Equilibrium Prices for Monetary Utility Functions*

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Abstract

This paper provides sufficient and necessary conditions for the existence of equilibrium pricing rules for monetary utility functions under convex consumption constraints. These utility functions are characterized by the assumption of a fully fungible numeraire asset ("cash"). Each agent's utility is nominally shifted by exactly the amount of cash added to his endowment. We find the individual maximum utility that each agent is eligible for in an equilibrium and provide a game theoretic point of view for the fair allocation of the aggregate utility.

Key words: existence of equilibrium prices, monetary utility functions, Pareto optimal allocation, convex consumption constraints

1 Introduction

Monetary utility functions are characterized by the assumption of a fully fungible numeraire asset ("cash") and the property that an agent's utility is nominally shifted by exactly the amount of cash added to his endowment. This "cash invariance" introduces the possibility of "rebalancing of the cash" without restrictions at any time.

Jouini, Schachermayer and Touzi [27] provide an existence result for Pareto optimal allocations in the case of law-invariant monetary utility functions, see also Filipović and Svindland [19] for the non-monotone case. We extend the framework of [27] and consider an infinite dimensional economy, where the agents are described by convex cash invariant unbounded below consumption sets and monetary utility functions. It is well known that

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equilibria do not exist in general in such a framework. Explicit counter examples are given in Delbaen [14] and [27].

On the other hand, existence results such as in [6, 9, 11, 33] do not apply for monetary utility functions: the existence theorems for equilibria in Bewley [6] assume bounded below consumption sets in L^∞. The no-arbitrage based existence theorem in Werner [33], for unbounded below consumption sets, only works in finite dimensional model spaces. But also the extension of Werner's results towards infinite dimensions in Dana and Le Van [11] does not apply in our framework. Indeed, one can show that the set of the so-called fair utility weight vectors in [11] is always empty for monetary utility functions: in the notation of [11] it follows that "$U(0)$" is of the form $\{v \in \mathbb{R}^m \mid \langle v, 1\rangle \leq c\}$ for some $c \geq 0$; therefore the polar cone "D" is generated by $(1, \ldots, 1)$ and thus has empty interior. Finally, monetary utility functions are not of mean variance type as in Dana [9]. We refer to [11] for a more thorough overview of the economics literature on the existence issue.

This leads us to develop another concept, that of an equilibrium pricing rule. Such pricing rules support uniformly asymptotic optimal allocations. We provide sufficient and necessary conditions for the existence of equilibrium pricing rules. This applies in particular if a financial market is assumed (Example 3.11). Moreover, in our setup, we show that the existence of an equilibrium is equivalent to the existence of a Pareto optimal allocation. The equilibrium pricing rules coincide with the super-gradients of the representative agent's utility function. We make essential use of the fact that this utility function does not depend on the individual agents' initial endowments (it is just the convolution of the individual utility functions) and that the set of super-gradients is not empty if it is finite valued. For each pricing rule we then can calculate the maximum utility that each agent is qualified for given his initial endowment. Moreover, we provide a game theoretic point of view for the fair allocation of the aggregate utility.

Assuming the space L^∞ of essentially bounded payoffs, the pricing rules are only finitely additive in general. We therefore provide necessary and sufficient conditions under which the equilibrium pricing rules are given as expectation operators and thus are σ-additive.

Monetary risk measures have recently attracted much attention in the mathematical finance community, see e.g. [3, 4, 21]. Monetary utility functions are, up to the sign, identical to convex risk measures. Heath and Ku [25] characterize Pareto optimality for convex risk measures in a simpler framework without constraints, similar to the setup in [4]. That characterization is extended in Burgert and Rüschendorf [7]. In particular, they introduce trading constraints described by a linear subspace. Acciaio [1] provides explicit calculations of Pareto optimal allocations for some particular monetary utility functions. Equilibria for positively homogeneous convex (this is, coherent) risk measures in connection with financial markets have also been recently considered in Cherny [8]. Our paper is more general and encompasses the above in that it considers general concave monetary utility functions including convex trading constraints. Moreover, we provide explicit conditions for the existence of equilibrium pricing rules.

Monetary utility functions can be seen as particular type of niveloid in the spirit of [15] and [28]. Within the traditional class of expected utility functions, only the affine and entropic utility functions are monetary. In fact, the latter is defined as the scaled logarithm

of the exponential utility (see [21]), which induces the same preference order as – and is thus equivalent to – the exponential utility function. However, extending the class of expected utility functions to multi-prior utility functions (see [24]) enlarges the intersection with the class of monetary utility functions. Eventually, all monetary utility functions belong to a general class of utility functions recently introduced by Maccheroni, Marinacci and Rustichini in [28]. A typical example of preferences induced by a monetary utility function is the monotone version of the mean-variance preferences of Markovitz and Tobin (see for instance [29]). A detailed study concerning equilibria and Pareto optimality for monetary utility functions under constraints, such as provided by this paper, is therefore needed. Notice that by switching the sign, which turns utility functions into risk measures, our results can also be used to do derive fair values for optimal risk exchanges subject to constraints (see for instance [17]).

The outline of the paper is as follows. In Section 2 we introduce the notion of a monetary utility function restricted to a convex cash invariant subset of L^∞. From convex duality theory there results a representation of any monetary utility function in terms of its conjugate function. It is key that the inf in the representation is always attained (Proposition 2.5). The section concludes with the definition and basic properties of supergradients of monetary utility functions.

Section 3 contains our main results. We introduce the economy and define equilibrium pricing rules. As illustrative example we consider the case with a financial market. We then provide sufficient and necessary conditions for the existence of (both, finitely additive and σ-additive) equilibrium pricing rules. Moreover, we show that the existence of an equilibrium is equivalent to the existence of a Pareto optimal allocation in our framework. Finally, we provide a game theoretic point of view on the optimality of an equilibrium.

Section 4 contains the key results on the constrained convolution of monetary utility functions, which are at the core for the proofs of the main theorems. It also briefly discusses the representative agent view.

In Section 5 we characterize Pareto optimality and resolve some issues regarding the market clearing.

For the sake of readability, some proofs are postponed to the appendix.

2 Monetary utility functions on subsets of L^∞

Throughout this paper, we fix a probability space $(\Omega, \mathcal{F}, P)$. All equalities and inequalities between random variables are always understood in the P-almost sure sense.

L^∞ and L^1 denote the Banach spaces of all essentially bounded and integrable random variables, respectively, where random variables which are P-almost surely equal are identified. $(L^\infty)^*$ denotes the dual space of L^∞. It is well known, that $(L^\infty)^*$ can be identified with the space of all bounded finitely additive signed measures on $(\Omega, \mathcal{F})$ which vanish on P-null sets. We shall write

$$(L^\infty)^*_+ := \{\mu \in (L^\infty)^* \mid \langle \mu, \xi \rangle \geq 0 \, \forall \xi \geq 0\} \quad \text{and} \quad \mathcal{P} := \left\{\mu \in (L^\infty)^*_+ \mid \langle \mu, 1 \rangle = 1\right\}. \quad (2.1)$$

$\mathcal{P}$ is the convex set of *pricing rules*.

A function $f : L^\infty \to \overline{\mathbb{R}} := [-\infty, \infty]$ is *proper* if $f < \infty$ and its *domain*

$$\mathrm{dom}(f) := \{\xi \in L^\infty \mid f(\xi) > -\infty\}$$

is non-empty (since in this paper we consider concave functions, the signs in this definition are different than for convex functions as in [16, 30]). For a set $M \subseteq L^\infty$, we define the *restriction of f to M* as

$$f^M(\xi) := \begin{cases} f(\xi), & \xi \in M \\ -\infty, & \text{else.} \end{cases}$$

The *conjugate function* of f is defined by

$$f^*(\mu) := \inf_{\xi \in L^\infty} (\langle \mu, \xi \rangle - f(\xi)), \quad \mu \in (L^\infty)^*,$$

and the set of *super-gradients* is denoted by

$$\partial f(\xi) = \{\mu \in (L^\infty)^* \mid f(\eta) \leq f(\xi) + \langle \mu, \eta - \xi \rangle \text{ for every } \eta \in L^\infty\}, \quad \xi \in L^\infty.$$

The following characterization is fundamental (see e.g. [16, Proposition 5.1])

$$\mu \in \partial f(\xi) \quad \Leftrightarrow \quad f(\xi) + f^*(\mu) = \langle \mu, \xi \rangle. \tag{2.2}$$

The conjugate function $f^* : (L^\infty)^* \to \overline{\mathbb{R}}$ is concave and $\sigma((L^\infty)^*, L^\infty)$-upper semi-continuous. Moreover, if f is proper, concave and $\sigma(L^\infty, (L^\infty)^*)$-upper semi-continuous then Fenchel's Theorem states that

$$f^{**}(\xi) := \inf_{\mu \in (L^\infty)^*} (\langle \mu, \xi \rangle - f^*(\mu)) = f(\xi),$$

see e.g. [16, Proposition 4.1, Chapter I].

Definition 2.1 *A proper function $U : L^\infty \to \overline{\mathbb{R}}$ is called* monetary utility function *if it is*

(i) *monotone: $U(X) \geq U(Y)$ if $X \geq Y$,*

(ii) *concave: $U(\lambda X + (1-\lambda)Y) \geq \lambda U(X) + (1-\lambda)U(Y)$ for all $\lambda \in [0,1]$,*

(iii) *cash invariant: $U(X+m) = U(X) + m$ for all $m \in \mathbb{R}$.*

Remark 2.2 For a monetary utility function U we thus have $\partial U(\xi) \subset \mathcal{P}$ for all $\xi \in L^\infty$. Indeed, for every $\mu \in \partial U(\xi)$ we have $\langle \mu, \eta \rangle \geq U(\xi+\eta) - U(\xi)$ for all $\eta \in L^\infty$. For $\eta \equiv c \in \mathbb{R}$ we obtain $c\langle \mu, 1 \rangle \geq c$, which implies $\langle \mu, 1 \rangle = 1$. Moreover, $0 \leq U(\xi+\eta) - U(\xi) \leq \langle \mu, \eta \rangle$ for all $\eta \geq 0$ implies $\mu \in (L^\infty)^*_+$.

It also follows from the monotonicity and cash invariance that U is $\mathbb{R}$-valued and Lipschitz continuous with respect to the L^∞-norm, and hence $\sigma(L^\infty, (L^\infty)^*)$-upper semi-continuous.

Remark 2.3 U is a monetary utility function if and only if $\rho = -U$ is a convex risk measure on L^∞. For a detailed discussion of convex risk measures we refer to [21] and references therein. Convex risk measures are a generalization of coherent risk measures, which were introduced to the mathematical finance literature in [3, 12].

Definition 2.4 *A non-empty set $M \subseteq L^\infty$ is called* cash invariant *if $X \in M$ implies $X + m \in M$ for all $m \in \mathbb{R}$.*

The following proposition summarizes the crucial properties of the restriction U^M of U to a cash-invariant set M. The robust representation (2.4) below is well known in the mathematical finance literature when U^M is a genuine monetary utility function, see e.g. [20, 22] and Theorem 4.15 in [21]. However, in view of Remark 2.2, U^M is monotone if and only if $M = L^\infty$. We thus give a full proof in Section A. We remark that monotonicity of risk measures is discussed in some detail in [18].

Proposition 2.5 *Let U be a monetary utility function and $M \subseteq L^\infty$ be a convex closed cash invariant set. Then U^M is proper, concave, cash invariant, and $\sigma(L^\infty, (L^\infty)^*)$-upper semi-continuous. The set of super-gradients satisfies*

$$\partial U^M(\xi) \begin{cases} \supset \partial U(\xi) \cap \mathcal{P} \neq \emptyset, & \xi \in M \\ = \emptyset, & \xi \notin M. \end{cases} \tag{2.3}$$

Hence U^M can be represented by

$$U^M(\xi) = \min_{\mu \in \mathcal{P}} (\langle \mu, \xi \rangle - U^{M*}(\mu)) \quad \forall \xi \in M. \tag{2.4}$$

Moreover,

$$\sup_{\langle \mu, \eta \rangle \leq \langle \mu, \xi \rangle} U^M(\eta) = \langle \mu, \xi \rangle - U^{M*}(\mu) \quad \forall \mu \in \mathcal{P}, \quad \forall \xi \in M. \tag{2.5}$$

Property (2.5) distinguishes monetary utility functions and is key for what follows below. Indeed, it is easy to find non-monetary examples (e.g. $\Omega = \{\omega\}$, $L^\infty \equiv \mathbb{R}$, $\mathcal{P} = \{1\}$ and $U^M(\xi) = \xi/2$) where (2.5) does not hold.

Remark 2.6 The pricing rules $\mu \in \mathcal{P}$ are, by definition, finitely additive. From an applied point of view, σ-additive pricing rules μ would be desirable, since these are in fact expectations with respect to absolutely continuous probability measures $Q \ll P$. In what follows, we identify $Q \ll P$ with its density $dQ/dP \in \mathcal{P}_\sigma := \{Z \in L^1_+ \mid E[Z] = 1\}$. With the usual embedding $L^1 \subset (L^1)^{**} = (L^\infty)^*$ one then has $\mathcal{P}_\sigma \subset \mathcal{P}$.

We can then improve the robust representation (2.4) as follows: suppose U is *continuous from below*, that is, for every increasing sequence $(\xi_n)_{n \in \mathbb{N}} \subset L^\infty$ which converges to 0 almost surely, it follows $U(\xi_n) \to U(0)$. Then for any $\xi \in M$ there exists a $Z \in \mathcal{P}_\sigma$ with

$$U^M(\xi) = E[Z\xi] - U^{M*}(Z).$$

Moreover, $\partial U^M(\xi) \supset \partial U(\xi) \cap \mathcal{P}_\sigma \neq \emptyset$. For a proof see Section C.

3 Equilibria and Pareto optimal allocations under constraints

We consider a pure exchange economy with n agents. Denote $I = \{1, \dots, n\}$. Agent i is described by a convex closed cash invariant consumption set $M_i \subseteq L^\infty$ and an initial endowment $X_i \in M_i$. The preferences of agent i are represented by a monetary utility function U_i. We write

$$X := X_1 + \cdots + X_n$$

for the aggregate endowment (market portfolio), and define $\mathbb{M} := M_1 \times \cdots \times M_n \subseteq (L^\infty)^n$.

Definition 3.1 *An allocation* $(\xi_1, \dots, \xi_n)$ *is called* attainable *if* $(\xi_1, \dots, \xi_n) \in \mathbb{M}$ *and the clearing condition* $\sum_{i=1}^n \xi_i \leq X$ *holds.*

Attainable allocations are the possible sharings of the aggregate endowment X among the n agents. Notice that $P[\xi_1 + \cdots + \xi_n < X] > 0$ is allowed, which amounts to say that markets clear up to a non-negative residual. In economic terms this is usually referred to as "free disposal". We provide an interpretation in the context of optimal capital and risk sharing in [17]. See also Remark 5.2 below for a more formal interpretation.

Definition 3.2 *An attainable allocation* $(\xi_1, \dots, \xi_n)$ *together with a pricing rule* $\mu \in \mathcal{P}$ *is called an* equilibrium *if*

$$U_i(\xi_i) = \sup_{\langle \mu, \eta \rangle \leq \langle \mu, X_i \rangle,\, \eta \in M_i} U_i(\eta) \quad \textit{and} \quad \langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle \quad i = 1, \dots, n. \tag{3.6}$$

Hence each agent i optimizes his utility subject to his consumption ($\xi_i \in M_i$) and budget ($\langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle$) constraints.

We now introduce the concept of a pricing rule which supports a uniformly asymptotic equilibrium.

Definition 3.3 *We call* $\mu \in \mathcal{P}$ *an* equilibrium pricing rule *if, for every* $\varepsilon > 0$*, there exists an* ε-equilibrium*, that is, an attainable allocation* $(\xi_1, \dots, \xi_n)$ *such that*

$$\langle \mu, X_i \rangle - \varepsilon \leq \langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle \quad \textit{and} \tag{3.7}$$

$$\sup_{\langle \mu, \eta \rangle \leq \langle \mu, X_i \rangle,\, \eta \in M_i} U_i(\eta) \leq U_i(\xi_i) + \varepsilon \quad \forall i = 1, \dots, n. \tag{3.8}$$

Note that ε-optimality (3.8) is obtained uniformly across all agents, while the budget constraints (3.7) apply. Moreover, $(\xi_1, \dots, \xi_n, \mu)$ is an equilibrium only if μ is an equilibrium pricing rule. Hence, in order that an equilibrium allocation ever be found, the agents have to trade with respect to an equilibrium pricing rule. It is therefore vital to have a characterization and existence result for the set of equilibrium pricing rules.

Definition 3.4 *The* $\mathbb{M}$-constrained convolution of $U_1, \dots, U_n$ *is defined by*

$$\square_{i \in I}^{\mathbb{M}} U_i(\xi) := \sup_{\sum_i \xi_i \leq \xi,\, \xi_i \in M_i} \sum_{i=1}^n U_i(\xi_i), \quad \xi \in L^\infty. \tag{3.9}$$

The unconstrained convolution of $U_1, \dots, U_n$ *(that is,* (3.9) *for* $M_i = L^\infty$*) is simply denoted by* $\square_{i \in I} U_i$*.*

Here is our main existence result.

Theorem 3.5 *The set of equilibrium pricing rules equals* $\partial\square^{\mathbb{M}}_{i\in I}U_i(X) \subset \mathcal{P}$. *Moreover, there exists an equilibrium pricing rule* $\mu \in \partial\square^{\mathbb{M}}_{i\in I}U_i(X)$ *if and only if one of the following equivalent conditions holds:*

$$\bigcap_{i=1}^{n} \operatorname{dom}\left(U_i^{M_i*}\right) \neq \emptyset, \quad \textit{or} \tag{3.10}$$

$$\square^{\mathbb{M}}_{i\in I}U_i(X) < \infty. \tag{3.11}$$

In this case, we can interpret $\square^{\mathbb{M}}_{i\in I}U_i$ *as utility function of the representative agent:*

$$\square^{\mathbb{M}}_{i\in I}U_i(X) = \sup_{\langle\mu,Y\rangle\leq\langle\mu,X\rangle} \square^{\mathbb{M}}_{i\in I}U_i(Y), \tag{3.12}$$

and the individual maximum utility values are given by

$$\sup_{\langle\mu,\eta\rangle\leq\langle\mu,X_i\rangle,\, \eta\in M_i} U_i(\eta) = \langle\mu, X_i\rangle - U_i^{M_i*}(\mu). \tag{3.13}$$

Proof. This follows from (2.2), (2.3), (2.5) and Lemmas 4.2–4.4 below. □

Equation (3.12) states that μ is an equilibrium pricing rule if and only if it makes the representative agent not wanting to trade away from its endowment X.

Note that the right hand side of (3.13) gives an *a priori* value for the individual maximum utility agent i is eligible for in an equilibrium.

Remark 3.6 It is shown in [19] that an equilibrium allocation exists if $(\Omega, \mathcal{F}, P)$ is atomless and $U_i^{M_i}$ is law-invariant (that is, $U_i^{M_i}(\xi) = U_i^{M_i}(\eta)$ for all $\xi \overset{(d)}{=} \eta$), for all $i = 1, \ldots, n$. However, there are situations where an equilibrium pricing rule exists, but an equilibrium allocation is not attained, see [27, 14] for examples.

The following corollaries give sufficient conditions for the existence of an equilibrium pricing rule in terms of the unconstrained utility functions U_i.

Corollary 3.7 *Any of the following two equivalent conditions implies existence of an equilibrium pricing rule:*

$$\bigcap_{i=1}^{n} \operatorname{dom}\left(U_i^*\right) \neq \emptyset, \quad \textit{or} \tag{3.14}$$

$$\square_{i\in I}U_i(X) < \infty. \tag{3.15}$$

Proof. Equivalence of (3.14) and (3.15) follows from Theorem 3.5 for $M_i = L^\infty$.

Moreover, $U_i^* \leq U_i^{M_i*} < \infty$ and therefore $\operatorname{dom}(U_i^*) \subseteq \operatorname{dom}(U_i^{M_i*})$. Hence the corollary follows from (3.10). □

Corollary 3.8 *Suppose $(\Omega, \mathcal{F}, P)$ is atomless and U_i is law-invariant for all $i = 1, \ldots, n$. Then there exists an equilibrium pricing rule.*

Proof. It is shown in [27] that under the stated assumptions, $1 \in \mathrm{dom}(U_i^*)$ for all i. Hence the claim follows from Corollary 3.7. $\square$

Here is a simple example where an equilibrium pricing rule does not exist.

Example 3.9 Let $n = 2$ and fix two random variables $D_1 \neq D_2$ with $D_i \geq 0$ and $E[D_i] = 1$. Define the monetary utility functions $U_i(\xi) := E[D_i \xi]$ and let $M_i = L^\infty$. Then $\mathrm{dom}(U_i^*) = \{D_i\}$, so that (3.10) is not satisfied.

Remark 3.10 In view of Section C we can improve the results of Theorem 3.5 and the subsequent corollaries as follows. If $\square_{i \in I}^{\mathbb{M}} U_i$ is continuous from below (see Remark 2.6) then there exists an equilibrium pricing rule $\mu \in \partial \square_{i \in I}^{\mathbb{M}} U_i(X) \cap \mathcal{P}_\sigma$ if and only if (3.10) or (3.11) holds.

A sufficient condition for the continuity from below of $\square_{i \in I}^{\mathbb{M}} U_i$ is that U_1 is continuous from below and $M_1 = L^\infty$. Indeed, let $(\xi_n)_{n \in \mathbb{N}} \subset L^\infty$ be an increasing sequence which converges to 0 almost surely. In view of (3.9) and the monotonicity of U_1 we deduce (c.f. [4])

$$\begin{aligned}
\sup_{k \in \mathbb{N}} \square_{i \in I}^{\mathbb{M}} U_i(\xi_k) &= \sup_{k \in \mathbb{N}} \sup_{\eta \in L^\infty} \left\{ U_1(\xi_k - \eta) + \square_{i \geq 2} U_i^{M_i}(\eta) \right\} \\
&= \sup_{\eta \in L^\infty} \sup_{k \in \mathbb{N}} \left\{ U_1(\xi_k - \eta) + \square_{i \geq 2} U_i^{M_i}(\eta) \right\} \\
&= \sup_{\eta \in L^\infty} \left\{ U_1(-\eta) + \square_{i \geq 2} U_i^{M_i}(\eta) \right\} = \square_{i \in I}^{\mathbb{M}} U_i(0).
\end{aligned}$$

On the other hand, if we drop $M_1 = L^\infty$ then $\square_{i \in I}^{\mathbb{M}} U_i$ is not continuous from below in general (even if U_1 is continuous from below). As an example, consider the one-dimensional linear subspaces $M_1 = M_2 = \mathbb{R} \subset L^\infty$. Then, for any monetary utility functions U_1, U_2 and $X \in L^\infty$, we have

$$\square_{i \in I}^{\mathbb{M}} U_i(X) = \sup_{\xi_1 + \xi_2 \leq X,\, \xi_i \in \mathbb{R}} (U_1(\xi_1) + U_2(\xi_2)) = \sup_{\xi_1 + \xi_2 \leq \mathrm{ess\,inf\,X},\, \xi_i \in \mathbb{R}} (\xi_1 + \xi_2) = \mathrm{ess\,inf\,X},$$

which is not continuous from below if $(\Omega, \mathcal{F}, P)$ is atomless.

The above setup is very general. For the sake of illustration we should have the following example with a financial market in mind.

Example 3.11 Assume there exists a financial market consisting of $m+1$ securities with discounted payoffs $S_0, S_1, \ldots, S_m \in L^\infty$, where $S_0 \equiv 1$ is the numeraire asset. We let

$$D := \left\{ \sum_{j=0}^{m} \beta_j S_j \mid \beta_j \in \mathbb{R} \right\}$$

denote the space of attainable payoffs by trading. The consumption set of agent i is then defined as

$$M_i = X_i + D, \tag{3.16}$$

that is, any attainable allocation is now of the form $\xi_i = X_i + \sum_{j=0}^m \beta_{i,j} S_j$.

Now suppose that (3.14) holds. Then Corollary 3.7 asserts the existence of equilibrium prices

$$s_j := \langle \mu, S_j \rangle \quad j = 0, \ldots, m$$

of the securities in the financial market, for some equilibrium pricing rule $\mu \in \partial \Box_{i \in I}^{\mathbb{M}} U_i(X)$. The budget constraint for agent i in (3.6) boils down to

$$\sum_{j=0}^m \beta_{i,j} s_j \leq 0, \quad i = 1, \ldots, n,$$

and by individual utility maximization (3.6), the economy will eventually come to an equilibrium.

This framework can be extended to include individual convex trading (such as short-selling) constraints by replacing D in (3.16) by

$$D_i := \left\{ \sum_{j=0}^m \beta_j S_j \mid \beta_0 \in \mathbb{R},\, (\beta_1, \ldots, \beta_m) \in W_i \right\},$$

for some closed convex set $W_i \subseteq \mathbb{R}^m$ (note that X_i may already contain shares of $S_0, \ldots, S_m$).

Alternatively, we may define the above consumption sets as

$$M_i = \{\xi \in X_i + D_i \mid \mathrm{E}\,[(\xi - \mathrm{E}\,[\xi])_-] \leq \kappa_i\}$$

for some risk bound $\kappa_i \geq 0$ such that $\mathrm{E}\,[(X_i - \mathrm{E}\,[X_i])_-] \leq \kappa_i$. The interpretation is clear: every agent faces individual trading constraints in terms of size (D_i) and riskiness (κ_i) of its portfolio.

This example can be further extended by replacing $\mathrm{E}\,[(\xi - \mathrm{E}\,[\xi])_-]$ by a so called generalized deviation measure $\mathcal{D}_i : L^\infty \to \mathbb{R}_+$, recently introduced in [31] (see also [18] for more examples), such that $\xi \mapsto \rho_i(\xi) := \mathrm{E}\,[-\xi] + \mathcal{D}_i(\xi)$ becomes a coherent risk measure. In any case, our results assert the existence of equilibrium prices.

3.1 Pareto optimality

Strongly linked to equilibrium is another optimality concept:

Definition 3.12 *An attainable allocation* $(\xi_1, \ldots, \xi_n)$ *is called* Pareto optimal *if for all attainable* $(\eta_1, \ldots, \eta_n)$ *we have*

$$U_i(\eta_i) \geq U_i(\xi_i) \quad \forall i = 1, \ldots, n \quad \textit{only if} \quad U_i(\eta_i) = U_i(\xi_i) \quad \forall i = 1, \ldots, n.$$

It is a classical result from the economic theory (Welfare Theorem) that an equilibrium is Pareto optimal. Conversely, a Pareto optimal allocation is an equilibrium up to transfer payments (see Dana and Jeanblanc [10] for the finite dimensional case, and Becker [5] for a survey of Welfare Theorems in infinite dimensions). Because of the cash-invariance of U_i and M_i we have in fact equivalence, as stated by the following theorem, the proof of which is given in Section B.

Theorem 3.13 *$(\xi_1, \ldots, \xi_n, \mu)$ is an equilibrium if and only if*

(i) *$(\xi_1, \ldots, \xi_n)$ is Pareto optimal and*

(ii) *$\mu \in \partial \Box^{\mathbb{M}}_{i \in I} U_i(X)$ and*

(iii) *$\langle \mu, \xi_i \rangle = \langle \mu, X_i \rangle$ for all $i = 1, \ldots, n$.*

Conversely, for every Pareto optimal allocation $(\xi_1, \ldots, \xi_n)$ and $\mu \in \partial \Box^{\mathbb{M}}_{i \in I} U_i(X)$ we have that

$$(\xi_1 + \langle \mu, X_1 - \xi_1 \rangle, \ldots, \xi_n + \langle \mu, X_n - \xi_n \rangle, \mu) \quad \textit{is an equilibrium.} \tag{3.17}$$

It is shown in Theorem 5.1 below, see (5.29), that the existence of a Pareto optimal allocation implies the existence of an equilibrium pricing rule $\mu \in \partial \Box^{\mathbb{M}}_{i \in I} U_i(X)$ and thus of an equilibrium.

3.2 Game theoretic view

In this section, we show that no subset (coalition) of agents can overturn the aggregate market utility of the entire economy. Indeed, we obtain a "fair allocation" of the aggregate market utility across the agents. Coherent utility function based allocation methods are further discussed in [13].

An allocation $(k_1, \ldots, k_n) \in \mathbb{R}^n$ is said to be in the *core of the game* with *characteristic function*

$$\{1, \ldots, n\} \supseteq S \mapsto \Box^{\mathbb{M}}_{i \in S} U_i \left(\sum_{i \in S} X_i \right) := \sup_{\sum_{i \in S} \eta_i \leq \sum_{i \in S} X_i,\, \eta_i \in M_i} \sum_{i \in S} U_i(\eta_i) \tag{3.18}$$

if

$$\sum_{i=1}^{n} k_i = \Box^{\mathbb{M}}_{i \in I} U_i(X) \quad \text{and} \tag{3.19}$$

$$\sum_{i \in S} k_i \geq \Box^{\mathbb{M}}_{i \in S} U_i \left(\sum_{i \in S} X_i \right) \quad \text{for all } S \subseteq \{1, \ldots, n\}. \tag{3.20}$$

Property (3.19) is simply the Pareto optimality, as we shall see in Theorem 5.1, Property (ii), below. We now obtain a stronger version:

Corollary 3.14 *Let $(\xi_1, \ldots, \xi_n, \mu)$ be an equilibrium. Then*

$$(U_1(\xi_1), \ldots, U_n(\xi_n)) = (\langle \mu, X_1 \rangle - U_1^{M_1 *}(\mu), \ldots, \langle \mu, X_n \rangle - U_n^{M_n *}(\mu)) \tag{3.21}$$

lies in the core of the game (3.18).

Proof. Property (3.19) and (3.21) follow from Theorem 3.13 and (2.5). For (3.20) we simply calculate

$$\begin{aligned}\sum_{i \in S} (\langle \mu, X_i \rangle - U_i^{M_i *}(\mu)) &= \sup_{\sum_{i \in S} \eta_i \leq \sum_{i \in S} X_i \,, \eta_i \in M_i} \sum_{i \in S} (\langle \mu, \eta_i \rangle - U_i^{M_i *}(\mu)) \\ &\geq \sup_{\sum_{i \in S} \eta_i \leq \sum_{i \in S} X_i \,, \eta_i \in M_i} \sum_{i \in S} U_i(\eta_i) = \Box_{i \in S}^{\mathbb{M}} U_i \left(\sum_{i \in S} X_i \right).\end{aligned}$$

□

For bounded below consumption sets, this "core equivalence" of equilibria is proved in Gabszewicz [23].

4 On the $\mathbb{M}$-constrained convolution

The unconstrained convolution, $\Box_{i \in I} U_i$, of $U_1, \ldots, U_n$ (see Definition 3.4) has been studied in e.g. [4]. By monotonicity of U_i one easily sees that

$$\Box_{i \in I} U_i(\xi) = \sup_{\sum_i \xi_i \leq \xi} \sum_{i=1}^n U_i(\xi_i) = \sup_{\sum_i \xi_i = \xi} \sum_{i=1}^n U_i(\xi_i). \tag{4.22}$$

However, Example 4.1 below shows that

$$\Box_{i \in I}^{\mathbb{M}} U_i(\xi) > \sup_{\sum_i \xi_i = \xi, \, \xi_i \in M_i} \sum_{i=1}^n U_i(\xi_i) \tag{4.23}$$

is possible, even if $\xi \in \sum_i M_i$. Hence a naive extension of (4.22) to the constrained case is not possible. On the other hand, it follows by inspection that we have equality in (4.23) if all $M_i \equiv M$ coincide and are linear, which is the assumption made in [7].

Example 4.1 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $L^\infty \cong \mathbb{R}^3$, and $n = 2$. Consider the linear independent random variables $X_0 = (1, 1, 1)$, $X_1 = (1, 0, 0)$, $X_2 = (-1, 0, 1)$, and the convex closed cash invariant sets

$$M_1 = \operatorname{span}\{X_0, X_1\}, \quad M_2 = \operatorname{span}\{X_0, X_2\}.$$

The aggregate endowment is $X = (0, 0, 1) = X_1 + X_2$. Note that this decomposition is unique. Let U_i be the worst case utility function, i.e. $U_i(\xi) = \min\{\xi(\omega_1), \xi(\omega_2), \xi(\omega_3)\}$. We then calculate

$$\sup_{\xi_1 + \xi_2 = X \,, \, \xi_i \in M_i} (U_1(\xi_1) + U_2(\xi_2)) = U(X_1) + U(X_2) = 0 - 1 = -1.$$

On the other hand, since $0 \leq X$ we get

$$\sup_{\xi_1+\xi_2 \leq X\,,\, \xi_i \in M_i} (U_1(\xi_1) + U_2(\xi_2)) \geq U_1(0) + U_2(0) = 0.$$

This proves (4.23).

The following key lemmas provide a characterization and the necessary and sufficient conditions under which the $\mathbb{M}$-constrained convolution is well-behaved. This is an extension of [27, Lemma 2.1].

Lemma 4.2 *The $\mathbb{M}$-constrained convolution $\square^{\mathbb{M}}_{i\in I} U_i : L^\infty \to \overline{\mathbb{R}}$ is monotone, concave and cash-invariant, and its conjugate function is*

$$(\square^{\mathbb{M}}_{i\in I} U_i)^*(\mu) = \begin{cases} \sum_{i=1}^n U_i^{M_i *}(\mu), & \text{if } \mu \in (L^\infty)^*_+ \\ -\infty, & \text{else.} \end{cases} \tag{4.24}$$

Proof. Monotonicity and concavity of $\square^{\mathbb{M}}_{i\in I} U_i$ is obvious (e.g. the proof of [30, Theorem 5.4] carries over to our setup). Cash invariance follows from cash invariance of $U_i^{M_i}$.

Let $\mu \in (L^\infty)^*$, then we have

$$\begin{aligned}
(\square^{\mathbb{M}}_{i\in I} U_i)^*(\mu) &= \inf_{\xi \in L^\infty} \left(\langle \mu, \xi \rangle - \square^{\mathbb{M}}_{i\in I} U_i(\xi) \right) = \inf_{\xi \in L^\infty} \left(\langle \mu, \xi \rangle - \sup_{\sum_{i=1}^n \xi_i \leq \xi\,,\, \xi_i \in M_i} \sum_{i=1}^n U_i(\xi_i) \right) \\
&\leq \inf_{\xi \in \sum_{i=1}^n M_i} \left(\inf_{\sum_{i=1}^n \xi_i = \xi\ \xi_i \in M_i} \sum_{i=1}^n (\langle \mu, \xi_i \rangle - U_i(\xi_i)) \right) \\
&= \sum_{i=1}^n \inf_{\xi_i \in M_i} (\langle \mu, \xi_i \rangle - U_i(\xi_i)) = \sum_{i=1}^n U_i^{M_i *}(\mu).
\end{aligned}$$

On the other hand, for $\mu \in (L^\infty)^*_+$, we have

$$\begin{aligned}
(\square^{\mathbb{M}}_{i\in I} U_i)^*(\mu) &= \inf_{\xi \in L^\infty} \left(\langle \mu, \xi \rangle - \square^{\mathbb{M}}_{i\in I} U_i(\xi) \right) \\
&\geq \inf_{\xi \in L^\infty} \left(\inf_{\sum_{i=1}^n \xi_i \leq \xi\,,\, \xi_i \in M_i} \sum_{i=1}^n (\langle \mu, \xi_i \rangle - U_i(\xi_i)) \right) \\
&= \sum_{i=1}^n \inf_{\xi_i \in M_i} (\langle \mu, \xi_i \rangle - U_i(\xi_i)) = \sum_{i=1}^n U_i^{M_i *}(\mu).
\end{aligned}$$

If $\mu \notin (L^\infty)^*_+$, then there exists $\xi \in L^\infty_+$ with $\langle \mu, \xi \rangle < 0$. But $\square^{\mathbb{M}}_{i\in I} U_i(n\xi) \geq 0$, for all $n \geq 1$, and hence

$$(\square^{\mathbb{M}}_{i\in I} U_i)^*(\mu) \leq \langle \mu, n\xi \rangle - \square^{\mathbb{M}}_{i\in I} U_i(n\xi) \leq n \langle \mu, \xi \rangle \quad \forall n \geq 1.$$

This proves (4.24). $\square$

Lemma 4.3 *The set of equilibrium pricing rules equals* $\partial\Box^{\mathbb{M}}_{i\in I}U_i(X) \subset \mathcal{P}$.

Proof. That $\partial\Box^{\mathbb{M}}_{i\in I}U_i(X) \subset \mathcal{P}$ follows from the monotonicity and cash invariance as in Remark 2.2.

Now let $\mu \in \mathcal{P}$ be an equilibrium pricing rule, and let $\varepsilon > 0$. By definition, there exists an attainable allocation $(\xi_1, \dots, \xi_n)$ satisfying (3.7) and (3.8). In view of (2.5) and (4.24), we thus have

$$\begin{aligned}\Box^{\mathbb{M}}_{i\in I}U_i(X) &\geq \sum_{i=1}^n U_i^{M_i}(\xi_i) \geq \sum_{i=1}^n \Big(\langle \mu, X_i\rangle - U_i^{M_i*}(\mu) - \varepsilon\Big)\\ &\geq \langle \mu, X\rangle - (\Box^{\mathbb{M}}_{i\in I}U_i)^*(\mu) - n\varepsilon \geq \Box^{\mathbb{M}}_{i\in I}U_i(X) - n\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\Box^{\mathbb{M}}_i U_i(X) = \langle \mu, X\rangle - (\Box^{\mathbb{M}}_{i\in I}U_i)^*(\mu)$, and hence $\mu \in \partial\Box^{\mathbb{M}}_{i\in I}U_i(X)$, see (2.2).

Conversely, let $\mu \in \partial\Box^{\mathbb{M}}_{i\in I}U_i(X) \subset \mathcal{P}$ and $\varepsilon > 0$. Then, in view of (2.2), $\Box^{\mathbb{M}}_i U_i(X) = \langle \mu, X\rangle - (\Box^{\mathbb{M}}_{i\in I}U_i)^*(\mu)$ and there exists an attainable allocation $(\xi'_1, \dots, \xi'_n)$ such that

$$\sum_{i=1}^n U_i(\xi'_i) + \varepsilon \geq \Box^{\mathbb{M}}_i U_i(X) = \langle \mu, X\rangle - (\Box^{\mathbb{M}}_{i\in I}U_i)^*(\mu) \geq \sum_{i=1}^n \Big(\langle \mu, X_i\rangle - U_i^{M_i*}(\mu)\Big), \quad (4.25)$$

where we have used (4.24). By rebalancing the cash, see (5.28) below, we can find an attainable allocation $(\xi''_1, \dots, \xi''_n)$ which satisfies (4.25) instead of $(\xi'_1, \dots, \xi'_n)$ and

$$\langle \mu, X_i\rangle - U_i^{M_i*}(\mu) \leq U_i(\xi''_i) + \frac{\varepsilon}{n} \leq \langle \mu, \xi''_i\rangle - U_i^{M_i*}(\mu) + \frac{\varepsilon}{n} \quad \forall i = 1, \dots, n. \quad (4.26)$$

Hence $\langle \mu, X_i\rangle \leq \langle \mu, \xi''_i\rangle + \frac{\varepsilon}{n}$. From $\sum_i X_i \geq \sum_i \xi''_i$ we conclude that

$$\langle \mu, X_i\rangle - \frac{\varepsilon}{n} \leq \langle \mu, \xi''_i\rangle \leq \langle \mu, X_i\rangle + \frac{(n-1)\varepsilon}{n} \quad \forall i = 1, \dots, n. \quad (4.27)$$

It is then clear from (4.26),(4.27) and (2.5) that $\xi_i = \xi''_i - \frac{(n-1)\varepsilon}{n}$ satisfy (3.7) and (3.8). Thus μ is an equilibrium pricing rule. □

Lemma 4.4 $\Box^{\mathbb{M}}_{i\in I}U_i$ *is proper – and thus a monetary utility function – if and only if* (3.10) *or* (3.11) *holds.*

Proof. It follows from the monotonicity and cash-invariance (Lemma 4.2) that $\Box^{\mathbb{M}}_{i\in I}U_i$ is proper if and only if (3.11) holds. Equivalence of (3.10) and (3.11) follows from (4.24) and the fact that

$$-\infty < \operatorname{ess\,inf}(\xi - X) + \sum_{i=1}^n U_i^{M_i}(X_i) \leq \Box^{\mathbb{M}}_{i\in I}U_i(\xi) \leq \langle \mu, \xi\rangle - (\Box^{\mathbb{M}}_{i\in I}U_i)^*(\mu), \quad \forall \xi \in L^\infty.$$

□

5 Characterization of Pareto optimality

We consider the setup of Section 3 and provide some preliminary results of independent interest on Pareto optimality, which will be used for the proof of Theorem 3.13. We also resolve the sub-clearing of markets by introducing a maximally risk averse dummy agent.

First, notice that Pareto optimal allocations are not unique in our setup. Indeed, for every attainable allocation $(\xi_1, \dots, \xi_n)$ we can arbitrarily *rebalance the cash* without changing the aggregate utility $\sum_{i=1}^n U_i(\xi_i)$. That is, for every constant vector of cash transitions $(c_1, \dots, c_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n c_i = 0$ we have that $(\xi_1 + c_1, \dots, \xi_n + c_n)$ is attainable and by the cash invariance of U_i

$$\sum_{i=1}^n U_i(\xi_i + c_i) = \sum_{i=1}^n U_i(\xi_i). \tag{5.28}$$

The following characterization result for Pareto optimality is an extension of [27, 7, 25].

Theorem 5.1 *Let $(\xi_1, \dots, \xi_n)$ be an attainable allocation and set $\xi_0 := X - \sum_{i=1}^n \xi_i \geq 0$. Then the following properties are equivalent:*

(i) *$(\xi_1, \dots, \xi_n)$ is Pareto optimal,*

(ii) *$\square_{i\in I}^{\mathbb{M}} U_i(X) = \sum_{i=1}^n U_i(\xi_i)$,*

(iii) *$\langle \mu, \xi_0 \rangle = 0$ and $U_i(\xi_i) = \langle \mu, \xi_i \rangle - U_i^{M_i *}(\mu)$, $\forall i = 1, \dots, n$, for some $\mu \in \mathcal{P}$,*

(iv) *$\langle \mu, \xi_0 \rangle = 0$ and $\mu \in \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$,*

(v) *$U_0(\xi_0) = 0$ and $\mu \in \partial U_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$, where $U_0(\xi) := \operatorname{ess\,inf} \xi$ is the* worst case utility function.

(vi) *$\mu \in \partial V_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$, where*

$$V_0(\xi) := \begin{cases} 0, & \xi \in L_+^\infty \\ -\infty, & \textit{else} \end{cases}$$

is the concave indicator function of L_+^∞.

Moreover, any of the above properties implies

$$\begin{aligned} \partial \square_{i\in I}^{\mathbb{M}} U_i(X) &= \partial U_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n) \\ &= \partial V_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n) \neq \emptyset. \end{aligned} \tag{5.29}$$

Proof. (i)⇒(ii): Suppose that $\sum_{i=1}^n U_i(\xi_i) < \square_{i\in I}^{\mathbb{M}} U_i(X)$. Then there exists an attainable allocation $(\eta_1, \dots, \eta_n)$ with $\sum_{i=1}^n U_i(\xi_i) < \sum_{i=1}^n U_i(\eta_i)$. By rebalancing the cash we can find an attainable allocation $(\eta_1', \dots, \eta_n')$ such that $U_1(\xi_1) < U_1(\eta_1')$ and $U_i(\xi_i) \leq U_i(\eta_i')$ for all $i = 2, \dots, n$. But then $(\xi_1, \dots, \xi_n)$ is not Pareto optimal.

(ii)⇒(i) follows from the definition of Pareto optimality.

(iii)⇒(ii): follows from

$$\square^{\mathbb{M}}_{i\in I} U_i(X) \geq \sum_{i=1}^{n} U_i(\xi_i) = \langle \mu, \xi_0 \rangle + \sum_{i=1}^{n} (\langle \mu, \xi_i \rangle - U_i^{M_i *}(\mu))$$
$$= \langle \mu, X \rangle - (\square^{\mathbb{M}}_{i\in I} U_i)^*(\mu) \geq \square^{\mathbb{M}}_{i\in I} U_i(X). \quad (5.30)$$

(ii)⇒(iii): property (ii) implies (3.11) and thus there exists a $\mu \in \partial \square^{\mathbb{M}}_{i\in I} U_i(X)$, by Lemma 4.4 and (2.3). Then $\mu \in \mathcal{P}$ by Lemma 4.3, and (2.2) yields

$$\square^{\mathbb{M}}_{i\in I} U_i(X) = \langle \mu, X \rangle - (\square^{\mathbb{M}}_{i\in I} U_i)^*(\mu) \geq \sum_{i=1}^{n} (\langle \mu, \xi_i \rangle - U_i^{M_i *}(\mu)) \geq \sum_{i=1}^{n} U_i(\xi_i) = \square^{\mathbb{M}}_{i\in I} U_i(X). \quad (5.31)$$

Hence $U_i(\xi_i) = \langle \mu, \xi_i \rangle - U_i^{M_i *}(\mu)$ for all $i = 1, \ldots, n$ and $\langle \mu, \xi_0 \rangle = 0$.

(iii)⇔(iv) follows from (2.2).

(iv)⇔(v): it is readily checked that, for $\mu \in \mathcal{P}$,

$$\langle \mu, \xi_0 \rangle = 0 \quad \Leftrightarrow \quad \mu \in \partial U_0(\xi_0) = \{\nu \in \mathcal{P} \mid \langle \nu, \xi_0 \rangle = U_0(\xi_0)\} \text{ and } U_0(\xi_0) = 0.$$

(iv)⇔(vi): follows from the fact that $\partial V_0(\xi_0) = \{\nu \in (L^\infty)^*_+ \mid \langle \nu, \xi_0 \rangle = 0\}$ (note that $\partial V_0(\xi) = \emptyset$ for $\xi \notin L^\infty_+$).

Moreover, (5.30) and (5.31) together with (2.2) and the above arguments imply (5.29).□

Remark 5.2 Property (vi) ((v)) says that markets clear in a Pareto optimal allocation by introducing a dummy agent with utility function V_0 (U_0). In fact, V_0 (U_0) is the most risk averse concave (monetary) utility function, in the sense that every concave (monetary) utility function W with $W(0) = 0$ satisfies $W \geq V_0$ ($W \geq U_0$), see [18] for a proof.

A Proof of Proposition 2.5

It follows by inspection that U^M is proper, concave and cash invariant. Since U^M is also monotone on M, we have that

$$\left| U^M(X) - U^M(Y) \right| \leq ||X - Y||_\infty \quad \text{for all } X, Y \in M.$$

Hence, for all $c \in \mathbb{R}$, the set $\left\{ X \in L^\infty \mid U^M(X) \geq c \right\}$ is L^∞-closed and, since convex, therefore $\sigma(L^\infty, (L^\infty)^*)$-closed (see [32, Theorem 3.12]). Hence U^M is $\sigma(L^\infty, (L^\infty)^*)$-upper semi-continuous.

From the monotonicity and cash invariance of U it follows that

$$U(\xi) = U^{**}(\xi) = \inf_{\mu \in \mathcal{P}} (\langle \mu, \xi \rangle - U^*(\mu)), \quad \xi \in L^\infty, \quad (A.32)$$

see e.g. [21]. To see that the inf is actually attained, let $\xi \in L^\infty$, and write $g(\mu) := \langle \mu, \xi \rangle - U^*(\mu)$. Since $|\langle \mu, \eta \rangle| \leq \langle \mu, 1 \rangle \|\eta\|_\infty = \|\eta\|_\infty$ for all $\eta \in L^\infty$, the Alaoglu Compactness

Theorem ([2, Theorem 5.93]) implies that $\mathcal{P}$ is $\sigma((L^\infty)^*, L^\infty)$-compact. On the other hand, g is $\sigma((L^\infty)^*, L^\infty)$-lower semi-continuous. Hence, by a generalization of Weierstrass' Theorem ([2, Theorem 2.40]), g attains a minimum on $\mathcal{P}$. In view of (2.2), we also have $\partial U(\xi) \cap \mathcal{P} \neq \emptyset$.

On the other hand, for every $\mu \in (L^\infty)^*$ we have

$$U^*(\mu) = \inf_{\xi \in L^\infty} (\langle \mu, \xi \rangle - U(\xi)) \leq \inf_{\xi \in L^\infty} (\langle \mu, \xi \rangle - U^M(\xi)) = U^{M*}(\mu).$$

From this we obtain, for $\xi \in M$ and $\mu \in \partial U(\xi)$,

$$U^M(\xi) = U(\xi) = \langle \mu, \xi \rangle - U^*(\mu) \geq \langle \mu, \xi \rangle - U^{M*}(\mu) \geq U^M(\xi),$$

showing that $U^M(\xi) = \langle \mu, \xi \rangle - U^{M*}(\mu)$. Hence, by (2.2) again, $\partial U(\xi) \subset \partial U^M(\xi)$. This proves (2.3) and (2.4).

Let $\mu \in \mathcal{P}$. Then $\langle \mu, \eta + \langle \mu, \xi - \eta \rangle \rangle = \langle \mu, \xi \rangle$. Hence

$$\begin{aligned} \sup_{\langle \mu, \eta \rangle \leq \langle \mu, \xi \rangle} U^M(\eta) &= \sup_{\eta \in L^\infty} U^M(\eta + \langle \mu, \xi - \eta \rangle) \\ &= \langle \mu, \xi \rangle - \inf_{\eta \in L^\infty} (\langle \mu, \eta \rangle - U^M(\eta)) = \langle \mu, \xi \rangle - U^{M*}(\mu), \end{aligned}$$

which is (2.5).

B Proof of Theorem 3.13

Let $(\xi_1, \ldots, \xi_n, \mu)$ be an equilibrium. Then μ is an equilibrium pricing rule and Property (ii) follows from Lemma 4.3. Moreover, $U_i^{M_i}(\xi_i) = U_i(\xi_i) = \sup_{\langle \mu, \zeta \rangle \leq \langle \mu, \xi_i \rangle} U_i^{M_i}(\zeta)$ and $\langle \mu, \xi_i \rangle = \langle \mu, X_i \rangle$, for all $i = 1, \ldots, n$. Indeed, otherwise by cash invariance $\xi_i + \langle \mu, X_i - \xi_i \rangle \in M_i$ would yield a strictly bigger utility than ξ_i. Hence $\langle \mu, \xi_0 \rangle = 0$, for ξ_0 as in Theorem 5.1, and (2.2) implies that $\mu \in \bigcap_{i=1}^n \partial U_i^{M_i}(\xi_i)$. Now Theorem 5.1 yields Pareto optimality of $(\xi_1, \ldots, \xi_n)$.

Conversely, let $(\xi_1, \ldots, \xi_n)$ be Pareto optimal and $\mu \in \partial \Box_{i \in I}^{\mathbb{M}} U_i(X)$. Then $\eta_i := \xi_i + \langle \mu, X_i - \xi_i \rangle \in M_i$ and $\xi_0 := X - \sum_{i=1}^n \xi_i$ satisfy, by Theorem 5.1,

$$\sum_{i=1}^n \eta_i = \sum_{i=1}^n \xi_i + \langle \mu, \xi_0 \rangle = \sum_{i=1}^n \xi_i \leq X$$

and $U_i(\eta_i) = U_i(\xi_i) + \langle \mu, X_i - \xi_i \rangle = \langle \mu, X_i \rangle - U_i^{M_i*}(\mu)$ for all $i = 1, \ldots, n$. Hence $(\eta_1, \ldots, \eta_n)$ is attainable and

$$U_i(\zeta_i) \leq \langle \mu, \zeta_i \rangle - U_i^{M_i*}(\mu) \leq \langle \mu, X_i \rangle - U_i^{M_i*}(\mu) = U_i(\eta_i)$$

for all $\zeta_i \in M_i$ with $\langle \mu, \zeta_i \rangle \leq \langle \mu, X_i \rangle$. Therefore $(\eta_1 \ldots, \eta_n, \mu)$ is an equilibrium, which proves (3.17).

That (i)–(iii) imply the equilibrium property is a simple consequence of (3.17).

C σ-additive pricing rules

The proof of Proposition 2.5 in Section A uses that $\mathcal{P}$ is $\sigma((L^\infty)^*, L^\infty)$-compact and the conjugate function U^* is $\sigma((L^\infty)^*, L^\infty)$-upper semi-continuous. Hence if we can provide sufficient conditions such that the *level sets*

$$\mathcal{Q}_\sigma^K = \{Q \in \mathcal{P}_\sigma \,|\, U^*(Q) \geq -K\}\,, \quad K \in \mathbb{N},$$

are $\sigma(L^1, L^\infty)$-compact and U is $\sigma(L^\infty, L^1)$-upper semi-continuous, then for any $\xi \in L^\infty$ we have $\partial U(\xi) \cap \mathcal{P}_\sigma \neq \emptyset$. It turns out that continuity from below of U is enough. This is the content of our next theorem, which is a generalization of Theorem 3.6 in [12] to the non-coherent case.

Theorem C.1 *For a monetary utility function U with conjugate function U^* the following properties are equivalent:*

(i) *U is continuous from below (see Remark 2.6);*

(ii) *U is $\sigma(L^\infty, L^1)$-upper semi-continuous and $\mathcal{Q}_\sigma^K$ is $\sigma(L^1, L^\infty)$-compact for all $K \in \mathbb{N}$.*

Proof. (i)⇒(ii): Monotonicity and concavity of U implies $U(\xi) \leq U(\xi + \eta) \leq 2U(\xi) - U(\xi - \eta)$ for all $\eta \geq 0$. Hence U is continuous from above and thus $\sigma(L^\infty, L^1)$-upper semi-continuous (see [21, Theorem 4.31]).

Now suppose that $\mathcal{Q}_\sigma^K$ is not $\sigma(L^1, L^\infty)$-compact for some $K \in \mathbb{N}$. Then, by the Dunford–Pettis Theorem (see e.g. Theorem A.67 in [21]), $\mathcal{Q}_\sigma^K$ is not uniformly integrable. Hence there exists a decreasing sequence A_n of elements in $\mathcal{F}$ which converges to $\emptyset$, such that

$$\limsup_{n\to\infty} \inf_{Z \in \mathcal{Q}_\sigma^K} E[-Z1_{A_n}] < 0\,.$$

We thus can find a constant $C \in \mathbb{N}$ such that $\limsup_{n\to\infty} \inf_{Z\in\mathcal{Q}_\sigma^K} E[-CZ1_{A_n}] \leq -K-1$. In view of (2.4), we infer

$$\begin{aligned} U(-C1_{A_n}) &\leq \inf_{Z\in\mathcal{Q}_\sigma^K} \{E[-CZ1_{A_n}] - U^*(Z)\} \\ &\leq -1. \end{aligned}$$

But the sequence $-C1_{A_n}$ increases to 0, which contradicts (i).

(ii)⇒(i): since U is $\sigma(L^\infty, L^1)$-upper semi-continuous, we conclude that that $\mathrm{dom}(U^*) \cap \mathcal{P}_\sigma$ is $\sigma((L^\infty)^*, L^\infty)$-dense in $\mathrm{dom}(U^*)$. Hence the $\sigma((L^\infty)^*, L^\infty)$-closure of $\mathcal{Q}_\sigma^K$ satisfies $\overline{\mathcal{Q}_\sigma^K} = \{\mu \in \mathcal{P} \,|\, U^*(\mu) \geq -K\}$.

Now fix an increasing sequence $(\xi_n)_{n\in\mathbb{N}} \subset L^\infty$ which converges to 0 almost surely. Let $\varepsilon > 0$. For $A_n = \{\omega \in \Omega \,|\, \xi_n(\omega) < -\varepsilon\}$ it follows that $P[A_n] \to 0$. Since $(\xi_n)_{n\in\mathbb{N}}$ is L^∞-bounded, there exists $K \in \mathbb{N}$ such that

$$\begin{aligned} U(\xi_n) - U(0) &= \inf_{Z\in\overline{\mathcal{Q}_\sigma^K}} \{E[Z\xi_n] - U^*(Z)\} - U(0) \\ &\geq \inf_{Z\in\mathcal{Q}_\sigma^K} E[Z(\xi_n 1_{A_n} + \xi_n 1_{A_n^c})] \\ &\geq -||\xi_0||_\infty \inf_{Z\in\mathcal{Q}_\sigma^K} E[Z1_{A_n}] - \varepsilon\,. \end{aligned}$$

The first inequality follows, because U^* takes only values in $[-\infty, -U(0)]$. By the Dunford–Pettis Theorem, $\inf_{Z \in \mathcal{Q}_\sigma^K} E[Z 1_{A_n}]$ tends to zero. The monotonicity of U implies $U(\xi_n) \to U(0)$. $\square$

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