Stochastic order-monotone uncertainty-averse preferences*

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Abstract

In this paper we derive a numerical representation for general complete preferences respecting the following two principles: a) more is better than less, b) averages are better than extremes. To be able to distinguish between risk and ambiguity we work in an Anscombe–Aumann framework. Our main result is a quasi-concave numerical representation for a class of preferences wide enough to accommodate Ellsberg- as well as Allais-type behavior. Instead of assuming the usual monotonicity we suppose that our preferences are monotone with respect to first order stochastic dominance. Preference for averages expresses uncertainty-aversion. We do not make independence assumptions of any form. In general, our preferences intertwine attitudes towards risk and ambiguity. But if one assumes a weak form of Savage's sure thing principle, there is separation between risk and ambiguity attitudes, and the representation decomposes into state-dependent preference functionals over the consequences and a quasi-concave functional aggregating the preferences of the decision maker in different states of the world.

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1 Introduction

The goal of this paper is to derive a numerical representation that can describe general complete preference relations that respect the following two principles:

a) more is better than less

b) averages are better than extremes.

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To be able to distinguish between risk and ambiguity we work in an Anscombe–Aumann framework of horse lotteries whose outcomes are roulette lotteries. a) is a natural property for any preference over scarce goods. The standard approach in the literature is to consider an abstract set of prizes and formalize a) by requiring a preference to be monotone with respect to the order it induces on the roulette lotteries. But this relates the rankings of roulette lotteries in different states of the world to each other and implies weak forms of the sure thing principle and state-independence. In this paper we consider monetary prizes and use the natural order among real numbers to describe a). This keeps our formulation of a) free of implicit assumptions that go beyond any pure notion of monotonicity and allows for general orderings of roulette lotteries in different states of the world as well as intertwinement of risk and ambiguity attitudes. The mathematical description of b) is that the preference is convex. It has been used as a definition of uncertainty-aversion by e.g. Schmeidler (1989), Gilboa and Schmeidler (1989), Maccheroni et al. (2006) or Cerreia-Vioglio et al. (2011), which have all generalized the subjective expected utility representation of Anscombe and Aumann (1963) by weakening the independence axiom. Here, we do not make independence assumptions of any form. This leads to a class of preferences with the following features:

- In general, they intertwine attitudes towards risk and ambiguity.
- If they separate risk and ambiguity attitudes, the induced preference orders on the roulette lotteries are state-dependent.
- They can accommodate the following two well-documented violations of expected utility theory: Ellsberg-type behavior, which is inconsistent with any preference order based on a single probability distribution over possible outcomes (see Ellsberg, 1961), and Allais-type behavior, which contradicts the independence axiom even in situations where uncertain events occur according to known objective probabilities (see Allais, 1953).

Allais- and Ellsberg-type behavior are two of the most extensively studied deviations from expected utility theory and have given rise to a large body of literature aiming at extending standard models of decision making under uncertainty. Most of the existing work concentrates on one of the two phenomena. Approaches that address both at the same time include Gul and Pesendorfer (2013), Dean and Ortoleva (2014) and Bommier (2015). The first work with Savage acts and subjective sources. The other two consider Anscombe–Aumann acts. But Dean and Ortoleva (2014) replace the standard mixture by an outcome mixture in the style of Ghirardato et al. (2003), while Bommier (2015) does not concentrate on convex preferences but instead, assumes a comonotonic sure-thing principle.

In this paper we use the standard mixture corresponding to the convex combination of acts viewed as mappings from the state space to a set of measures. More precisely, we work with the collection L of Anscombe–Aumann acts $f : S \to P_c$, where $S = \{1, \ldots, m\}$ is a finite state space and P_c denotes the set of all Borel probability measures on $\mathbb{R}$ with compact support. Anscombe–Aumann acts have been used in more general form. But an element $f \in L$ has the straightforward interpretation of a monetary payoff subject to two different types of uncertainty. If the state of the world is $s \in S$, the payoff is distributed according to $f_s \in P_c$. In the terminology of Anscombe and Aumann (1963), the draw of the true state $s \in S$ is a *horse lottery* and the payoff according to f_s a *roulette lottery*.

An individual might have beliefs on which of the elements of S is the true state of the world. But there are no objective probabilities associated with them. Such a situation is also referred to as *Knightian uncertainty, model uncertainty* or *ambiguity*. In case the true state is $s \in S$, f yields a payoff governed by the distribution $f_s \in P_c$. This kind of uncertainty is usually called *measurable uncertainty* or *risk*.

For a number $\alpha \in [0,1]$, the mixture $\alpha f + (1-\alpha)g$ of two acts $f, g \in L$ is defined to be the act with consequences $\alpha f_s + (1-\alpha)g_s$, $s \in S$, where $\alpha f_s + (1-\alpha)g_s$ is the convex combination of f_s and g_s in the vector space M_c of signed Borel measures of bounded variation with compact support on $\mathbb{R}$. So L is a convex subset of the vector space M_c^S consisting of all functions $f : S \to M_c$. The following two sets can be embedded in L:

- Roulette lotteries over $\mathbb{R}$: P_c
- Horse lotteries over $\mathbb{R}$: $H := \{h : S \to \mathbb{R}\} = \mathbb{R}^S$.

A roulette lottery $\mu \in P_c$ can be identified with the constant act f^μ defined by $f^\mu_s = \mu$, $s \in S$, and a horse lottery $h \in H$ with the deterministic act $f_s = \delta_{h(s)}$, $s \in S$, where $\delta_{h(s)}$ denotes the Dirac measure at $h(s)$. However, for $h, h' \in H$ the mixture of the deterministic acts $\delta_h, \delta_{h'} \in L$ does not correspond to the convex combination of h and h' in $\mathbb{R}^S$; that is, in general, $\alpha\delta_{h(s)} + (1-\alpha)\delta_{h'(s)}$ is different from $\delta_{\alpha h(s)+(1-\alpha)h'(s)}$. We refer to the latter as the deterministic mixture of acts $h \in H$.

In the whole paper $\succcurlyeq$ is a *complete preference* on L; that is, it satisfies

(A1) Weak order:
$\succcurlyeq$ is a transitive binary relation on L with the property that for all $f, g \in L$, $f \succcurlyeq g$ or $f \preccurlyeq g$.

As usual, $\sim$ and $\succ$ denote the symmetric and asymmetric part of $\succcurlyeq$, respectively. We formalize principle a) above as follows:

(A2) Monotonicity with respect to first order stochastic dominance:
For all $f, g \in L$ satisfying $f_s \trianglerighteq_1 g_s$ for every $s \in S$, one has $f \succcurlyeq g$,

where $\trianglerighteq_1$ denotes first order stochastic dominance[1] between lotteries in P_c. In the general theory, acts typically take values in an abstract set, and the usual monotonicity condition is as follows:

(SM) Standard monotonicity:[2] For all $f, g \in L$ satisfying $f_s \succcurlyeq_* g_s$ for every $s \in S$, one has $f \succcurlyeq g$,

where $\mu \succcurlyeq_* \nu$ means $f^\mu \succcurlyeq f^\nu$ for the constant acts f^μ and f^ν corresponding to μ and ν. But (SM) has consequences that go beyond pure notions of monotonicity. For instance, it implies[3] the following weak versions of Savage's sure thing principle P2 and ordinal event independence P3 (Savage, 1954):

[1] That is, $\mu \trianglerighteq_1 \nu$ means $\mu\,[x,\infty) \geq \nu\,[x,\infty)$ for all $x \in \mathbb{R}$, or equivalently, $\int_{\mathbb{R}} h d\mu \geq \int_{\mathbb{R}} h d\nu$ for all nondecreasing continuous functions $h : \mathbb{R} \to \mathbb{R}$.

[2] (SM) has been used in e.g. Schmeidler (1989), Gilboa and Schmeidler (1989), Maccheroni et al. (2006) and Cerreia-Vioglio et al. (2011). Anscombe and Aumann (1963) also assumed (SM) but with respect to a preference on the roulette lotteries that is given from the beginning.

[3] That (P3') is a consequence of (SM) is clear. Moreover, if (SM) holds, then $\mu 1_A + f 1_{A^c} \succ \nu 1_A + f 1_{A^c}$ implies $\mu \succ_* \nu$, from which it follows that $\mu 1_A + g 1_{A^c} \succcurlyeq \nu 1_A + g 1_{A^c}$. This shows that (SM) implies (P2').

(P2') For all $\mu, \nu \in P_c$, $f, g \in L$ and $A \subseteq S$, $\mu 1_A + f 1_{A^c} \succ \nu 1_A + f 1_{A^c}$ implies $\mu 1_A + g 1_{A^c} \succcurlyeq \nu 1_A + g 1_{A^c}$

(P3') For all $\mu, \nu \in P_c$, $f \in L$ and $A \subseteq S$, $\mu \succcurlyeq_* \nu$ implies $\mu 1_A + f 1_{A^c} \succcurlyeq \nu 1_A + f 1_{A^c}$.

Axiom (A2) has the advantage that it does not mix notions of monotonicity with implicit assumptions of separability or state-independence. Moreover, it makes it possible to derive general representation results under weak continuity assumptions that are easy to test.

A preference is said to exhibit risk-aversion if it is averse to mean preserving spreads, or equivalently, monotone with respect to the concave order. Under (A2), this is equivalent[4] to

(A2') Monotonicity with respect to second order stochastic dominance:
If $f, g \in L$ satisfy $f_s \trianglerighteq_2 g_s$ for all $s \in S$, then $f \succcurlyeq g$,

where $\trianglerighteq_2$ denotes second order stochastic dominance[5].

If principle b) is understood with respect to the standard mixture of Anscombe–Aumann acts, it corresponds to

(A3) Convexity: If $f, g, h \in L$ satisfy $f \succcurlyeq h$ and $g \succcurlyeq h$, then $\alpha f + (1-\alpha) g \succcurlyeq h$ for all $\alpha \in (0,1)$,

which means that the upper contour sets of $\succcurlyeq$ are convex. If (A1) holds, (A3) is equivalent to the simpler condition: for all $f, g \in L$ satisfying $f \succcurlyeq g$, one has $\alpha f + (1-\alpha) g \succcurlyeq g$ for every $\alpha \in (0,1)$.

Schmeidler (1989), as well as e.g., Gilboa and Schmeidler (1989), Maccheroni et al. (2006) and Cerreia-Vioglio et al. (2011) used (A3) as definition of uncertainty-aversion. It is important to note that it differs from the following convexity with respect to deterministic mixtures of deterministic acts:

(A3') d-convexity: If $f, g, h \in H = \mathbb{R}^S$ satisfy $\delta_f \succcurlyeq \delta_h$ and $\delta_g \succcurlyeq \delta_h$, then $\delta_{\alpha f + (1-\alpha) g} \succcurlyeq \delta_h$ for all $\alpha \in (0,1)$.

A related notion of convexity naturally arises in applications in optimal asset allocation. Asset prices under model uncertainty can be modeled with the set K of all bounded measurable functions $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ on a measurable space equipped with different probability measures $\mathbb{P}_1, \dots, \mathbb{P}_m$. If an asset $X \in K$ is assessed purely based on its distributions μ_s^X under the measures $\mathbb{P}_s$, $s = 1, \dots, m$, it is enough to consider its image under the mapping $\psi(X) := (\mu_1^X, \dots, \mu_m^X) \in L$. Convex combinations $\alpha X + (1-\alpha) Y$ of elements in K describe portfolio diversification. So, adapting Definition 2 of Dekel (1989) to our setup, we say a preference relation $\succcurlyeq$ on L exhibits diversification if the following holds:

(A3") D-convexity[6]: For every specification of $(\Omega, \mathcal{F}, \mathbb{P}_1, \dots, \mathbb{P}_m)$, the induced preference $X \succcurlyeq_\psi Y$, given by $\psi(X) \succcurlyeq \psi(Y)$, is convex on K; that is, for all $X, Y, Z \in K$ satisfying $\psi(X) \succcurlyeq \psi(Z)$ and $\psi(Y) \succcurlyeq \psi(Z)$, one has $\psi(\alpha X + (1 - \alpha Y)) \succcurlyeq \psi(Z)$ for every $\alpha \in (0,1)$.

[4]See, e.g., Proposition 2.1 in Dana (2005).

[5]Defined by $\int_{\mathbb{R}} h d\mu \geq \int_{\mathbb{R}} h d\nu$ for all nondecreasing concave functions $h : \mathbb{R} \to \mathbb{R}$.

[6]Note that depending on the form of $(\Omega, \mathcal{F}, \mathbb{P}_1, \dots, \mathbb{P}_m)$, the image of the mapping $\psi : K \to L$ might not be all of L. But one has $\psi(K) = L$ if $(\Omega, \mathcal{F}, \mathbb{P}_1, \dots, \mathbb{P}_m)$ is rich enough; for instance if Ω is the unit interval $(0,1]$ with the Borel σ-algebra and $\mathbb{P}_s$ are the uniform distributions on $((s-1)/m, s/m]$, $s = 1, \dots m$. So in contrast to (A3'), (A3") is a condition on the full preference order $\succcurlyeq$ and not only its restriction to a subset of L.

(A3”) is stronger[7] than (A3’). On the other hand, it follows[8] from the arguments in the proof of Proposition 3 in Dekel (1989) that (A3”) follows from (A2’) and (A3).

Our main result is a numerical representation for preferences satisfying (A1)–(A3) together with a weak semicontinuity condition. It is of the form

$$V(f) = \inf_{u \in I^S} A\left(u, \langle u, f\rangle\right), \tag{1.1}$$

where

- I is the set of continuous nondecreasing functions $h : \mathbb{R} \to \mathbb{R}$
- $\langle u, f\rangle := \sum_{s\in S} \int_{\mathbb{R}} u_s df_s$
- $A : I^S \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is a function that is nondecreasing in the second argument.

For stochastic order-monotone preferences that are uncertainty-averse and risk-averse, we derive a representation of the form

$$V(f) = \inf_{u \in I_c^S} A\left(u, \langle u, f\rangle\right), \tag{1.2}$$

for the set of nondecreasing concave functions $I_c \subseteq I$ and a mapping $A : I_c^S \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ nondecreasing in the second argument. In general, (1.1) and (1.2) intertwine risk and ambiguity attitudes. But under a suitable version of the sure thing principle and a condition ensuring that state-wise certainty equivalents exist, (1.1) simplifies to

$$V(f) = \inf_{u \in I^S} A\left(u, \sum_{s\in S} u_s(c_s(f_s))\right), \tag{1.3}$$

where c_s are certainty equivalents of the preferences on P_c induced by $\succcurlyeq$ in the different states of the world $s \in S$. If in addition, $\succcurlyeq$ is d-convex, it is representable as

$$V(f) = \inf_{p\in\Delta} B\Big(p, \sum_{s\in S} p_s c_s(f_s)\Big), \tag{1.4}$$

where Δ is the set of all probability measures on the state space S and $B : \Delta \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ a function that is nondecreasing in the second argument. We show that as a special case, (1.4) includes risk-averse uncertainty-averse preferences which separate attitudes towards risk and ambiguity. Similarly, if $\succcurlyeq$ satisfies a sure thing principle and in every state $s \in S$, the von Neumann–Morgenstern axioms for preferences over roulette lotteries hold, it can be represented as

$$V(f) = \inf_{p\in\Delta} B\Big(p, \sum_{s\in S} p_s \int_{\mathbb{R}} u_s df_s\Big), \tag{1.5}$$

[7] Choose for instance, $\Omega = \{1, \ldots, m\}$, $\mathcal{F} = 2^\Omega$ and $\mathbb{P}_s = \delta_s$, $s = 1, \ldots, m$. Then the induced preference $\succcurlyeq_\psi$ is convex on K if and only if $\succcurlyeq$ is d-convex.

[8] See Lemma 4.5 below for details.

for nondecreasing right-continuous functions $u_s : \mathbb{R} \to \mathbb{R}$. This is a variant of the representation derived by Cerreia-Vioglio et al. (2011). The main difference is that in (1.5), u_s can depend on the state s while in Cerreia-Vioglio et al. (2011) it does not. Like the preference of Cerreia-Vioglio et al. (2011), (1.5) can accommodate Ellsberg-type behavior. But since it is affine on P_c, it cannot explain Allais' paradox. The representations (1.1)–(1.4) on the other hand, can cope with both.

In the special case where S consists of only one element, L reduces to the roulette lotteries P_c, and (1.1) becomes

$$V(\mu) = \inf_{u \in I} D\Big(u, \int_{\mathbb{R}} u d\mu\Big), \quad \mu \in P_c, \tag{1.6}$$

for a function $D : I \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ that is nondecreasing in $\int_{\mathbb{R}} u d\mu$. For every $u \in I$, $\int_{\mathbb{R}} u d\mu$ is an expected utility. But (1.6) takes different evaluation functions $u \in I$ into account and weighs them according to D. Maccheroni (2002) and Cerreia-Vioglio (2009) derived versions of (1.6) with no monotonicity but stronger continuity assumptions. Under monotonicity and slightly different continuity assumptions, (1.6) was previously derived by Drapeau and Kupper (2013) as well as Cheridito et al. (2013).

Coming back to our general representation result, is easy to see that every preference $\succcurlyeq$ with a representation of the form (1.1) fulfills (A1)–(A3). But to derive (1.1) from (A1)–(A3), an additional continuity assumption is needed. If, for instance, one requires $\succcurlyeq$ to be $\sigma(L, C^S)$-upper semicontinuous, where C is the set of all continuous functions $h : \mathbb{R} \to \mathbb{R}$, it follows from general convex duality arguments that a representation of the form (1.1) exists. However, $\sigma(L, C^S)$-upper semicontinuity is a technical condition, which is hard to test in practice. In its place we use the following axiom:

(A4) Upper semicontinuity:
a) For all $f, g, h \in L$ with $f \succcurlyeq g \succ h$, there exists an $\alpha \in (0, 1)$ such that $g \succ \alpha f + (1 - \alpha)h$.
b) For all $f, g \in L$, $T_\varepsilon f \succcurlyeq g$ for each $\varepsilon > 0$ implies $f \succcurlyeq g$, where T_ε denotes the ε-shift defined by $(T_\varepsilon f)_s(E) := f_s(E - \varepsilon)$ for the Borel sets E in $\mathbb{R}$.

(A4.a) is a one-sided version of the classical

(AA) Archimedean axiom:
For all $f, g, h \in L$ with $f \succ g \succ h$, there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h$.

(A4.b) means that the preference is upper semicontinuous under translation. For instance, if $\mu = \sum_{i=1}^n \lambda_i \delta_{x_i}$ is a simple lottery with finitely many possible payoffs $x_i \in \mathbb{R}$ and probabilities $\lambda_i \geq 0$ summing up to 1, then $T_\varepsilon \mu = \sum_{i=1}^n \lambda_i \delta_{x_i + \varepsilon}$. That is, T_ε shifts the payoffs x_i by ε and keeps the probabilities λ_i unchanged. (A4.b) says that if an act f state-wise shifted to the right by an arbitrary small amount $\varepsilon > 0$ is weakly preferred to another act g, then f itself is weakly preferred to g. In particular, (A4.a) and (A4.b) are both one-dimensional semicontinuity conditions with a normative appeal that can be tested in experiments.

The rest of the paper is organized as follows. Section 2 contains our main result, which shows that every preference on L satisfying the axioms (A1)–(A4) has a numerical representation of the form (1.1). As a special case we derive the representation (1.2) for risk-averse preferences fulfilling

(A1)–(A4). In Section 3 we concentrate on the case where the state space S contains just one element. Then $\succcurlyeq$ becomes a preference over the roulette lotteries P_c, and the general representation (1.1) takes on the form (1.6). We show that very simple non-affine specifications of (1.6) are enough to accommodate Allais-style behavior. We also derive a von Neumann–Morgenstern representation under weak continuity assumptions. In Section 4 we introduce additional axioms guaranteeing that a preference over L separates attitudes towards risk and ambiguity. For such preferences, numerical representations of the form (1.3)–(1.5) are derived. All proofs are given in the appendix.

2 General representation results

In this section we provide two representation results for uncertainty-averse preferences on L. Our main result, Theorem 2.1, gives a numerical representation for general preferences satisfying (A1)–(A4). Corollary 2.3 provides the analog for risk-averse preferences.

We denote by $\mathcal{A}$ the set of all functions $A : I^S \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying the condition

(R1) $A(u, x)$ is nondecreasing and right-continuous in x,

and by $\mathcal{A}^{\min}$ those functions $A \in \mathcal{A}$ which also fulfill

(R2) $A(u, x)$ is quasi-convex in (u, x)

(R3) $\sup_x A(u, x) = \sup_x A(u', x)$ for all $u, u' \in I^S$

(R4) $A^-(u, x) := \sup_{y<x} A(u, y)$ is $\sigma(I^S, L)$-lower semicontinuous in u

(R5) $A(\lambda u, x) = A(u, x/\lambda)$ for all $\lambda \in \mathbb{R}_+ \setminus \{0\}$.

It is straightforward to check that for every function $A \in \mathcal{A}$,

$$V(f) = \inf_{u \in I^S} A(u, \langle u, f \rangle) \tag{2.1}$$

defines a quasi-concave functional $V : L \to \mathbb{R} \cup \{\pm\infty\}$ respecting first order stochastic dominance $\trianglerighteq_1$. As a consequence, the corresponding preference relation satisfies (A1)–(A3). Furthermore, the following holds:

Theorem 2.1. *For a preference $\succcurlyeq$ on L, the following are equivalent:*

(i) *$\succcurlyeq$ satisfies* (A1)–(A4)

(ii) *$\succcurlyeq$ has a numerical representation of the form* (2.1) *for a function $A \in \mathcal{A}$.*

Moreover, if (ii) *holds, there exists a unique function $\hat{A} \in \mathcal{A}^{\min}$ inducing the same preference functional V as A, and this $\hat{A}$ satisfies $\hat{A} \leq A$.*

Remark 2.2. Theorem 2.1 shows that there is a one-to-one correspondence between preference functionals V of the form (2.1) and functions $A \in \mathcal{A}^{\min}$. But of course, it is possible that different preference functionals in the class (2.1) represent the same preference order on L; for instance, V and $\exp(V)$. In particular, the aggregator A in (2.1) can be monotonically transformed so that it takes values in any non-trivial closed subinterval of $\mathbb{R}$, such as for instance, $[0, \infty]$ or $[0, 1]$.

The following is a version of Theorem 2.1 for risk-averse preferences. Denote by $\mathcal{A}_c$ the set of all functions $A : I_c^S \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying (R1) and by $\mathcal{A}_c^{\min}$ the subset of those $A \in \mathcal{A}_c$ satisfying (R1)–(R5).

Corollary 2.3. *For a preference $\succcurlyeq$ on L, the following are equivalent:*

(i) $\succcurlyeq$ *satisfies* (A1), (A2'), (A3), (A4)

(ii) $\succcurlyeq$ *has a numerical representation of the form* $V(f) = \inf_{u \in I_c^S} A(u, \langle u, f \rangle)$ *for a function* $A \in \mathcal{A}_c$.

Moreover, if (ii) *holds, there exists a unique function* $\hat{A} \in \mathcal{A}_c^{\min}$ *inducing the same numerical representation V as A, and one has $\hat{A} \le A$.*

3 Preferences on roulette lotteries

If there is only one state of the world $s \in S$, the set L reduces to the roulette lotteries P_c, and the axioms (A1)–(A4), (A2') become

(a1) Weak order:
$\succcurlyeq$ is a transitive binary relation on P_c with the property that for all $\mu, \nu \in P_c$, either $\mu \succcurlyeq \nu$ or $\mu \preccurlyeq \nu$.

(a2) Monotonicity with respect to first order stochastic dominance:
For all $\mu, \nu \in P_c$ satisfying $\mu \trianglerighteq_1 \nu$, one has $\mu \succcurlyeq \nu$.

(a3) Convexity:
If $\mu, \nu, \eta \in P_c$ satisfy $\mu \succcurlyeq \eta$ and $\nu \succcurlyeq \eta$, then $\alpha\mu + (1-\alpha)\nu \succcurlyeq \eta$ for all $\alpha \in (0,1)$.

(a4) Upper semicontinuity:
a) For all $\mu, \nu, \eta \in P_c$ with $\mu \succcurlyeq \nu \succ \eta$, there exists an $\alpha \in (0,1)$ such that $\nu \succ \alpha\mu + (1-\alpha)\eta$.
b) For all $\mu, \nu \in P_c$, $T_\varepsilon \mu \succcurlyeq \nu$ for each $\varepsilon > 0$ implies $\mu \succcurlyeq \nu$.

(a2') Monotonicity with respect to second order stochastic dominance:
For all $\mu, \nu \in P_c$ satisfying $\mu \trianglerighteq_2 \nu$, one has $\mu \succcurlyeq \nu$.

3.1 Numerical representation of convex preferences over roulette lotteries

Denote by $\mathcal{D}$ the set of all functions $D : I \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying condition (R1) and by $\mathcal{D}^{\min}$ the subset of those $D \in \mathcal{D}$ with the properties (R1)–(R5). Analogously, $\mathcal{D}_c$ is the set of all functions $D : I_c \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying (R1) and $\mathcal{D}_c^{\min}$ the subset of those $D \in \mathcal{D}_c$ fulfilling (R1)–(R5). The following is an immediate consequence of Theorem 2.1 and Corollary 2.3:

Corollary 3.1. *For a preference $\succcurlyeq$ on P_c, the implications* (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) $\Leftrightarrow$ (iv) *hold among the conditions*

(i) $\succcurlyeq$ *satisfies* (a1)–(a4)

(ii) $\succcurlyeq$ *has a numerical representation of the form* $V(\mu) = \inf_{u \in I} D\left(u, \int_{\mathbb{R}} u d\mu\right)$ *for a function* $D \in \mathcal{D}$

(iii) $\succcurlyeq$ *satisfies* (a1), (a2'), (a3), (a4)

(iv) $\succcurlyeq$ *has a numerical representation of the form* $V(\mu) = \inf_{u \in I_c} D\left(u, \int_{\mathbb{R}} u d\mu\right)$ *for a function* $D \in \mathcal{D}_c$

Moreover, if (ii) *holds, there exists a unique* $\hat{D} \in \mathcal{D}^{\min}$ *inducing the same preference functional* V *as* D*, and this* $\hat{D}$ *satisfies* $\hat{D} \leq D$*. Similarly, if* (iv) *holds, there exists a unique* $\hat{D} \in \mathcal{D}_c^{\min}$ *inducing the same numerical representation* V *as* D*, and one has* $\hat{D} \leq D$*.*

3.2 Allais paradox

In this subsection we show that a simple non-affine specification of the representation (iv) in Corollary 3.1 can resolve Allais' paradox. Various experiments have shown that most people prefer A to B and D to C, where A is a lottery that pays \$1 million for sure, B a lottery paying nothing with probably 1%, \$1 million with probability 89% and \$5 million with probability 10%, C a lottery paying nothing with probability 89% and \$1million with probability 11%, and D a lottery paying nothing with probability 90% and \$5 million with probability 10% . This contradicts the independence axiom for preferences over roulette lotteries:

(ia) Independence axiom:
For all $\mu, \nu, \eta \in P_c$, $\mu \succ \nu$ implies $\alpha\mu + (1-\alpha)\eta \succ \alpha\nu + (1-\alpha)\eta$ for every $\alpha \in (0,1)$,

which is satisfied by any preference with a von Neumann–Morgenstern representation of the form $V(\mu) = \int_{\mathbb{R}} u d\mu$; see Allais (1953). This has given rise to a number of alternatives to excepted utility theory such as prospect theory (Kahneman and Tversky, 1979), weighted expected utility (Chew and MacCrimmon, 1979), rank-dependent utility (Quiggin, 1982) and cumulative prospect theory (Tversky and Kahneman, 1992). However, Allais-style behavior is also consistent with e.g. the following version of (iv) in Corollary 3.1:

$$V(\mu) := \min_{i=1,2} V_i(\mu) \quad \text{for } V_i := \int_{\mathbb{R}} u_i d\mu,$$

where $u_1(x) = x$ and u_2 is a continuous function such that

$$\begin{aligned} u_2(0) &= 100,000 \\ u_2(1,000,000) &= 1,000,000 \\ u_2(5,000,000) &= 1,050,000. \end{aligned}$$

Indeed, $V_1(A) = V_2(A) = 1,000,000$, $V_1(B) = 1,390,000$, $V_2(B) = 991,000$, $V_1(C) = 110,000$, $V_2(C) = 199,000$, $V_1(D) = 500,000$, $V_2(D) = 195,000$, and therefore,

$$V(A) = 1,000,000 > V(B) = 991,000 \quad \text{and} \quad V(C) = 110,000 < V(D) = 195,000.$$

3.3 Certainty equivalents

Preferences with a representation as in Corollary 3.1 do not always admit certainty equivalents. But the next proposition shows that they do if the following holds:

(ce) Existence of unique certainty equivalents:
a) $\delta_x \succ \delta_y$ for all $x, y \in \mathbb{R}$ such that $x > y$
b) if $x \in \mathbb{R}$ and $\mu \in P_c$ are such that $\delta_x \succ \mu$, there exists an $\varepsilon > 0$ such that $\delta_{x-\varepsilon} \succcurlyeq \mu$.

(ce.a) means that the preference restricted to deterministic outcomes is strictly monotone, and (ce.b) is a one-dimensional lower semicontinuity condition. Both have normative appeal and can be tested. Also, it is clear that any complete preference on P_c having the monotonicity property (a2) must satisfy (ce) if it admits unique certainty equivalents.

Proposition 3.2. *Let $\succcurlyeq$ be a preference on P_c satisfying* (a1)–(a4) *and* (ce). *Then the mapping*

$$c(\mu) := \inf \{x \in \mathbb{R} : \delta_x \succcurlyeq \mu\} \tag{3.1}$$

provides unique certainty equivalents and can be written as

$$c(\mu) = \inf_{u \in I} D\Big(u, \int_{\mathbb{R}} u d\mu\Big) \tag{3.2}$$

for a unique function $D \in \mathcal{D}^{\min}$. If in addition, $\succcurlyeq$ satisfies (a2'), *it has a representation of the form*

$$c(\mu) = \inf_{u \in I_c} D\Big(u, \int_{\mathbb{R}} u d\mu\Big) \tag{3.3}$$

for a unique $D \in \mathcal{D}_c^{\min}$.

3.4 Von Neumann–Morgenstern representations

For later use, we here derive a von Neumann–Morgenstern representation of the form

$$V(\mu) = \int_{\mathbb{R}} u d\mu \tag{3.4}$$

for a nondecreasing upper semicontinuous function $u : \mathbb{R} \to \mathbb{R}$. Note that if u is not continuous, there exist lotteries $\mu \in P_c$ without a certainty equivalent. Moreover, if u is not strictly increasing there are numbers $x > y$ such that $\delta_x \sim \delta_y$. But of course, a preference with representation (3.4) always satisfies (ia) together with

(aa) Archimedean axiom:
For all $\mu, \nu, \eta \in P_c$, with $\mu \succ \nu \succ \eta$, there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha\mu + (1-\alpha)\eta \succ \nu \succ \beta\mu + (1-\beta)\eta.$$

If u does not attain its infimum and supremum, the preference induced by (3.4) also satisfies

(ub) Unboundedness: For all $x \in \mathbb{R}$, there exist $y, z \in \mathbb{R}$ such that $\delta_y \succ \delta_x \succ \delta_z$.

As a consequence of Corollary 3.1, the following version of the representation result of von Neumann and Morgenstern (1947) can be derived.

Proposition 3.3. *Every preference order on P_c satisfying* (a1), (a2), (a4.b)*,* (ia) *and* (aa) *has a numerical representation of the form* (3.4) *for a nondecreasing right-continuous function $u : \mathbb{R} \to \mathbb{R}$, and the representation is unique up to strictly increasing affine transformations. If in addition, the preference satisfies* (ce)*, u is continuous. If the preference fulfills* (a2')*, u is concave. If* (ub) *holds, then u does not attain its infimum or supremum.*

Instead of making the usual strong continuity assumptions, Proposition 3.3 derives the von Neumann–Morgenstern representation (3.4) for complete preferences satisfying the independence axiom (ia) from the monotonicity condition (a2) and the one-dimensional continuity assumptions (aa) and (a4.b). In Delbaen et al. (2011), conditions are given that imply a von Neumann–Morgenstern representation with a nondecreasing u that is not necessarily right-continuous.

4 Separation of risk and ambiguity attitudes

In this section we introduce additional axioms, which guarantee that a preference $\succcurlyeq$ over L separates attitudes towards risk and ambiguity. In the following three subsections we discuss different special cases.

4.1 Separability and state-independence

The following is a separability condition slightly weaker than Savage's sure thing principle P2:

(A5) Separability:
For all $s \in S$, $\mu, \nu \in P_c$ and $f, g \in L$, $\mu 1_s + f 1_{S \setminus s} \succcurlyeq \nu 1_s + f 1_{S \setminus s}$ implies $\mu 1_s + g 1_{S \setminus s} \succcurlyeq \nu 1_s + g 1_{S \setminus s}$.

If (A5) holds, then for every state $s \in S$, $\mu 1_s + f 1_{S \setminus s} \succcurlyeq \nu 1_s + f 1_{S \setminus s}$ defines a preference $\succcurlyeq_s$ among $\mu, \nu \in P_c$ that does not depend on f.

If in addition to (A5), the preference $\succcurlyeq$ fulfills

(A6) State-wise existence of unique certainty equivalents:
For every $s \in S$, the induced preference $\succcurlyeq_s$ satisfies (ce),

we denote by $c(f)$ the vector of state-wise certainty equivalents $c_s(f_s)$, $s \in S$.

If (A5) holds together with

(A7) State-independence:
For all $s, s' \in S$, $\mu, \nu \in P_c$ and $f \in L$, $\mu 1_s + f 1_{S \setminus s} \succcurlyeq \nu 1_s + f 1_{S \setminus s}$ implies $\mu 1_{s'} + f 1_{S \setminus s'} \succcurlyeq \nu 1_{s'} + f 1_{S \setminus s'}$,

then $\succcurlyeq_s$ does not depend on the state $s \in S$. (A7) is a weak version of Savage's ordinal event independence axiom P3.

Our first representation result for preferences separating risk and ambiguity attitudes is as follows:

Proposition 4.1. *A preference $\succcurlyeq$ on L satisfying* (A1)–(A6) *has a representation of the form*

$$V(f) = \inf_{u \in I^S} A\Big(u, \sum_{s \in S} u_s(c_s(f_s))\Big), \tag{4.1}$$

where A is a function in $\mathcal{A}^{\min}$, and for every $s \in S$, $c_s : P_c \to \mathbb{R}$ is a mapping that provides unique certainty equivalents for the induced preference $\succcurlyeq_s$ and is of the form

$$c_s(\mu) = \inf_{u \in I} D_s\Big(u, \int_{\mathbb{R}} u d\mu\Big)$$

for a unique function $D_s \in \mathcal{D}^{\min}$. Moreover, if $\succcurlyeq$ satisfies (A1)–(A7)*, then c_s and D_s do not depend on the state $s \in S$.*

Remark 4.2. It is easy to see that a preference $\succcurlyeq$ on L with a representation of the form (4.1) satisfies (A1), (A2) and (A4). If in addition, it is strictly monotone on $H = \mathbb{R}^S \subseteq L$, it also fulfills (A5)–(A6). On the other hand, $\succcurlyeq$ does not necessarily have the convexity property (A3). However, it follows from Theorem 2.1 that (A3) holds if one has

$$V(f) = \inf_{u \in I^S} A\Big(u, \sum_{s \in S} u_s(c_s(f_s))\Big) = \inf_{u \in I^S} A\left(u, \langle u, f \rangle\right) \quad \text{for all } f \in L.$$

4.2 d-convexity, D-convexity and risk-aversion

Denote by Δ the set of all probability measures on S, by $\mathcal{B}$ the family of all functions $B : \Delta \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying (R1) with respect to Δ instead of I^S and by $\mathcal{B}^{\min}$ the subset of those $B \in \mathcal{B}$ fulfilling (R1)–(R4). The following result gives a representation for d-convex convex preferences separating risk and ambiguity attitudes.

Proposition 4.3. *A preference $\succcurlyeq$ on L satisfying* (A1)–(A6) *and* (A3') *has a representation of the form*

$$V(f) = \inf_{p \in \Delta} B\Big(p, \sum_{s \in S} p_s c_s(f_s)\Big), \tag{4.2}$$

where B is a function in $\mathcal{B}^{\min}$, and for every $s \in S$, $c_s : P_c \to \mathbb{R}$ is a mapping providing unique certainty equivalents for the induced preference $\succcurlyeq_s$ of the form

$$c_s(\mu) = \inf_{u \in I} D_s\Big(u, \int_{\mathbb{R}} u d\mu\Big) \tag{4.3}$$

for a unique function $D_s \in \mathcal{D}^{\min}$. Moreover, B is the only function in $\mathcal{B}^{\min}$ inducing the numerical representation V, and $B \le \tilde{B}$ for every $\tilde{B} \in \mathcal{B}$ leading to the same V. If $\succcurlyeq$ also satisfies (A7)*, then c_s and D_s do not depend on the state $s \in S$.*

Remark 4.4. Any preference $\succcurlyeq$ on L with a representation (4.2) for a function $B \in \mathcal{B}$ and unique certainty equivalent mappings c_s of the form (4.3) satisfies (A1), (A2) and (A4). Moreover, if $\succcurlyeq$ is strictly monotone on $H = \mathbb{R}^S \subseteq L$, then it also fulfills (A5)–(A6). The convexity property (A3) does not always hold. But if the mappings $c_s : P_c \to \mathbb{R}$ are concave, it follows from the quasi-concavity of $x \mapsto \inf_{p\in\Delta} B(p, \sum_{s\in S} p_s x_s)$ that $\succcurlyeq$ satisfies (A3).

The next lemma shows that a risk-averse uncertainty-averse preference exhibits diversification. It follows from the same arguments as Proposition 3 of Dekel (1989).

Lemma 4.5. *Every preference $\succcurlyeq$ on L with the properties* (A2') *and* (A3) *satisfies* (A3").

Since D-convexity implies d-convexity, one obtains from Proposition 4.3 and Lemma 4.5 the following

Corollary 4.6. *A preference $\succcurlyeq$ on L satisfying* (A1), (A2'), (A3), (A4), (A5), (A6) *has a representation of the form*

$$V(f) = \inf_{p\in\Delta} B\Big(p, \sum_{s\in S} p_s c_s(f_s)\Big) \tag{4.4}$$

where B is a function in $\mathcal{B}^{\min}$, and for every $s \in S$, $c_s : P_c \to \mathbb{R}$ is a mapping providing unique certainty equivalents for the induced preference $\succcurlyeq_s$ of the form

$$c_s(\mu) = \inf_{u\in I_c} D_s\Big(u, \int_{\mathbb{R}} u d\mu\Big)$$

for a unique function $D_s \in \mathcal{D}_c^{\min}$. Moreover, B is the only function in $\mathcal{B}^{\min}$ inducing V, and $B \le \tilde{B}$ for every $\tilde{B} \in \mathcal{B}$ generating the same V. If in addition, $\succcurlyeq$ satisfies (A7)*, then c_s and D_s do not depend on the state $s \in S$.*

4.3 Risk-independence

A special subclass of preferences on L with the properties (A1)–(A4) are those which satisfy (A5) together with the condition

(A8) Risk-independence, Archimedean property and unboundedness:
For every $s \in S$, the induced preference $\succcurlyeq_s$ satisfies (ia), (aa) and (ub).

Proposition 4.7. *A preference $\succcurlyeq$ on L satisfying* (A1)–(A5) *and* (A8) *has a representation of the form*

$$V(f) = \inf_{p\in\Delta} B\Big(p, \sum_{s\in S} p_s \int_{\mathbb{R}} u_s df_s\Big), \tag{4.5}$$

where B is an element of $\mathcal{B}^{\min}$, and for every $s \in S$, $u_s : \mathbb{R} \to \mathbb{R}$ is a nondecreasing right-continuous function not attaining its infimum or supremum such that $\int_{\mathbb{R}} u_s d\mu$ represents the induced preference $\succcurlyeq_s$ on P_c. Every u_s is unique up to strictly increasing affine transformation, and for for given u_s, $s \in S$, B is the smallest of all functions in $\mathcal{B}$ generating the same V. If in addition, (A2') *holds, then all u_s are concave. If $\succcurlyeq$ satisfies* (A7)*, then u_s does not depend on the state $s \in S$.*

Remark 4.8. If for every $s \in S$, $u_s : \mathbb{R} \to \mathbb{R}$ is a nondecreasing right-continuous function, then the mapping $\varphi : L \to \mathbb{R}^S$, given by $\varphi_s(f) := \int_{\mathbb{R}} u_s df_s$, is affine. Moreover, for any $B \in \mathcal{B}$, $x \mapsto \inf_{p\in\Delta} B(p, \sum_{s\in S} p_s x_s)$ defines a quasi-concave functional on $\mathbb{R}^S$. As a consequence, a preference $\succcurlyeq$ on L with a numerical representation of the form (4.5) for a function $B \in \mathcal{B}$, satisfies (A1)–(A4). Moreover, if it is strictly monotone on $H = \mathbb{R}^S \subseteq L$, then all the functions u_s must be strictly increasing, and $\succcurlyeq$ fulfills (A5) as well as (A8).

In particular, a preference over L that is strictly increasing on $H = \mathbb{R}^S \subseteq L$ has a representation of the form (4.5) if and only if it satisfies (A1)–(A5) and (A8).

A Proofs of Theorem 2.1 and Corollary 2.3

The most difficult part in the proof of Theorem 2.1 is to show the implication (i) $\Rightarrow$ (ii). The crucial step in the derivation is to prove that the upper contour sets of the preference $\succcurlyeq$ are closed in the $\sigma(L, C^S)$-topology. We recall that a Fréchet space is a locally convex topological vector space X which is complete with respect to a translation-invariant metric generating the topology. The absolute polar U° of a subset $U \subseteq X$ is the following set in the topological dual X^* of X:

$$U^\circ := \{x^* \in X^* : |\langle x^*, x\rangle| \le 1 \text{ for all } x \in U\}.$$

It follows from the Banach–Dieudonné theorem (see e.g. Schaefer and Wolff, 1986) that a convex set Y in X^* is $\sigma(X^*, X)$-closed if and only if $Y \cap U^\circ$ is $\sigma(X^*, X)$-closed for every 0-neighborhood U in X.

Denote by C the set of all continuous functions $h : \mathbb{R} \to \mathbb{R}$. The subspace C_b of bounded continuous functions with the supremum norm $\|h\|_\infty := \sup_{x\in\mathbb{R}} |h(x)|$ is a Banach space. Moreover, $\|h\|_k := \left\|h 1_{[-k,k]}\right\|_\infty$, $k \in \mathbb{N}$, is a sequence of seminorms on C generating the translation-invariant metric

$$d(h, h') := \sum_{k\ge1} 2^{-k} \frac{\|h - h'\|_k}{1 + \|h - h'\|_k},$$

under which C becomes a Fréchet space. More generally, C^S is a Fréchet space with respect to the metric

$$d^S(u, u') := \sum_{s\in S} \sum_{k\ge1} 2^{-k} \frac{\|u_s - u'_s\|_k}{1 + \|u_s - u'_s\|_k}.$$

Every continuous linear functional $\varphi : C \to \mathbb{R}$ has a unique representation $\varphi(h) = \langle h, \mu\rangle := \int_{\mathbb{R}} u d\mu$ for a signed Borel measure $\mu \in M_c$. So the topological dual of C can be identified with M_c and the topological dual of C^S with M_c^S, where $f \in M_c^S$ acts on C^S like

$$\langle u, f\rangle := \sum_{s\in S} \int_{\mathbb{R}} u_s df_s.$$

It is well-known that the Lévy metric, given by

$$\rho(\mu, \nu) := \inf\{\varepsilon \in \mathbb{R}_+ : \mu(-\infty, x - \varepsilon] - \varepsilon \le \nu(-\infty, x] \le \mu(-\infty, x + \varepsilon] + \varepsilon\},$$

induces the $\sigma(P_c, C_b)$-topology on P_c. For $k \in \mathbb{N}$, we denote by $P(k)$ the set of all probability measures $\mu \in P_c$ with support in $[-k, k]$. Then one has the following

Lemma A.1. *Fix $k \in \mathbb{N}$, $\mu \in P(k)$, $\alpha \in (0, 1]$ and $\varepsilon > 0$. Let (μ^j) be a sequence in $P(k)$ converging to μ with respect to the Lévy metric ρ. Then $\alpha\delta_k + (1-\alpha)T_\varepsilon\mu \trianglerighteq_1 \mu^j$ for j large enough.*

Proof. Let (μ^j) be a sequence in $P(k)$ converging to μ in $P(k)$ with respect to ρ. Denote $G(x) = \mu(x, \infty)$, $H(x) = \mu[x, \infty)$ and $H^j(x) = \mu^j[x, \infty)$. The point $x^* := \sup\{x : H(x) = 1\}$ is in $[-k, k]$, and

$$H^{\alpha,\varepsilon}(x) := \alpha 1_{\{x \le k\}} + (1-\alpha)H(x-\varepsilon) = 1$$

for all $x \le (x^* + \varepsilon) \wedge k$. In particular, $H^{\alpha,\varepsilon}(x) \ge H^j(x)$, $x \le (x^* + \varepsilon) \wedge k$ for all j. Now assume there exists a sequence (x_n) in $[x^* + \varepsilon, k]$ such that

$$H^{\alpha,\varepsilon}(x_n) < H(x_n - 1/n) + 1/n.$$

By passing to a subsequence one can assume that $x_n \to \bar{x} \in [x^* + \varepsilon, k]$. Then

$$\alpha(1 - H(\bar{x})) \le (1-\alpha)G(\bar{x} - \varepsilon) + \alpha - H(\bar{x}) \le \liminf_n H^{\alpha,\varepsilon}(x_n) - H(x_n - 1/n) - 1/n \le 0.$$

But this implies $H(\bar{x}) = 1$, contradicting $x^* < \bar{x}$. Therefore, there exists an $h > 0$ such that

$$H^{\alpha,\varepsilon}(x) \ge H(x-h) + h \quad \text{for all } x \in [x^* + \varepsilon, k],$$

from which it follows that $H^{\alpha,\varepsilon} \ge H^j$, and therefore, $\alpha\delta_k + (1-\alpha)T_\varepsilon\mu \trianglerighteq_1 \mu^j$ for j large enough. □

Lemma A.2. *Every subset $E \subseteq L$ satisfying the following four properties is $\sigma(L, C^S)$-closed.*

(i) *If $f \in L$ satisfies $f_s \trianglerighteq_1 g_s$, $s \in S$, for some $g \in E$, then $f \in E$*

(ii) *Convexity*

(iii) *For all $f, g \in L$, $\{\alpha \in [0, 1] : \alpha f + (1-\alpha)g \in E\}$ is a closed subset of $[0, 1]$*

(iv) *If $f \in L$ satisfies $T_\varepsilon f \in E$ for all $\varepsilon > 0$, then $f \in E$.*

Proof. Since E is convex and L is a $\sigma(M_c^S, C^S)$-closed subset of M_c^S, it follows from the Banach–Dieudonné theorem that E is $\sigma(L, C^S)$-closed if and only if $E \cap U^\circ$ is $\sigma(L, C^S)$-closed for every 0-neighborhood U in C^S. The sets

$$U_{k,l} := \left\{ u \in C^S : \sum_{s \in S} \|u_s\|_k \le 1/l \right\}, \quad k, l \in \mathbb{N},$$

form a neighborhood base of 0 in C^S. Therefore, it is enough to show that $E \cap U_{k,l}^\circ$ is $\sigma(L, C^S)$-closed for all $k, l \in \mathbb{N}$. However,

$$U_{k,l}^\circ \cap L = \{f \in L : |\langle u, f\rangle| \le 1 \text{ for all } u \in U_{k,l}\} \subseteq L_k := \{f \in L : \operatorname{supp}(f_s) \subseteq [-k, k] \text{ for all } s \in S\}.$$

So it suffices to show that $E \cap L_k$ is a $\sigma(L, C^S)$-closed subset of L_k for all $k \in \mathbb{N}$. To do that, fix $k \in \mathbb{N}$ and assume $E \cap L_k \neq \emptyset$. Since $E \cap L_k$ is $\sigma(L, C^S)$-closed if and only if it is $\sigma(L, C_b^S)$-closed, and $\sigma(L, C_b^S)$ is metrizable, it is enough to show that $f \in L_k$ belongs to $E \cap L_k$ if it is the $\sigma(L, C_b^S)$-limit of a sequence (f^j) in $E \cap L_k$. In this case, for every $s \in S$, (f_s^j) is a sequence in $P(k)$ converging to $f_s \in P(k)$ with respect to $\sigma(P_c, C_b)$, and therefore also in the Lévy metric ρ. Choose $\alpha \in (0, 1]$ and $\varepsilon > 0$. It follows from Lemma A.1 that $f_s^{\alpha,\varepsilon} := \alpha\delta_k + (1-\alpha)T_\varepsilon f_s \trianglerighteq_1 f_s^j$ for j large enough. By assumption (i), $f^{\alpha,\varepsilon}$ belongs to E. So it follows from assumption (iii) that $T_\varepsilon f$ is in E, and since E has property (iv), f belongs to E. □

Lemma A.3. *For a convex subset E of L the following are equivalent:*

(i) *For all $f \in E$ and $g \notin E$, there exists an $\alpha \in (0, 1)$ such that $\alpha f + (1-\alpha)g \notin E$*

(ii) *For all $f, g \in L$, $\{\alpha \in [0, 1] : \alpha f + (1-\alpha)g \in E\}$ is a closed subset of $[0, 1]$.*

Proof. The implication (ii) ⇒ (i) is clear. To show (i) ⇒ (ii), let (α^j) be a sequence in $[0, 1]$ converging to α such that $\alpha^j f + (1-\alpha^j)g \in E$ for all j. Assume $\alpha f + (1-\alpha)g \notin E$. If infinitely many α^j are below α, choose j_0 such that $\alpha^{j_0} < \alpha$. Since E is convex, one has $\beta f + (1-\beta)g \in E$ for all $\beta \in [\alpha^{j_0}, \alpha)$. On the other hand, if condition (i) holds, there must exist a $\beta \in (\alpha^{j_0}, \alpha)$ such that $\beta f + (1-\beta)g \notin E$, a contradiction. This shows that $\alpha f + (1-\alpha)g \in E$ and (ii) holds. If infinitely many α^j are above α, the proof works analogously. □

Proof of Theorem 2.1

(ii) ⇒ (i): If a preference $\succcurlyeq$ on L has a representation of the form $V(f) = \inf_{u \in I^S} A(u, \langle u, f\rangle)$ for a function $A \in \mathcal{A}$, it can easily be checked that it satisfies (A1)–(A3). Moreover, if $f, g, h \in L$ satisfy $f \succcurlyeq g \succ h$, there exists an $u \in I^S$ such that $A(u, \langle u, f\rangle) \geq V(f) \geq V(g) > A(u, \langle u, h\rangle)$. Since $A(u, x)$ is non-decreasing and right-continuous in x, there exists an $\alpha \in (0, 1)$ such that

$$V(g) > A(u, \langle u, \alpha f + (1-\alpha)h\rangle) \geq V(\alpha f + (1-\alpha)h).$$

This shows that $\succcurlyeq$ fulfills (A4.a). Similarly, if f, g are acts in L such that $T_\varepsilon f \succcurlyeq g$ for all $\varepsilon > 0$, then

$$A(u, \langle u, T_\varepsilon f\rangle) \geq V(g) \quad \text{for all } u \in I^S \text{ and } \varepsilon > 0.$$

Therefore,

$$V(f) = \inf_{u \in I^S} A(u, \langle u, f\rangle) \geq V(g),$$

showing that $\succcurlyeq$ satisfies (A4.b).

(i) ⇒ (ii): If $\succcurlyeq$ has the properties (A1)–(A4), one obtains from Lemma A.3 that the upper contour sets $\{f \in L : f \succcurlyeq g\}$, $g \in L$, satisfy conditions (i)–(iv) of Lemma A.2. Therefore, they are $\sigma(L, C^S)$-closed. So it follows from the convex duality arguments in the proof of Theorem 7 in Drapeau and Kupper (2013) that $\succcurlyeq$ has a representation of the form $V(f) = \inf_{u \in I^S} A(u, \langle u, f\rangle)$ for a mapping $\hat{A} \in \mathcal{A}^{\min}$. Moreover, $\hat{A}$ is the only mapping in $\mathcal{A}^{\min}$ inducing V, and $\hat{A} \leq A$ for all $A \in \mathcal{A}$ inducing the same V. This completes the proof of the theorem. □

Proof of Corollary 2.3
The implication (ii) $\Rightarrow$ (i) follows as in the proof of Theorem 2.1. To show (i) $\Rightarrow$ (ii), one notices that (A2') is stronger than (A2). So if (i) holds, one obtains from Theorem 2.1 that the upper contour sets $\{f \in L : f \succcurlyeq g\}$, $g \in L$, are $\sigma(L, C^S)$-closed. Moreover, I_c is the polar cone generated by second order stochastic dominance. Therefore, (ii) and the rest of the corollary follow as in Theorem 2.1 from the proof of Theorem 7 in Drapeau and Kupper (2013) by replacing I with I_c. □

B Proofs of the results in Section 3

Proof of Proposition 3.2
Let $\succcurlyeq$ be a preference on P_c satisfying (a1)–(a4) and (ce). By (ce.a), a given $\mu \in P_c$ can have at most one certainty equivalent. Moreover, it follows from (a2) and (ce.a) that there exist $x, y, z \in \mathbb{R}$ such that $\delta_x \succcurlyeq \mu \succcurlyeq \delta_y \succ \delta_z$, guaranteeing that the mapping (3.1) is real-valued. Due to (a4.b), one has $\delta_{c(\mu)} \succcurlyeq \mu$, and by (ce.b), it cannot be that $\delta_{c(\mu)} \succ \mu$. This shows $\delta_{c(\mu)} \sim \mu$, and hence, $c(\mu)$ is the unique certainty equivalent of μ. Since $\succcurlyeq$ satisfies (a1)–(a4), it follows from the proof of Theorem 2.1 that the upper contour sets

$$\{\mu \in P_c : c(\mu) \geq x\} = \{\mu \in P_c : \mu \succcurlyeq \delta_x\}, \quad x \in \mathbb{R},$$

are $\sigma(P_c, C)$-closed. So the mapping $c : P_c \to \mathbb{R}$ is monotone with respect to $\trianglerighteq_1$, quasi-concave and $\sigma(P_c, C)$-lower semicontinuous. Therefore it follows from the second part of the proof of Theorem 7 in Drapeau and Kupper (2013) that it has a representation of the form (3.2). If (a2') holds, then c is monotone with respect to $\trianglerighteq_2$. So, since I_c is the polar cone generated by second order stochastic dominance, $\succcurlyeq$ is representable as in (3.3). □

Proof of Proposition 3.3
Since $\succcurlyeq$ satisfies (ia) and (aa), it follows from the von Neumann–Morgenstern theorem that it has an affine representation $V : P_c \to \mathbb{R}$ that is unique up to strictly increasing affine transformations, (see e.g. Föllmer and Schied, 2004). As a consequence, it satisfies (a3) and (a4.a), and one obtains from Corollary 2.3 that the upper contour sets of $\succcurlyeq$ are $\sigma(P_c, C)$-closed. Since $V : P_c \to \mathbb{R}$ is affine, its image $V(P_c)$ is an interval. Therefore, it follows from

$$\{\mu \in P_c : V(\mu) \geq V(\nu)\} = \{\mu \in P_c : \mu \succcurlyeq \nu\}$$

that V is $\sigma(P_c, C)$-upper semicontinuous. In particular, the function $u : \mathbb{R} \to \mathbb{R}$ given by $u(x) := V(\delta_x)$, is nondecreasing and right-continuous. By affinity, one has $V(\mu) = \int_{\mathbb{R}} u(x)\mu(dx)$ for every $\mu \in P_c$ which is a finite convex combination of Dirac measures. A general $\mu \in P_c$ can be approximated from above in the Lévy metric by a sequence (μ^j) of finite convex combinations of Dirac measures. Due to upper semicontinuity, one has

$$V(\mu) = \lim_j V(\mu^j) = \lim_j \int_{\mathbb{R}} u(x)\mu^j(dx) = \int_{\mathbb{R}} u(x)\mu(dx).$$

This proves the representation (3.4). If $\succcurlyeq$ satisfies (ce), it has certainty equivalents, from which it follows that u must be continuous. If $\succcurlyeq$ fulfills (a2'), u must be concave. Finally, if the preference satisfies (ub), u cannot attain its infimum or supremum. □

C Proofs of the results in Section 4

Proof of Proposition 4.1

If $\succcurlyeq$ is a preference relation on L with the properties (A1)–(A6), it follows from Theorem 2.1 that it has a representation of the form

$$V(f) = \inf_{u \in I^S} A(u, \langle u, f \rangle)$$

for a function $A \in \mathcal{A}^{\min}$. Due to (A5),

$$\mu \succcurlyeq_s \nu :\Leftrightarrow \mu 1_s + f 1_{S \setminus s} \succcurlyeq \nu 1_s + f 1_{S \setminus s}$$

defines for every $s \in S$, a complete preference relation on P_c that does not depend on f. Since (A6) holds, it follows from Proposition 3.2 that $\succcurlyeq_s$ has a unique certainty equivalent mapping $c_s : P_c \to \mathbb{R}$. Due to (A1)–(A4), $\succcurlyeq_s$ satisfies (a1)–(a4). So one obtains from Proposition 3.2 that it is of the form

$$c_s(\mu) = \inf_{u \in I} D_s \left(u, \int u d\mu \right)$$

for a unique function $D_s \in \mathcal{D}^{\min}$. Denote by $c : L \to \mathbb{R}^S$ the mapping defined by $c_s(f) := c_s(f_s)$ and note that if $f, g \in L$ are two acts satisfying $f_s \succcurlyeq_s g_s$ for all $s \in S$, one has

$$f \succcurlyeq g 1_1 + f 1_{S \setminus 1} \succcurlyeq g 1_{\{1,2\}} + f 1_{S \setminus \{1,2\}} \succcurlyeq \cdots \succcurlyeq g 1_{S \setminus m} + f 1_m \succcurlyeq g.$$

It follows that

$$V(f) = V(\delta_{c(f)}) = \inf_{u \in I^S} A\Big(u, \sum_{s \in S} u_s(c_s(f_s))\Big).$$

□

Proof of Proposition 4.3

As in the proof of Proposition 4.1, it follows from (A1)–(A6) that $\succcurlyeq$ has a representation

$$V(f) = \inf_{u \in I^S} A\left(u, \langle u, f \rangle\right)$$

for a function $A \in \mathcal{A}^{\min}$ and induces for every $s \in S$, a complete preference $\succcurlyeq_s$ on P_c with a unique certainty equivalent mapping $c_s : P_c \to \mathbb{R}$ of the form

$$c_s(\mu) = \inf_{u \in I} D_s \left(u, \int u d\mu \right)$$

for a unique $D_s \in \mathcal{D}^{\min}$. The mapping $v : \mathbb{R}^S \to \mathbb{R} \cup \{\pm\infty\}$ defined by $v(x) := V(\delta_x)$ is monotone with respect to the component-wise order on $\mathbb{R}^S$ and upper semicontinuous. By (A3'), it is quasi-convex. So it follows from Theorem 6 in Drapeau and Kupper (2013) that it is representable as

$$v(x) = \inf_{p \in \Delta} B\left(p, \sum_{s \in S} p_s x_s\right)$$

for a unique function $B \in \mathcal{B}^{\min}$. Moreover, $B \leq \tilde{B}$ for any $\tilde{B} \in \mathcal{B}$ inducing the same mapping v. Since $f \sim c(f)$ for any act $f \in L$, one obtains

$$V(f) = V(c(f)) = \inf_{p \in \Delta} B\left(p, \sum_{s \in S} p_s c_s(f_s)\right).$$

It is clear that if (A7) holds, then c_s and D_s do not depend on $s \in S$. □

Proof of Lemma 4.5
Consider a measurable space $(\Omega, \mathcal{F})$ equipped with m probability measures $\mathbb{P}_s$, $s \in S$. Let $X, Y, Z : \Omega \to \mathbb{R}$ be bounded measurable functions such that $\psi(X), \psi(Y) \succcurlyeq \psi(Z)$, where $\psi(X)$ denotes the vector $(\mu_1^X, \ldots, \mu_m^X)$ of distributions of X under $\mathbb{P}_1, \ldots, \mathbb{P}_m$. Since $\mu_s^{\alpha X + (1-\alpha)Y} \trianglerighteq_2 \alpha \mu_s^X + (1-\alpha)\mu_s^Y$, it follows from (A2') and (A3) that

$$\psi(\alpha X + (1-\alpha)Y) \succcurlyeq \alpha\psi(X) + (1-\alpha)\psi(Y) \succcurlyeq \psi(Z).$$

□

Proof of Corollary 4.6
By Lemma 4.5, $\succcurlyeq$ satisfies (A3") and therefore also (A3'). Now the corollary follows as Proposition 4.3 except that the certainty equivalents c_s are representable as

$$c_s(\mu) = \inf_{u \in I_c} D_s\Big(u, \int_{\mathbb{R}} u d\mu\Big)$$

for unique functions $D_s \in \mathcal{D}_c^{\min}$ due to Corollary 3.1.

Proof of Proposition 4.7
Due to (A1)–(A5), one obtains from Theorem 2.1 that $\succcurlyeq$ has a representation

$$V(f) = \inf_{u \in I^S} A\left(u, \langle u, f \rangle\right)$$

for a mapping $A \in \mathcal{A}^{\min}$ and induces for every $s \in S$, a complete preference $\succcurlyeq_s$ on P_c. Since $\succcurlyeq_s$ satisfies (a1), (a2), (a4), (ia), (aa) and (ub), it follows from Proposition 3.3 that it has a representation of the form $\int_{\mathbb{R}} u_s d\mu$ for a nondecreasing right-continuous function $u_s : \mathbb{R} \to \mathbb{R}$ that does not attain its infimum or supremum and is unique up to strictly increasing affine transformations. Define the function $\varphi : L \to \mathbb{R}^S$ by $\varphi_s(f) := \int_{\mathbb{R}} u_s df_s$ and the vectors $a, b \in (\mathbb{R} \cup \{\pm\infty\})^S$ by

$$a_s := \inf_{x \in \mathbb{R}} u_s(x), \quad b_s := \sup_{x \in \mathbb{R}} u_s(x).$$

Then $\operatorname{Im}(\varphi) = (a, b)$. Since $\varphi(f) = \varphi(g)$ implies $f \sim g$, the function $v : (a, b) \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$v(x) := V(f), \quad \text{where } x = \varphi(f),$$

is well-defined. Moreover, v is monotone with respect to the component-wise order on $\mathbb{R}^S$, and since φ is affine, the superlevel sets

$$C^z := \{x \in (a,b) : v(x) \geq z\} = \varphi\{f \in L : V(f) \geq z\}, \quad z \in \mathbb{R},$$

are convex. Now consider a sequence (x^j) in C^z converging to some $x \in (a,b)$. By passing to a subsequence, one can assume that $|x^j - x| \leq R2^{-j}/j > 0$ for $R := \inf_s(b_s - x_s) > 0$. Then $y = x + \sum_j j(x^j - x)^+$ is in (a,b), and for all $\lambda \in [0,1)$, one has

$$\lambda x + (1-\lambda)y \geq x + (1-\lambda)j(x^j - x)^+ \geq x^j \in C^z \quad \text{if } j \geq \frac{1}{1-\lambda}.$$

So, by monotonicity, $\lambda x + (1-\lambda)y$ belongs to C^z for all $\lambda \in [0,1)$. Choose $f, g \in L$ such that $x = \varphi(f)$ and $y = \varphi(g)$. Then one obtains from the $\sigma(L, C^S)$-upper semicontinuity of V that the set

$$\{\lambda \in [0,1] : \lambda x + (1-\lambda)y \in C^z\} = \{\lambda \in [0,1] : V(\lambda f + (1-\lambda)g) \geq z\}$$

is closed. It follows that x is in C^z, which shows that C^z is relatively closed in (a,b). Denote by $\hat{v} : \mathbb{R}^S \to \mathbb{R} \cup \{\pm\infty\}$ the minimal monotone quasi-concave lower semicontinuous function dominating v on (a,b). Its superlevel sets $D^z := \left\{x \in \mathbb{R}^S : \hat{v}(x) \geq z\right\}$ are given by

$$D^z = \bigcap_{z' < z} \operatorname{cl}(C^{z'} + \mathbb{R}^S_+).$$

Since C^z is monotone and relatively closed in (a,b), one has

$$\operatorname{cl}\left(C^z + \mathbb{R}^S_+\right) \cap (a,b) = \operatorname{cl}\left((C^z + \mathbb{R}^S_+) \cap (a,b)\right) \cap (a,b) = \operatorname{cl}(C^z) \cap (a,b) = C^z.$$

Therefore,

$$D^z \cap (a,b) = \bigcap_{z' < z} C^{z'} = C^z.$$

which shows that $\hat{v}(x) = v(x)$ for $x \in (a,b)$. By Theorem 6 of Drapeau and Kupper (2013), $\hat{v}$ is representable as $\hat{v}(x) = \inf_{p \in \Delta} B(p, \sum_s p_s x_s)$ for a unique element $B \in \mathcal{B}^{\min}$, and $B \leq B'$ for every $B' \in \mathcal{B}$ inducing the same $\hat{v}$. Finally, if $\tilde{B}$ is a function in $\mathcal{B}$ such that $v(x) = \inf_{p \in \Delta} \tilde{B}(p, \sum_s p_s x_s)$ for $x \in (a,b)$, then $\tilde{v}(x) = \inf_{p \in \Delta} \tilde{B}(p, \sum_s p_s x_s)$, $x \in \mathbb{R}^S$, is a monotone quasi-concave upper semicontinuous extension of v. Therefore, $\hat{v} \leq \tilde{v}$ and $B \leq \tilde{B}$.

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