then eventually $|a_\eta| \leq |a| + e + |b|$ and obviously $|b| \leq |a| + e + |b|$, whence eventually

$$|a_\eta b_\eta - ab| \leq (|a| + e + |b|) \, |b_\eta - b| + |a_\eta - a| \, (|a| + e + |b|) \overset{so}{\to} 0$$

and $a_\eta b_\eta \overset{so}{\to} ab$. ■

6. Dual A-modules

Let F be a nonempty subset of A-linear forms on E and $w = \sigma(E, F)$ the weak topology generated by F on E once A is endowed with the strong order topology.[22] The following "omnibus theorem" regroups the fundamental properties of w. Recall the F is said to be *total* if and only if for each $x \neq y$ in E, there exists $f \in F$ such that $f(x) \neq f(y)$.

THEOREM 5. *Let A be a Stonean algebra, E be an A-module, F a nonempty set of A-linear forms $f : E \to A$, and $w = \sigma(E, F)$. Then:*

(i) $x_\eta \xrightarrow{w} x \iff f(x_\eta) \xrightarrow{so} f(x)$ *for all* $f \in F$;

(ii) *w is Hausdorff if and only if F is total;*

(iii) *for every $x_o \in E$, the sets*
$$V_{x_o}(N, r) = \left\{ x \in E : \sup_{f \in N} |f(x) - f(x_o)| \leq r \right\}$$
where N is a finite subset of F and $r \in A^{++}$ form a neighborhood basis for w at x_o, and they are A-convex;

(iv) *the module operations of sum and scalar product are continuous;*

[22] Thus w is the weakest topology on E that makes all the functions $f \in F$ continuous.

(v) *the submodule of* $\mathrm{Hom}_A(E,A)$ *generated by* F *coincides with the set of all* A*-linear forms that are* w*-continuous.*

As usual, $\mathrm{Hom}_A(E,A)$ denotes the module of all A-linear forms from E to A.

6.1. Proof of Theorem 5. Point (i) is true for any weak topology.

Assume F is total. If $x_\eta \xrightarrow{w} x$ and $x_\eta \xrightarrow{w} y$, then $f(x_\eta) \xrightarrow{so} f(x)$ and $f(x_\eta) \xrightarrow{so} f(y)$ for all $f \in F$, since the strong order topology on A is Hausdorff, this implies $f(x) = f(y)$ for all $f \in F$, and totality of F implies $x = y$. Thus w-convergent nets have unique limits, that is, w is Hausdorff.

Conversely, if F is not total, there are $x \neq y$ such that $f(x) = f(y)$ for all $f \in F$ and the constant net $x_\eta \equiv x$ w-converges both to x and y, and w cannot be Hausdorff. This proves (ii).

Denote by $\mathcal{P}_0(F)$ the class of all nonempty and finite subsets of F. For each $(N,r) \in \mathcal{P}_0(\mathcal{F}) \times A^{++}$ and each $x_o \in E$, it is easy to check that

$$\begin{aligned} V_{x_o}(N,r) &= \{x \in E : |f(x) - f(x_o)| \leq r \quad \forall f \in N\} = \bigcap_{f\in N} \{x \in E : |f(x) - f(x_o)| \leq r\} \\ &= \bigcap_{f\in N} \left\{x \in E : f(x) \in \bar{B}(f(x_o), r)\right\} = \bigcap_{f\in N} f^{-1}\left(\bar{B}(f(x_o), r)\right) \end{aligned}$$

As a consequence, for each (N,r), $V_{x_o}(N,r) \supseteq \left(\bigcap_{f\in N} f^{-1}(B(f(x_o), r))\right) \ni x_o$; therefore $V_{x_o}(N,r)$ is a neighborhood of x_o in w;[23] $V_{x_o}(N,r)$ is also A-convex (and hence $\mathbb{R}$-convex), in fact, let $x, y \in V_{x_o}(N,r)$ and $a \in [0,e]$, then for every $f \in N$

$$\left.\begin{array}{l} f(x_o) - r \leq f(x) \leq f(x_o) + r \\ f(x_o) - r \leq f(y) \leq f(x_o) + r \end{array}\right\} \implies f(x_o) - r \leq af(x) + (e-a) f(y) \leq f(x_o) + r$$

that is, $|f(ax + (e-a)y) - f(x_o)| \leq r$ and $ax + (e-a)y \in V_{x_o}(N,r)$.

Next we show that the class of neighborhoods $\mathcal{V}_{x_o} = \{V_{x_o}(N,r) : (N,r) \in \mathcal{P}_0(F) \times A^{++}\}$ is actually a decreasing net in the class $\mathcal{N}^w_{x_o}$ of all neighborhoods of x_o.

Lemma 5. *For every* $x_o \in E$*, the relation*

$$(N_1, r_1) \succsim (N_2, r_2) \iff N_1 \supseteq N_2 \text{ and } r_1 \leq r_2$$

is a direction on $\mathcal{P}_0(F) \times A^{++}$ *such that*

$$(N_1, r_1) \succsim (N_2, r_2) \implies V_{x_o}(N_1, r_1) \subseteq V_{x_o}(N_2, r_2).$$

Moreover, each net $\left\{x_{(N,r)}\right\}$ *in* E*, such that* $x_{(N,r)} \in V_{x_o}(N,r)$ *for all* $(N,r) \in \mathcal{P}_0(F) \times A^{++}$*,* w*-converges to* x_o*.*

Proof. Clearly, $\succsim$ is a preorder, moreover,

$$(N_1 \cup N_2, r_1 \wedge r_2) \succsim (N_1, r_1), (N_2, r_2) \quad \forall (N_1, r_1), (N_2, r_2) \in \mathcal{P}_0(F) \times A^{++}$$

so that $\succsim$ is a direction and $\{V_{x_o}(N,r)\}_{(N,r)}$ is a net in $\mathcal{N}^w_{x_o}$.

If $(N_1, r_1) \succsim (N_2, r_2)$, then $N_1 \supseteq N_2$ and $r_1 \leq r_2$, thus for each $x \in V_{x_o}(N_1, r_1)$

$$\sup_{f\in N_2} |f(x) - f(x_o)| \leq \sup_{f\in N_1} |f(x) - f(x_o)| \leq r_1 \leq r_2$$

so that $x \in V_{x_o}(N_2, r_2)$, and $V_{x_o}(N_1, r_1) \subseteq V_{x_o}(N_2, r_2)$, that is, $\{V_{x_o}(N,r)\}_{(N,r)}$ is decreasing.

Let $\left\{x_{(N,r)}\right\}$ be a net in E such that $x_{(N,r)} \in V_{x_o}(N,r)$ for all (N,r). Fix $f \in F$, for each $d \gg 0$ there exists

$$(N_d, r_d) := (\{f\}, d) \in \mathcal{P}_0(F) \times A^{++}$$

[23] $B(f(x_o), r)$ is open in A and each f is continuous.

such that for all $(N,r) \succsim (N_d, r_d)$, since $x_{(N,r)} \in V_{x_o}(N,r) \subseteq V_{x_o}(\{f\}, d)$,

$$\left|f\left(x_{(N,r)}\right) - f(x_o)\right| \leq d$$

that is, $f\left(x_{(N,r)}\right) \xrightarrow{so} f(x_o)$. Since this is true for all $f \in F$, then $x_{(N,r)} \xrightarrow{w} x_o$. ■

LEMMA 6. *Let (X,τ) be a topological space, $x_o \in X$, $\{V_\lambda\}_{\lambda\in\Lambda}$ a decreasing net of neighborhoods of x_o. The following conditions are equivalent:*

(i) *$\{V_\lambda : \lambda \in \Lambda\}$ is a neighborhood basis at x_o;*
(ii) *Each net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X such that $x_\lambda \in V_\lambda$ for all $\lambda \in \Lambda$ converges to x_o.*

PROOF. (i) implies (ii). Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in X such that $x_\lambda \in V_\lambda$ for all $\lambda \in \Lambda$. For each $U \in \mathcal{N}^\tau_{x_o}$ there exists $\bar{\lambda} = \bar{\lambda}(U)$ such that $V_{\bar{\lambda}} \subseteq U$. Since $\{V_\lambda\}_{\lambda\in\Lambda}$ is decreasing, $x_\lambda \in V_\lambda \subseteq V_{\bar{\lambda}} \subseteq U$ for all $\lambda \succsim \bar{\lambda}$, that is, $x_\lambda \xrightarrow{\tau} x_o$.

(ii) implies (i). Assume each net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X such that $x_\lambda \in V_\lambda$ for all $\lambda \in \Lambda$ converges to x_o, and *per contra* $\{V_\lambda : \lambda \in \Lambda\}$ is not a neighborhood basis. Then there exists a neighborhood U of x_o such that $U^c \cap V_\lambda \neq \varnothing$ for all $\lambda \in \Lambda$. Choose arbitrarily $x_\lambda \in U^c \cap V_\lambda$ for all $\lambda \in \Lambda$. Then $x_\lambda \xrightarrow{\tau} x_o$ and it never meets $U \in \mathcal{N}^\tau_{x_o}$, which is absurd. ■

Summing up: for every $x_o \in E$, $\mathcal{V}_{x_o} = \{V_{x_o}(N,r) : (N,r) \in \mathcal{P}_0(F) \times A^{++}\}$ is a neighborhood basis for w at x_o consisting of A-convex sets and (iii) holds.

Next we show that operations are continuous, thus also (iv) holds:

- if $(x_\eta, y_\eta) \longrightarrow (x,y)$ in the $w \times w$ topology on $E \times E$, then

$$\begin{aligned} x_\eta \xrightarrow{w} x \text{ and } y_\eta \xrightarrow{w} y &\implies f(x_\eta) \xrightarrow{so} f(x) \text{ and } f(y_\eta) \xrightarrow{so} f(y) \quad \forall f \in F \\ &\implies f(x_\eta) + f(y_\eta) \xrightarrow{so} f(x) + f(y) \quad \forall f \in F \\ &\implies f(x_\eta + y_\eta) \xrightarrow{so} f(x+y) \quad \forall f \in F \\ &\implies x_\eta + y_\eta \xrightarrow{w} x + y. \end{aligned}$$

- if $(a_\eta, x_\eta) \to (a,x)$ in the $so \times w$ topology on $A \times E$, then

$$\begin{aligned} a_\eta \xrightarrow{so} a \text{ and } x_\eta \xrightarrow{w} x &\implies a_\eta \xrightarrow{so} a \text{ and } f(x_\eta) \xrightarrow{so} f(x) \quad \forall f \in F \\ &\implies a_\eta f(x_\eta) \xrightarrow{so} af(x) \quad \forall f \in F \\ &\implies f(a_\eta x_\eta) \xrightarrow{so} f(ax) \quad \forall f \in F \\ &\implies a_\eta x_\eta \xrightarrow{w} ax. \end{aligned}$$

Finally, we turn to point (v). Clearly, AF is a is a submodule of the module $\mathrm{Hom}^w_A(E,A)$ of all w-continuous A-linear forms and $0 \in AF$. Conversely, let $f \neq 0$ be a w-continuous A-linear form. Since f is continuous at 0, there exist $k \in \mathbb{N}$, $f_1, f_2, ..., f_k \in F$, and $r \gg 0$ such that

$$f(\{x \in E : |f_i(x)| \leq r \quad \forall i = 1,...,k\}) = f(V_0(\{f_1, f_2, ..., f_k\}, r)) \subseteq \bar{B}(0,e) \tag{6.1}$$

but then for all $y \in \bigcap_{i=1}^k \ker f_i$, $y \in V_0(\{f_1, f_2, ..., f_k\}, r)$, and so $f(y) \subseteq \bar{B}(0,e)$, that is,

$$f\left(\bigcap_{i=1}^k \ker f_i\right) \subseteq \bar{B}(0,e)$$

but $f\left(\bigcap_{i=1}^k \ker f_i\right)$ is a submodule of A, therefore $f\left(\bigcap_{i=1}^k \ker f_i\right) = \{0\}$ (because A is Archimedean) and $\bigcap_{i=1}^k \ker f_i \subseteq \ker f$. Then

$$f_i(y) = f_i(z) \quad \forall i = 1,...,k$$

implies $f(y) = f(z)$. The map

$$\begin{array}{rccl} T: & E & \to & A^k \\ & y & \mapsto & (f_1(y), f_2(y), ..., f_k(y)) \end{array}$$

is an A-module homomorphism and so $T(E)$ is a submodule of A^k. Moreover, the map $\varphi : T(E) \to A$ defined by $\varphi(f_1(y), f_2(y), ..., f_k(y)) = f(y)$ is a well defined A-linear form. Next we show that φ is bounded by the A-sublinear function

$$p(a_1, a_2, ..., a_k) = r^{-1} \sup_{i=1...k} |a_i|$$

on A^k. We know that,

$$|f_j(x)| \leq r \quad \forall j = 1, ..., k \implies |f(x)| \leq e. \tag{6.2}$$

Given a generic $y \in E$ and a generic $n \in \mathbb{N}$,

$$\begin{aligned} |f(y)| &= \left| \frac{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r}{r} f\left(\frac{r}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} y \right) \right| \\ &= \left| \frac{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r}{r} \right| \left| f\left(\frac{r}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} y \right) \right| \end{aligned}$$

but for each $j = 1, ..., k$

$$\left| f_j\left(\frac{r}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} y \right) \right| = \frac{|f_j(y)|}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} r$$

and $|f_j(y)| \leq \sup_{i=1...k} |f_i(y)| + \frac{1}{n} r \in A^{++}$, multiplication by $\left(\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r\right)^{-1}$ yields

$$\frac{|f_j(y)|}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} \leq e \text{ and } \left| f_j\left(\frac{r}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} y \right) \right| \leq r \text{ for all } j = 1, ..., k$$

but then, by (6.2), $\left| f\left(\frac{r}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} y \right) \right| \leq e$ and

$$|f(y)| = \left| \frac{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r}{r} \right| \left| f\left(\frac{r}{\sup_{i=1...k} |f_i(y)| + \frac{1}{n} r} y \right) \right| \leq r^{-1} \sup_{i=1...k} |f_i(y)| + \frac{1}{n} e.$$

Since A is Archimedean, then $|f(y)| \leq r^{-1} \sup_{i=1...k} |f_i(y)|$ for all $y \in E$. Since for every $(a_1, a_2, ..., a_k) \in T(E)$ there exists $y \in E$ such that $(a_1, a_2, ..., a_k) = (f_1(y), f_2(y), ..., f_k(y))$,[24]

$$\left| \varphi \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} \right| = \left| \varphi \begin{pmatrix} f_1(y) \\ f_2(y) \\ \vdots \\ f_k(y) \end{pmatrix} \right| = |f(y)| \leq r^{-1} \sup_{i=1...k} |f_i(y)| = p \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}.$$

By Theorem 2 there exists an A-linear extension $\psi : A^k \to A$ of φ. Therefore, for every $x \in E$,

$$\begin{aligned} f(x) = \varphi \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_k(x) \end{pmatrix} &= \psi \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_k(x) \end{pmatrix} = \psi \begin{pmatrix} f_1(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + ... + \psi \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f_k(x) \end{pmatrix} \\ &= \psi \begin{pmatrix} e \\ 0 \\ \vdots \\ 0 \end{pmatrix} f_1(x) + ... + \psi \begin{pmatrix} 0 \\ 0 \\ \vdots \\ e \end{pmatrix} f_k(x) = \sum_{i=1}^{k} b_i f_i(x) \end{aligned}$$

[24]For notational convenience we indifferently use rows or columns to denote the elements of A^k.

where $b_i = \psi\left(\left(\delta_{ij}e\right)_{j=1}^{k}\right)$ and δ_{ij} is the Kronecker delta, that is, $f = \sum_{i=1}^{k} b_i f_i \in AF$.

7. Normed A-modules

In this section the strong order is used to define a topology on a normed A-module. Like in the previous one, this topology turns out to be very well behaved in terms of continuity of A-linear forms, and the analogy with (normed) vector spaces remains completely faithful.

The definition of a norm $\|\cdot\| : E \to A^+$ on an A-module E is again formally identical to the usual one, where the real field $\mathbb{R}$ is replaced by A, and it generates a topology with basis

$$B_E(x,r) = \{y \in E : \|y-x\| \ll r\} \quad \forall x \in E \text{ and } \forall r \gg 0$$

$\bar{B}_E(x,r)$ is defined analogously (with $\leq$ instead of $\ll$). It is easy to show that:

- $B_E(x,r) = \{y \in E : \|y-x\| \in B_A(0,r)\} = x + B_E(0,r)$;
- $x \in B_E(x,r)$,
- for each $y \in B_E(x,r)$ we have $B_E(y, r - \|y-x\|) \subseteq B_E(x,r)$,
- if $r_1, r_2 \gg 0$ and $r_1 \leq r_2$, then $B_E(x,r_1) \subseteq B_E(x,r_2)$,
- for each $y \in B_E(x_1,r_1) \cap B_E(x_2,r_2)$, $y \in B_E(y, (r_1 - \|y-x_1\|) \wedge (r_2 - \|y-x_2\|)) \subseteq B_E(x_1,r_1) \cap B_E(x_2,r_2)$,
- $\{B_E(x,r)\}_{r \gg 0}$ and $\{\bar{B}_E(x,r)\}_{r \gg 0}$ are neighborhood bases for the norm topology at x;
- $x_\eta \overset{\|\cdot\|}{\to} x \iff \|x_\eta - x\| \overset{so}{\to} 0$;
- the norm topology is Hausdorff;
- the module operations of sum and scalar product are continuous;
- if e is a strong order unit, then

$$\|x\|_e = \min\{\lambda \geq 0 : \|x\| \leq \lambda e\} \quad \forall x \in E$$

  is a (real valued) norm on E and it generates the the same topology as $\|\cdot\|$.[25]

Proposition 5. *Let A be Stonean algebra (with the strong order topology), E be a normed A-module (with the norm topology), and $f : E \to A$ be an A-linear form. The following properties are equivalent:*

(i) *f is continuous;*
(ii) *f is continuous at 0;*
(iii) *there exists $r \gg 0$ such that*

$$|f(x)| \leq r\,\|x\| \quad \forall x \in E;$$

(iv) *f is* bounded*, that is, there exists $a \in A$ such that*

$$|f(x)| \leq a\,\|x\| \quad \forall x \in E.$$

Proof. We only prove that (ii) implies (iii), the rest being routine. For every $x \in E$, $n \in \mathbb{N}$, and $r \gg 0$, we have the following chain of implications

$$\|x\| + n^{-1}e \gg \|x\| \implies e \gg \|x\|\left(\|x\| + n^{-1}e\right)^{-1} \implies r \gg \left\|r\left(\|x\| + n^{-1}e\right)^{-1} x\right\|.$$

By continuity at 0, there exists $r \gg 0$ such that $f(B_E(0,r)) \subseteq B_A(0,e) \subseteq \bar{B}_A(0,e)$, but we have just shown that $r\left(\|x\| + n^{-1}e\right)^{-1} x \in B_E(0,r)$ for all $x \in E$ and $n \in \mathbb{N}$, then

$$\left|f\left(r\left(\|x\| + n^{-1}e\right)^{-1} x\right)\right| \leq e \implies r\left(\|x\| + n^{-1}e\right)^{-1} |f(x)| \leq e \implies |f(x)| \leq r^{-1}\left(\|x\| + n^{-1}e\right)$$

which yields $|f(x)| \leq r^{-1}\|x\|$ because A is Archimedean. ∎

[25] By definition $\|x\|_e$ is the $\|\cdot\|_\infty$ of $\|x\| \in A$, it is actually a norm on E, when the latter is regarded as a vector space, and $x_\eta \overset{\|\cdot\|}{\to} x \iff \|x_\eta - x\| \overset{so}{\to} 0 \iff \|x_\eta - x\| \overset{\|\cdot\|_\infty}{\to} 0 \iff \|(\|x_\eta - x\|)\|_\infty \to 0 \iff \|x_\eta - x\|_e \to 0$.

Following the usual analogy, it is easy to show that the set of all bounded A-linear forms on a normed A-module E form an A-module $E^{\sim}$ that can be normed by setting

$$\|f\|^{\sim} = \min\left\{a \in A^{+} : |f(x)| \leq a\|x\| \quad \forall x \in E\right\} = \sup_{\|x\| \leq e} |f(x)|$$

for all $f \in E^{\sim}$.[26] At this point, we can state, without proof, the corresponding version of the Hahn Extension Theorem.

Proposition 6. *Let A be Stonean algebra, E be a normed A-module, $L \subseteq E$ be a submodule, and $f : L \to A$ be a continuous A-linear form. Then there exists a continuous A-linear form $g : E \to A$ such that $g_{|L} = f$ and $\|g\|^{\sim}_{E^{\sim}} = \|f\|^{\sim}_{L^{\sim}}$.*

The only non-routine observation in the proof is that the topology inherited by L as a submodule of E coincides with the one generated by the restriction to L of the norm of E.

We conclude with a pioneer result of Haydon, Levy, and Raynaud [**HLR**] showing that when the Stonean algebra $A = L_0(\mathcal{G})$ is considered, a fifth equivalent point can be added to Proposition 5. Before stating it we recall that

$$d_{L_0(\mathcal{G})}(a, b) = \int_{\Omega} (|a-b| \wedge 1)\, dP \quad \forall a, b \in A$$

is a metric on A that induces the topology of convergence in probability.

Proposition 7 (Haydon-Levy-Raynaud). *Let $A = L_0(\mathcal{G})$ and E be a normed A-module. Then the translation invariant metric*

$$d_E(x, y) = d_{L_0(\mathcal{G})}(\|x-y\|, 0) \quad \forall x, y \in E$$

induces a linear topology on E,[27] and the following properties are equivalent for an A-linear form $f : E \to A$:

(iv) *f is bounded;*
(v) *f is d_E-$d_{L_0(\mathcal{G})}$-continuous.*

8. Modules of random variables

Like in our opening example, the focus of this final section is on $L_0(\mathcal{G})$-submodules of $L_0(\mathcal{F})$ which we call *modules of random variables*. They form a large family of "concrete" A-modules that proved to be useful in financial modelling.[28]

Remark 3. *As we already observed, $A = L_0(\mathcal{G})$ is a Stonean algebra, moreover:*

- *$e = 1$;*
- *$A^{++} = \{a \in L_0(\mathcal{G}) : a(\omega) > 0$ for almost all $\omega \in \Omega\}$;*[29]
- *$C_e = \{1_G : G \in \mathcal{G}\}$.*[30]

The $L_0(\mathcal{G})$-linearity of conditional expectations and the duality results of Section 6 suggest the following definition.

[26] The verification of the above claims builds on a remarkable property of Stonean algebras: if $B \subseteq A$ is nonempty and bounded above (resp. below) and $c \in A^{+}$, then $c \sup B = \sup(cB)$ (resp. $c \inf B = \inf(cB)$). In fact, the map $a \mapsto ca$ is a positive orthomorphism of A ([**AlBu**, Theorem 2.62]), as such it is an order continuous ([**AlBu**, Theorem 2.44]) lattice homomorphism ([**AlBu**, page 115]), and therefore it preserves arbitrary suprema and infima ([**AlBu**, page 106]).

[27] It also makes the module scalar product continuous: if $a_n \xrightarrow{d_{L_0(\mathcal{G})}} a$ in A and $x_n \xrightarrow{d_E} x$, then $a_n x_n \xrightarrow{d_E} ax$.

[28] Like in the cited [**HaRi**] and in the more recent Frittelli and Maggis [**FrMa**] and Filipovic, Kupper, and Vogelpoth [**FKV2**].

[29] With the usual abuse of notation, $a(\cdot)$ denotes the generic representative of the equivalence class a. We tacitly choose real-valued and $\mathcal{G}$-measurable representatives in what follows.

[30] Here the converse abuse is performed by writing a representative instead of the corresponding equivalence class.

DEFINITION 3. *A pair (L, L') of modules of random variables is called a* conditional dual pair *if and only if:*

(i) $E^{\mathcal{G}} |xy| \in L_0(\mathcal{G})$ *for all* $(x, y) \in L \times L'$*;*
(ii) $E^{\mathcal{G}}(xy) = 0$ *for all* $y \in L'$ *implies* $x = 0$*;*
(iii) $E^{\mathcal{G}}(xy) = 0$ *for all* $x \in L$ *implies* $y = 0$.

In this case, we set

$$\langle x, y\rangle^{\mathcal{G}} = E^{\mathcal{G}}(xy) \quad \forall (x, y) \in L \times L'$$

and by identyfing $y \in L'$ with

$$\begin{array}{rccc} \langle \cdot, y\rangle^{\mathcal{G}}: & L & \to & L_0(\mathcal{G}) \\ & x & \mapsto & \langle x, y\rangle^{\mathcal{G}} \end{array}$$

L' can be seen as a total submodule of $\mathrm{Hom}_{L_0(\mathcal{G})}(L, L_0(\mathcal{G}))$.[31] By the previous Theorem 5, $L' = \mathrm{Hom}^{w}_{L_0(\mathcal{G})}(L, L_0(\mathcal{G}))$ when L is endowed with the $\sigma(L, L')$ topology w.

COROLLARY 1. *If (L, L') is a conditional dual pair, then $\sigma(L, L')$ is Hausdorff, and for every $\sigma(L, L')$-continuous $L_0(\mathcal{G})$-linear form π on L, there exists one and only one $y \in L'$ such that*

$$\pi(x) = E^{\mathcal{G}}(xy) \quad \forall x \in L. \tag{8.1}$$

Conversely, (8.1) defines a $\sigma(L, L')$-continuous $L_0(\mathcal{G})$-linear form π on L, for every $y \in L'$.

In particular, when the conditional dual pair $\left(L_p^{\mathcal{G}}(\mathcal{F}), L_q^{\mathcal{G}}(\mathcal{F})\right)$ with $1 \le p < \infty$ is considered, the above result shows that $L_q^{\mathcal{G}}(\mathcal{F})$ is not only the "strong" dual of $L_p^{\mathcal{G}}(\mathcal{F})$, as shown in Theorem 1, but also its "weak" dual. This suggests an even deeper analogy between classical L_p-spaces and conditional L_p-spaces. The final part of this paper is specifically devoted to investigate this analogy in greater detail.

8.1. The conjugate space of $L_p^{\mathcal{G}}(\mathcal{F})$, $1 \le p < \infty$. We can now complete the conditional "Riesz"representation Theorem 1.

THEOREM 6. *Let $p \in [1, \infty)$ and q be the conjugate exponent of p. The operator*

$$\begin{array}{rccc} I: & L_q^{\mathcal{G}}(\mathcal{F}) & \to & L_p^{\mathcal{G}}(\mathcal{F})^{\sim} \\ & y & \mapsto & \langle \cdot, y\rangle^{\mathcal{G}} \end{array}$$

is a module isomorphism such that $\|y\|_q^{\mathcal{G}} = \|I(y)\|^{\sim}$ for all $y \in L_q^{\mathcal{G}}(\mathcal{F})$.

PROOF. Theorem 1 guarantees that I is well defined and onto and it is easy to check it is a homomorphism. Next we show that it preserves the $L_0(\mathcal{G})$-norms of the two spaces.

Let $y \in L_q^{\mathcal{G}}(\mathcal{F})$ and set $I_y = I(y)$. By the conditional Hölder inequality

$$|I_y(x)| \le E^{\mathcal{G}} |xy| \le \|y\|_q^{\mathcal{G}} \|x\|_p^{\mathcal{G}} \quad \forall x \in L_p^{\mathcal{G}}(\mathcal{F})$$

and hence $\|I_y\|^{\sim} \le \|y\|_q^{\mathcal{G}}$ for all $p \in [1, \infty)$.

If $p > 1$, let $z = |y|^{q-1} \operatorname{sgn}(y)$ and notice that $|z| = |y|^{q-1}$, therefore

$$|z|^p = |y|^q \text{ and } zy = |y|^{q-1} \operatorname{sgn}(y)\, y = |y|^q$$

then $E^{\mathcal{G}} |z|^p = E^{\mathcal{G}} |y|^q \in L_0(\mathcal{G})$ and so $z \in L_p^{\mathcal{G}}(\mathcal{F})$, moreover

$$|I_y(z)| = \left|E^{\mathcal{G}}(zy)\right| = E^{\mathcal{G}} |y|^q .$$

Now if $a \in L_0(\mathcal{G})^+$ is such that $|I_y(x)| \le a \|x\|_p^{\mathcal{G}}$ for all $x \in L_p^{\mathcal{G}}(\mathcal{F})$, then taking $x = z$ we obtain

$$a \left(E^{\mathcal{G}} |y|^q\right)^{\frac{1}{p}} = a \|z\|_p^{\mathcal{G}} \ge |I_y(z)| = E^{\mathcal{G}} |y|^q .$$

---

[31] Notice that point (i) of the definition guarantees that $\langle \cdot, \cdot\rangle^{\mathcal{G}}$ is well defined; while points (ii) and (iii) are automatically satisfied if L' and L contain the (equivalence classes of) indicators of all elements of $\mathcal{F}$.

But notice that, given $a, b \in L_0(\mathcal{G})^+$,

$$ab^{\frac{1}{p}} \geq b \iff a \geq b^{\frac{1}{q}}$$

which, for $b = E^{\mathcal{G}} |y|^q$, delivers $a \geq \|y\|_q^{\mathcal{G}}$. The arbitrariness of a delivers

$$\|I_y\|^{\sim} = \inf \left\{ a \in L_0(\mathcal{G})^+ : |I_y(x)| \leq a \|x\|_p^{\mathcal{G}} \quad \forall x \in L_p^{\mathcal{G}}(\mathcal{F}) \right\} \geq \|y\|_q^{\mathcal{G}}.$$

Else $p = 1$. Let $D = \left\{ a \in L_0(\mathcal{G})^+ : |I_y(x)| \leq a \|x\|_1^{\mathcal{G}} \quad \forall x \in L_1^{\mathcal{G}}(\mathcal{F}) \right\}$, we already observed that $\|y\|_\infty^{\mathcal{G}} \in D$. Assume, per contra, that there exists $a \in D$ such that $\|y\|_\infty^{\mathcal{G}} \nleq a$, then for all $x \in L_1^{\mathcal{G}}(\mathcal{F})$

$$|I_y(x)| \leq a \|x\|_1^{\mathcal{G}} \text{ and } |I_y(x)| \leq \|y\|_\infty^{\mathcal{G}} \|x\|_1^{\mathcal{G}}$$

imply $|I_y(x)| \leq a \|x\|_1^{\mathcal{G}} \wedge \|y\|_\infty^{\mathcal{G}} \|x\|_1^{\mathcal{G}} = \left(a \wedge \|y\|_\infty^{\mathcal{G}} \right) \|x\|_1^{\mathcal{G}}$ and so $b = a \wedge \|y\|_\infty^{\mathcal{G}} \in D$ and $b < \|y\|_\infty^{\mathcal{G}}$. Since $b \in L_0(\mathcal{G})$, $b \geq |y|$ would imply $b \geq \|y\|_\infty^{\mathcal{G}} > b$, then $F = \{\omega \in \Omega : b(\omega) < |y(\omega)|\} \in \mathcal{F}$ and $1_F b < 1_F |y|$. Set $z = 1_F \operatorname{sgn}(y) \in L_1^{\mathcal{G}}(\mathcal{F})$ and notice that $|z| = 1_F$, then

$$\begin{aligned} |I_y(z)| &= \left| E^{\mathcal{G}} (1_F \operatorname{sgn}(y) y) \right| = E^{\mathcal{G}} (1_F |y|) \\ \text{(but } E^{\mathcal{G}} \text{ is strictly positive, so)} &> E^{\mathcal{G}} (1_F b) = b E^{\mathcal{G}} (1_F) = b \|z\|_1^{\mathcal{G}} \end{aligned}$$

which contradicts $b \in D$. Therefore $\|y\|_\infty^{\mathcal{G}}$ is the minimum of D, that is, $\|y\|_\infty^{\mathcal{G}} = \|I_y\|^{\sim}$.

Finally, I is injective because $I(y) = 0$ implies $\|y\|_q^{\mathcal{G}} = \|I(y)\|^{\sim} = 0$, and so $\ker(I) = \{0\}$.■

8.2. Metric completeness of $L_p^{\mathcal{G}}(\mathcal{F})$, $1 \leq p < \infty$. In this section we consider $L_p^{\mathcal{G}}(\mathcal{F})$ endowed with its Lévy metric

$$d_p(x, y) = d_{L_0(\mathcal{G})} \left(\|x - y\|_p^{\mathcal{G}}, 0 \right) = E\psi \left(\|x - y\|_p^{\mathcal{G}} \right) \quad \forall x, y \in L_p^{\mathcal{G}}(\mathcal{F})$$

where $\psi(t) = t \wedge 1$ for all $t \in \mathbb{R}^+$. By the conditional Hölder inequality $\|x\|_1^{\mathcal{G}} \leq \|x\|_r^{\mathcal{G}} \leq \|x\|_p^{\mathcal{G}}$ if $1 \leq r \leq p < \infty$. Therefore by the conditional Jensen inequality and since ψ is concave and monotone, for every $x \in L_p^{\mathcal{G}}(\mathcal{F})$,

$$E\psi \left(\|x\|_p^{\mathcal{G}} \right) \geq E\psi \left(\|x\|_1^{\mathcal{G}} \right) = E\psi \left(E^{\mathcal{G}} |x| \right) \geq E\psi \left(E^{\mathcal{G}} (|x| \wedge 1) \right) \geq E \left(E^{\mathcal{G}} \psi (|x| \wedge 1) \right) = E\psi(|x|)$$

and so

$$d_p(x, y) = E\psi \left(\|x - y\|_p^{\mathcal{G}} \right) \geq E\psi(|x - y|) = d_{L_0(\mathcal{F})}(x, y) \quad \forall x, y \in L_p^{\mathcal{G}}(\mathcal{F}).$$

We can conclude that every Cauchy sequence in $L_p^{\mathcal{G}}(\mathcal{F})$ is a Cauchy sequence in $L_0(\mathcal{F})$.

THEOREM 7. *For every $p \in [1, \infty)$, $\left(L_p^{\mathcal{G}}(\mathcal{F}), d_p \right)$ is a Frechet lattice.*

PROOF. Let $p \in [1, \infty)$. We already observed that $L_p^{\mathcal{G}}(\mathcal{F})$ is a submodule and hence a vector subspace of $L_0(\mathcal{F})$, and the topology induced by d_p is linear (see Proposition 7).

Moreover if $x \in L_p^{\mathcal{G}}(\mathcal{F})$ and $y \in L_0(\mathcal{F})$, then

$$|y| \leq |x| \implies E^{\mathcal{G}} |y|^p \leq E^{\mathcal{G}} |x|^p \tag{8.2}$$

in turn this means that $y \in L_p^{\mathcal{G}}(\mathcal{F})$, showing that $L_p^{\mathcal{G}}(\mathcal{F})$ is an ideal in $L_0(\mathcal{F})$. Inequality (8.2) also shows that every open ball centered at 0 in $\left(L_p^{\mathcal{G}}(\mathcal{F}), d_p \right)$ is solid, thus $\left(L_p^{\mathcal{G}}(\mathcal{F}), d_p \right)$ is a locally solid Riesz space. At this point we only have to prove completeness.

Let $\{y_n\}$ be a Cauchy sequence in $L_p^{\mathcal{G}}(\mathcal{F})$, it is enough to prove that it admits a subsequence that converges in $L_p^{\mathcal{G}}(\mathcal{F})$. Since $\{y_n\}$ is also a Cauchy sequence in $L_0(\mathcal{F})$, then it converges in probability to some $x \in L_0(\mathcal{F})$ and it admits a subsequence that converges to x a.s., but such a subsequence is a Cauchy sequence in $L_p^{\mathcal{G}}(\mathcal{F})$. Therefore, it suffices to prove that if $\{x_n\}$ is a

Cauchy sequence in $L_p^{\mathcal{G}}(\mathcal{F})$ that a.s. converges to $x \in L_0(\mathcal{F})$ it also converges to x in $L_p^{\mathcal{G}}(\mathcal{F})$. For every $n \in \mathbb{N}$, by the conditional Fatou Lemma

$$E\psi\left(\sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p}\right) = E\psi\left(\sqrt[p]{E^{\mathcal{G}}\left[\lim_{m\to\infty}\left|x_n - x_m\right|^p\right]}\right) \le E\psi\left(\sqrt[p]{\underline{\lim}_{m\to\infty}E^{\mathcal{G}}\left|x_n - x_m\right|^p}\right).$$

Moreover,

$$\varphi = \psi \circ \sqrt[p]{\,} : \mathbb{R}^+ \to [0,1]$$

is continuous and increasing, and so $\varphi\left(\underline{\lim}_{m\to\infty}t_m\right) = \underline{\lim}_{m\to\infty}\varphi\left(t_m\right)$ for any sequence $\{t_m\} \subseteq \mathbb{R}^+$. Thus, for every $n \in \mathbb{N}$, by the unconditional Fatou Lemma

$$\begin{aligned}0 &\le E\psi\left(\sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p}\right) \le E\varphi\left(\underline{\lim}_{m\to\infty}E^{\mathcal{G}}\left|x_n - x_m\right|^p\right) = E\left[\underline{\lim}_{m\to\infty}\varphi\left(E^{\mathcal{G}}\left|x_n - x_m\right|^p\right)\right]\\ &\le \underline{\lim}_{m\to\infty}E\varphi\left(E^{\mathcal{G}}\left|x_n - x_m\right|^p\right) = \underline{\lim}_{m\to\infty}d_p\left(x_n, x_m\right)\end{aligned}$$

and the latter quantity vanishes as $n \to \infty$ since $\{x_n\}$ is a Cauchy sequence in $L_p^{\mathcal{G}}(\mathcal{F})$.

In particular, for each $\varepsilon \in (0, 1/3)$, there is $n = n_\varepsilon \in \mathbb{N}$ such that $E\left(\sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p} \wedge 1\right) < \varepsilon^2$, and by the Markov inequality

$$\begin{aligned}P\left\{\omega \in \Omega : \sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p(\omega)} > \varepsilon\right\} &= P\left\{\omega \in \Omega : \sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p(\omega)} \wedge 1 > \varepsilon\right\}\\ &\le \frac{E\left(\sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p} \wedge 1\right)}{\varepsilon} \le \varepsilon.\end{aligned}$$

If, per contra, $E^{\mathcal{G}}|x|^p \notin L_0(\mathcal{G})$, then there exists $G \in \mathcal{G}$ such that $P(G) > 0$ and $E^{\mathcal{G}}|x|^p(\omega) = \infty$ for all $\omega \in G$. Let $\varepsilon = P(G)/6$ and $n = n_\varepsilon$, by the conditional Minkowski inequality and since $x_n \in L_p^{\mathcal{G}}(\mathcal{F})$, there exists $W \in \mathcal{F}$ with $P(W) = 1$ such that

$$\sqrt[p]{E^{\mathcal{G}}|x|^p(\omega)} \le \sqrt[p]{E^{\mathcal{G}}\left|x_n\right|^p(\omega)} + \sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p(\omega)}$$

and $\sqrt[p]{E^{\mathcal{G}}\left|x_n\right|^p(\omega)} \in \mathbb{R}$, for all $\omega \in W$. But then $\sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p(\omega)} = \infty$ for all $\omega \in G \cap W$, and since $P(G \cap W) = P(G) = 6\varepsilon$ it follows that

$$\varepsilon \ge P\left\{\omega \in \Omega : \sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p(\omega)} > \varepsilon\right\} \ge P\left\{\omega \in \Omega : \sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p(\omega)} = \infty\right\} \ge 6\varepsilon$$

a contradiction. We conclude that $E^{\mathcal{G}}|x|^p \in L_0(\mathcal{G})$, $x \in L_p^{\mathcal{G}}(\mathcal{F})$, and

$$d_p\left(x_n, x\right) = E\psi\left(\sqrt[p]{E^{\mathcal{G}}\left|x_n - x\right|^p}\right) \to 0$$

as $n \to \infty$, as wanted. ∎

Corollary 2. *Let $p \in [1, \infty)$ and q be the conjugate exponent of p. A map $\pi : L_p^{\mathcal{G}}(\mathcal{F}) \to L_0(\mathcal{G})$ is a positive $L_0(\mathcal{G})$-linear form if and only if there exists $y \in L_q^{\mathcal{G}}(\mathcal{F})^+$ such that*

$$\pi(x) = E^{\mathcal{G}}(xy) \quad \forall x \in L_p^{\mathcal{G}}(\mathcal{F}). \tag{8.3}$$

Proof. We only prove sufficiency. Notice that $\pi : L_p^{\mathcal{G}}(\mathcal{F}) \to L_0(\mathcal{G})$ is a positive linear operator between Frechet lattices, therefore by Aliprantis and Border [**AlBo**, Theorem 9.6], π is $\left(L_p^{\mathcal{G}}(\mathcal{F}), d_p\right)$-$\left(L_0(\mathcal{G}), d_{L_0(\mathcal{G})}\right)$-continuous. By Proposition 7, π is bounded. As observed in the proof of Theorem 1, since π is positive, it can be represented in the sense of (8.3) by $y \in L_q^{\mathcal{G}}(\mathcal{F})^+$. ∎

First notice that by Theorem 6, y is unique. More importantly the argument we just discussed leads to the following, very general remark.

Remark 4. *Let E be a normed $L_0(\mathcal{G})$-module such that the metric d_E is complete, and E^+ be a d_E-closed convex cone such that $E = E^+ - E^+$. If $f : E \to L_0(\mathcal{G})$ is convex and monotone, then, by Borwein* [**Bor**, Corollary 2.4]*, f is d_E-$d_{L_0(\mathcal{G})}$-continuous.*

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