Recursiveness of indifference prices and translation-invariant preferences*

Patrick Cheridito[†]
Princeton University
Princeton, NJ 08544, USA

Michael Kupper[‡]
Vienna Institute of Finance
1190 Vienna, Austria

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Abstract

We consider an economic agent with dynamic preferences over a set of uncertain monetary payoffs. We assume that preferences are updated in a time-consistent way as more information is becoming available. Our main result is that the agent's indifference prices are recursive if and only if the preferences are translation-invariant. The proof is based on a characterization of time-consistency of dynamic preferences in terms of indifference sets. As a special case, we obtain that expected utility leads to recursive indifference prices if and only if absolute risk aversion is constant, that is, the Bernoulli utility function is linear or exponential.

Keywords Dynamic utility functions, time-consistency, translation-invariant preferences, indifference prices, indifference sets.

JEL Classification D81, D9, G12, G13

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1 Introduction

The standard way of pricing financial assets is by taking expectations of discounted payoffs under a probability measure $\mathbb{Q}$ which is equivalent to the objective probability measure $\mathbb{P}$. This leads to prices that are linear in the payoffs and discounted price processes which are martingales under $\mathbb{Q}$. In particular, prices are recursive, that is, the price of a future payoff can be calculated directly or in two steps backwards in time; both give the same result. As a consequence, in discrete-time models, prices can be computed by backwards induction, and in continuous time, provided that state variables are Markovian, they can be expressed as solutions to partial differential equations. However, while in complete

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markets the pricing measure is unique, there exist infinitely many of them in incomplete markets, and often it is not clear which one should be used.

An alternative valuation method for incomplete market situations is indifference pricing. Since its introduction by Hodges and Neuberger [15] it has been extended to different setups and used for the valuation of various products from equity options and credit derivatives to real options and complex insurance contracts. An indifference price is the maximal amount of money for which a given economic agent would be willing to buy an uncertain future payoff or the least amount for which s/he would be willing to sell it. In contrast to prices obtained by taking expectations, indifference prices are not linear. In particular, they depend on whether an asset is bought or sold; see for instance, Henderson and Hobson [14] for an introduction to indifference pricing, examples and further literature. Almost all studies of indifference pricing so far have assumed that investor preferences are given by expected utility and many of them that the Bernoulli utility function is exponential. It has been shown by Rouge and El Karoui [24], Mania and Schweizer [19], Becherer [3] in different setups that in the expected exponential utility case, the dynamics of indifference prices can be described by the solution of a non-linear backward stochastic differential equation, from which it follows that they are recursive. Musiela and Zariphopoulou [20, 21] have shown that indifference prices corresponding to expected exponential utility can be obtained through a family of non-linear pricing operators satisfying the semigroup property, also yielding that they are recursive. In Klöppel and Schweizer [16] it is shown that indifference prices stay recursive when preferences are given by time-consistent dynamic convex risk measures.

In this paper we investigate the question whether indifference prices are always recursive, or if not, what is the largest class of preferences that lead to recursive indifference prices. To keep technicalities at a minimum we work in a discrete-time setup and with a finite probability space. Our main result, Theorem 3.4 shows that indifference prices are recursive if and only if preferences are translation-invariant. As a special case we obtain that expected utility leads to recursive indifference prices if and only if absolute risk aversion is constant, that is, the Bernoulli utility function is linear or exponential.

The structure of the paper is as follows: In Section 2 we specify conditional preferences through utility functions and discuss properties of certainty equivalents and indifference prices. In Section 3 we provide different characterizations of time-consistency of dynamic utility functions and show that time-consistent preferences are uniquely determined by the initial preference order. Then we state our main result, which shows that indifference prices are recursive if and only if preferences are translation-invariant. Its proof is given in the appendix in a framework where the only investment opportunity is a money market account. In Section 4 we discuss the case of a more general financial market.

2 Conditional preferences

For the sake of simplicity we consider a finite sample space $\Omega = \{\omega_1, \dots \omega_N\}$, whose elements describe all possible states of the world. Time is discrete and runs through the set $\{0, 1, \ldots, T\}$ for a finite horizon $T \in \mathbb{N}$. The evolution of information is modelled by a filtration $(\mathcal{F}_t)_{t=0}^T$. Since Ω is finite, every $\mathcal{F}_t$ is an algebra of subsets of Ω generated by finitely many non-empty, disjoint sets $A_t^1, \ldots, A_t^{N_t}$ such that $\Omega = A_t^1 \cup \cdots \cup A_t^{N_t}$. We call $A_t^1, \ldots, A_t^{N_t}$ the atoms of $\mathcal{F}_t$ and assume $\Omega = A_0^1$ and that the partitions have the following strong refinement property: for all $t \leq T-1$, every time t atom A_t^k splits into at least two parts at time $t+1$. One can think of this information structure in terms of an event tree in which every non-terminal node has at least two descendants. By $L(\mathcal{F}_t)$ we denote the set of all $\mathcal{F}_t$-measurable functions from Ω to $\mathbb{R}$. An element of $L(\mathcal{F}_T)$ is understood as an uncertain monetary payoff at time T. If it is in $L(\mathcal{F}_t)$, then its value is known by time t. We assume that there exists a money market account where money can be lent to and borrowed from at the same risk-free rate and use it as numeraire, that is, payoffs are expressed in multiples of one dollar put into the money market account at time zero. Equalities and inequalities between uncertain payoffs are understood ω-wise. For instance, $X \geq Y$ means $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$.

2.1 Utility functions

We consider an agent whose preferences at time t are given by a function $U_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)$. In the event A_t^k, the agent prefers X to Y if $U_t(X) > U_t(Y)$ on A_t^k.

We call U_t a utility function at time t if it has the following three properties:

(LP) **Local property:** If $X, Y \in L(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$ are such that $1_A X = 1_A Y$, then $1_A U_t(X) = 1_A U_t(Y)$.

(SM) **Strict monotonicity:** For all $X \in L(\mathcal{F}_T)$, $\varepsilon > 0$ and $\omega \in \Omega$:

$$U_t(X + \varepsilon 1_{\{\omega\}}) > U_t(X) \quad \text{on the } \mathcal{F}_t\text{-atom } A_t^k \text{ containing the state } \omega.$$

(C) **Continuity:** U_t is continuous with respect to the norm $||X||_\infty := \max_{\omega \in \Omega} |X(\omega)|$, $X \in L(\mathcal{F}_T)$.

The economic interpretation of (LP) is that in an event $A \in \mathcal{F}_t$, the utility $U_t(X)$ of a payoff $X \in L(\mathcal{F}_T)$ only depends on values X can take in states of the world contained in A. (LP) and (SM) imply

(M) **Monotonicity:** $U_t(X) \geq U_t(Y)$ for all $X, Y \in L(\mathcal{F}_T)$ such that $X \geq Y$.

It is natural to assume strict monotonicity (SM) if in each event A_t^k, the agent believes that every state $\omega \in A_t^k$ is possible. It will ensure that certainty equivalents and indifference prices are unique. (C) is a technical assumption. Together with (SM), it will guarantee existence of certainty equivalents and indifference prices.

If U_t satisfies the

(T) **Translation property:** $U_t(X+m) = U_t(X)+m \quad$ for all $X \in L(\mathcal{F}_T)$ and $m \in L(\mathcal{F}_t)$,

then the corresponding preference order is translation-invariant, that is, the agent prefers X to Y if and only if s/he prefers $X+m$ to $Y+m$ for all $m \in L(\mathcal{F}_t)$.

Other properties that we will play a role in this paper are:

(LS) **Loss sensitivity:** For all $X, Y \in L(\mathcal{F}_T)$ and $\omega \in \Omega$, there exists $m \in \mathbb{R}$ such that

$$U_t(X + m1_{\{\omega\}}) \le U_t(Y) \quad \text{on the } \mathcal{F}_t\text{-atom } A_t^k \text{ containing the state } \omega.$$

(CQC) **Conditional quasi-concavity:** $U_t(\lambda X + (1-\lambda)Y) \ge \min\{U_t(X), U_t(Y)\}$ for all $X, Y \in L(\mathcal{F}_T)$ and $\lambda \in L(\mathcal{F}_t)$ such that $0 \le \lambda \le 1$.

(CC) **Conditional concavity:** $U_t(\lambda X + (1-\lambda)Y) \ge \lambda U_t(X) + (1-\lambda)U_t(Y)$ for all $X, Y \in L(\mathcal{F}_T)$ and $\lambda \in L(\mathcal{F}_t)$ such that $0 \le \lambda \le 1$.

If U_t satisfies (CQC), then diversification increases utility in the sense that $\lambda X + (1-\lambda)Y$ is weakly preferred to X and Y for all $X, Y \in L(\mathcal{F}_T)$ with $U_t(X) = U_t(Y)$ and $\lambda \in L(\mathcal{F}_t)$ such that $0 \le \lambda \le 1$. (CC) is obviously stronger than (CQC). On the other hand, one has

Lemma 2.1 *For any function $U_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)$, the following hold:*

(1) *If U_t satisfies* (T) *and* (CQC)*, then it also satisfies* (CC)

(2) *If U_t satisfies* (SM) *and* (CC)*, then it also satisfies* (LS)

Proof.
(1) If $U_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)$ satisfies (T) and (CQC), then

$$\begin{aligned} & U_t(\lambda X + (1-\lambda)Y) - \lambda U_t(X) - (1-\lambda)U_t(Y) \\ = \;& U_t(\lambda[X - U_t(X)] + (1-\lambda)[Y - U_t(Y)]) \\ \ge \;& \min\{U_t(X - U_t(X)), U_t(Y - U_t(Y))\} = 0 \end{aligned}$$

for all $X, Y \in L(\mathcal{F}_T)$ and $\lambda \in L(\mathcal{F}_t)$ such that $0 \le \lambda \le 1$. This shows that U_t satisfies (CC).
(2) For fixed $X \in L(\mathcal{F}_T)$ and an $\mathcal{F}_t$-atom A_t^k, denote the value of $U_t(X)$ on A_t^k by $U_{t,k}(X)$. Then, by (SM) and (CC), $m \mapsto U_{t,k}(X + m1_{\{\omega\}})$ is for every $\omega \in A_t^k$ a strictly increasing concave function from $\mathbb{R}$ to $\mathbb{R}$. Therefore, $\lim_{m \to -\infty} U_{t,k}(X + m1_{\{\omega\}}) = -\infty$. In particular, U_t satisfies (LS). □

A probability measure $\mathbb{Q}$ on Ω is given by the weights $q_1, \dots, q_N$ it gives the states $\omega_1, \dots, \omega_N$. We call $\mathbb{Q}$ strictly positive if $q_1, \dots, q_N > 0$.

Examples 2.2
Consider a function $U_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)$ of the form

$$U_t(X) = \min_{\mathbb{Q} \in \mathcal{Q}} \{\mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t] + c(\mathbb{Q})\} \tag{2.1}$$

for a non-empty set $\mathcal{Q}$ of strictly positive probability measures on Ω, a mapping c from $\mathcal{Q}$ to $L(\mathcal{F}_t)$ and a strictly increasing, continuous function u from $\mathbb{R}$ to $\mathbb{R}$. Then U_t has the properties (LP), (SM) and (C). If $\lim_{x \to -\infty} u(x) = -\infty$, then U_t is loss sensitive (LS). If

u is concave, then U_t satisfies (CC). Preference functionals of the form (2.1) are studied in Maccheroni et al. [17, 18]. In the special case $c \equiv 0$, (2.1) reduces to a multi-prior conditional expectation

$$\min_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t]\,; \tag{2.2}$$

see Gilboa and Schmeidler [13] for an axiomatization in the unconditional case and Epstein and Schneider [9] for the conditional and dynamic case. If $\mathcal{Q}$ consists of only one element $\mathbb{Q}$, (2.2) is simply a conditional expected utility

$$\mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t]\,.$$

For $u(x) = x$, the mapping U_t in (2.1) has the translation property (T), and $-U_t$ is a convex monetary risk measure; see Föllmer and Schied [10, 11] for the unconditional case and Cheridito et al. [4, 5] for the conditional and dynamic case. If $u(x) = x$ and $c \equiv 0$, then $-U_t$ is a coherent risk measure; see Artzner et al. [1], Delbaen [6, 7] for the unconditional case and Artzner et al. [2], Delbaen [8], Riedel [22], Roorda et al. [23] for conditional and dynamic coherent risk measures.

2.2 Incremental utilities

If at time t, our agent is already holding a portfolio with time T payoff $V \in L(\mathcal{F}_T)$ when considering another payoff $X \in L(\mathcal{F}_T)$, the question is how $U_t(X+V)$ compares to $U_t(V)$. We call V the agent's endowment and define

Definition 2.3 *The incremental utility with respect to a utility function U_t at time t and endowment $V \in L(\mathcal{F}_T)$ is given by*

$$U_t^V(X) := U_t(V+X) - U_t(V)\,, \quad X \in L(\mathcal{F}_T)\,. \tag{2.3}$$

Clearly, if U_t has any of the properties (LP), (SM), (C), (T), (LS), (CQC), (CC), then so does U_t^V. Moreover, $U_t^V(0) = 0$, and U_t^0 induces the same conditional preference order on $L(\mathcal{F}_T)$ as U_t.

2.3 Certainty equivalents

Definition 2.4 *Let U_t be a utility function at time t. The certainty equivalent $C_t^V(X)$ of $X \in L(\mathcal{F}_T)$ with respect to U_t and endowment $V \in L(\mathcal{F}_T)$ is defined as the unique $m \in L(\mathcal{F}_t)$ that satisfies*

$$U_t^V(m) = U_t^V(X)\,.$$

Note that $C_t^V(X)$ always exists and is unique because U_t^V has the properties (SM) and (C).

Proposition 2.5 *For every utility function U_t at time t and $V \in L(\mathcal{F}_T)$, the corresponding certainty equivalent C_t^V has the following properties:*

(1) *For all* $X, Y \in L(\mathcal{F}_T)$*:* $U_t^V(X) \geq U_t^V(Y) \;\Leftrightarrow\; C_t^V(X) \geq C_t^V(Y)$*.*

(2) C_t^V *satisfies* (LP)*,* (SM)*,* (C) *and* $C_t^V(m) = m$ *for all* $m \in L(\mathcal{F}_t)$*.*

(3) *If* U_t *satisfies* (LS)*, then so does* C_t^V*.*

(4) *If* U_t *satisfies* (CQC)*, then so does* C_t^V*.*

(5) *If* U_t *satisfies* (T)*, then* $U_t^V = C_t^V$*.*

(6) *For all* $X, Y, W \in L(\mathcal{F}_T)$*:*
$C_t^V(X) \geq C_t^V(Y) \;\Leftrightarrow\; C_t^W(V - W + X) \geq C_t^W(V - W + Y)$*.*

(7) *If* C_t^V *satisfies* (T) *for some* $V \in L(\mathcal{F}_T)$*, then* C_t^W *satisfies* (T) *for all* $W \in L(\mathcal{F}_T)$*.*

(8) *If* C_t^V *satisfies* (CC)*, then it also satisfies* (T)*.*

Proof. (1)–(5) are obvious. (6) can be derived from (1) and the definition of U_t^V as follows:

$$\begin{aligned} & C_t^V(X) \geq C_t^V(Y) \Leftrightarrow U_t^V(X) \geq U_t^V(Y) \Leftrightarrow U_t(V+X) \geq U_t(V+Y) \\ \Leftrightarrow \quad & U_t^W(V-W+X) \geq U_t^W(V-W+X) \Leftrightarrow C_t^W(V-W+X) \geq C_t^W(V-W+X) \,. \end{aligned}$$

To show (7), let $W, X \in L(\mathcal{F}_T)$ and $m \in L(\mathcal{F}_t)$. It follows from $C_t^W(X) = C_t^W(C_t^W(X))$ and (6) that $C_t^V(W - V + X) = C_t^V(W - V + C_t^W(X))$. If C_t^V satisfies (T), then

$$\begin{aligned} & C_t^V(W-V+X+m) = C_t^V(W-V+X) + m \\ = \quad & C_t^V(W-V+C_t^W(X)) + m = C_t^V(W-V+C_t^W(X)+m) \,, \end{aligned}$$

which by (6) and (2) implies $C_t^W(X+m) = C_t^W(C_t^W(X)+m) = C_t^W(X) + m$.

As for (8), assume that C_t^V satisfies (CC). Then, for all $X \in L(\mathcal{F}_T)$, $m \in L(\mathcal{F}_t)$ and $\lambda \in (0,1)$,

$$C_t^V(X+m) \geq \lambda C_t^V\left(\frac{X}{\lambda}\right) + (1-\lambda) C_t^V\left(\frac{m}{1-\lambda}\right) = \lambda C_t^V\left(\frac{X}{\lambda}\right) + m \,.$$

Since C_t^V has the continuity property (C), we can let λ tend towards 1 to conclude $C_t^V(X + m) \geq C_t^V(X) + m$. This also shows $C_t^V(\tilde{X} + \tilde{m}) \geq C_t^V(\tilde{X}) + \tilde{m}$ for $\tilde{X} = X + m$ and $\tilde{m} = -m$, and (8) is proved. □

Properties (1) and (2) show that C_t^V is a utility function at time t that induces the same conditional preference order on $L(\mathcal{F}_T)$ as U_t^V. Since $C_t^V(0) = 0$, it follows from (LP) that

$$C_t^V(1_A X) = 1_A C_t^V(X) \quad \text{for all } X \in L(\mathcal{F}_t) \text{ and } A \in \mathcal{F}_t \,.$$

Properties (6) and (7) will be needed in the proof of the main result, Theorem 3.4, below.

2.4 Indifference prices

Definition 2.6 *The indifference bid price $b_t^V(X)$ of $X \in L(\mathcal{F}_T)$ with respect to a utility function U_t at time t and endowment $V \in L(\mathcal{F}_T)$ is the unique $m \in L(\mathcal{F}_t)$ such that*

$$U_t^V(X - m) = 0\,. \tag{2.4}$$

$b_t^V(X)$ exists and is unique because U_t^V satisfies (SM), (C) and $U_t^V(0) = 0$. The defining equality (2.4) means

$$U_t(V + X - b_t^V(X)) = U_t(V)\,;$$

in other words, $b_t^V(X)$ is the maximal price that an agent with utility function U_t and endowment V can pay for X at time t without loosing utility. The indifference ask price at time t corresponding to U_t and endowment $V \in L(\mathcal{F}_T)$ is given by $a_t^V(X) = -b_t^V(-X)$. It satisfies

$$U_t(V - X + a_t^V(X)) = U_t(V)$$

and is the minimal price for which the agent can sell X at time t without loosing utility.

Proposition 2.7 *Let b_t^V be the indifference price and C_t^V the certainty equivalent with respect to a utility function U_t at time t and endowment $V \in L(\mathcal{F}_T)$. Then b_t^V has the following properties:*

(1) *b_t^V satisfies* (LP), (SM), (C), (T) *and $b_t^V(m) = m$ for all $m \in L(\mathcal{F}_t)$.*

(2) *If U_t satisfies* (LS), *then so does b_t^V.*

(3) *If U_t satisfies* (CQC), *then b_t^V satisfies* (CC) *and $a_t^V \geq b_t^V$.*

(4) *If U_t satisfies* (T), *then $U_t^V = C_t^V = b_t^V$.*

(5) *If C_t^V satisfies* (T), *then $C_t^V = b_t^V$.*

(6) *For all $X \in L(\mathcal{F}_T)$: $b_t^V(X) = -C_t^{V+X}(-X)$ and $a_t^V(X) = C_t^{V-X}(X)$.*

Proof. The proof of (1), (2), (4) and (5) is straightforward. The first part of (3) follows from the fact that

$$\begin{aligned} & U_t^V(\lambda X + (1-\lambda)Y - \lambda b_t^V(X) - (1-\lambda) b_t^V(Y)) \\ \geq \; & \min\left\{U_t^V(X - b_t^V(X)), U_t^V(Y - b_t^V(Y))\right\} = 0 \end{aligned}$$

for all $X, Y \in L(\mathcal{F}_T)$ and $\lambda \in L(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$. To see that $a_t^V \geq b_t^V$, note that $0 = b_t^V(0) \geq \frac{1}{2} b_t^V(X) + \frac{1}{2} b_t^V(-X)$, and therefore, $a_t^V(X) \geq b_t^V(X)$. To show (6), we fix $X \in L(\mathcal{F}_T)$. $b_t^V(X)$ satisfies $C_t^V(X - b_t^V(X)) = C_t^V(0)$. By (6) of Proposition 2.5, this is equivalent to $-b_t^V(X) = C_t^{V+X}(-b_t^V(X)) = C_t^{V+X}(-X)$, which proves $b_t^V(X) = -C_t^{V+X}(-X)$. Since $C_t^V(-X + a_t^V(X)) = C_t^V(0)$, it follows from (6) of Proposition 2.5 that $a_t^V(X) = C_t^{V-X}(a_t^V(X)) = C_t^{V-X}(X)$. □

As for certainty equivalents, the local property (LP) for b_t^V takes the particular form

$$b_t^V(1_A X) = 1_A b_t^V(X) \quad \text{for all } X \in L(\mathcal{F}_T) \text{ and } A \in \mathcal{F}_t\,.$$

3 Time-consistency and recursiveness

Definition 3.1 *A dynamic utility function is a family $(U_t)_{t=0}^T$ of utility functions at times $t = 0, \dots, T$. We call a dynamic utility function $(U_t)_{t=0}^T$ time-consistent if for all $X, Y \in L(\mathcal{F}_T)$ and $t \leq T-1$,*

$$U_{t+1}(X) \geq U_{t+1}(Y) \quad \textit{implies} \quad U_t(X) \geq U_t(Y)\,. \tag{3.1}$$

If $(U_t)_{t=0}^T$ is not time-consistent, then there exist $X, Y \in L(\mathcal{F}_T)$ and $t \leq T-1$ such that $U_{t+1}(X) \geq U_{t+1}(Y)$ everywhere on Ω, but $U_t(X) < U_t(Y)$ on at least one $\mathcal{F}_t$-atom A_t^k. This means that at time t, in the event A_t^k, the agent prefers Y to X while s/he knows that at time $t+1$, s/he will weakly prefer X to Y in every state of the world.

Dynamic consistency conditions equal or similar to (3.1) have been studied in various contexts; see for instance, Cheridito et al. [4] and the references therein.

In the following lemma we give equivalent conditions for time-consistency of a dynamic utility function.

Lemma 3.2 *Let $(U_t)_{t=0}^T$ be a dynamic utility function with certainty equivalents $(C_t^V)_{t=0}^T$, $V \in L(\mathcal{F}_T)$. Then for fixed $0 \leq s < t \leq T$, the following are equivalent:*

(1) *For all $X, Y \in L(\mathcal{F}_T)$, $U_t(X) \geq U_t(Y)$ implies $U_s(X) \geq U_s(Y)$*

(2) *For all $X, Y, V \in L(\mathcal{F}_T)$, $U_t^V(X) \geq U_t^V(Y)$ implies $U_s^V(X) \geq U_s^V(Y)$*

(3) *$C_s^0(X) = C_s^0(C_t^0(X))$ for all $X \in L(\mathcal{F}_T)$*

(4) *$C_s^V(X) = C_s^V(C_t^V(X))$ for all $X, V \in L(\mathcal{F}_T)$*

Proof.
(1) $\Leftrightarrow$ (2):
First, assume (1) and let $X, Y, V \in L(\mathcal{F}_T)$ such that $U_t^V(X) \geq U_t^V(Y)$. This is equivalent to $U_t(V+X) \geq U_t(V+Y)$. Hence, it follows from (1) that $U_s(V+X) \geq U_s(V+Y)$, which is equivalent to $U_s^V(X) \geq U_s^V(Y)$. This proves (1) $\Rightarrow$ (2). (1) follows from (2) since U_t and U_s induce the same preference orders as U_t^0 and U_s^0, respectively.
(2) $\Leftrightarrow$ (4):
Assume (2) and let $X, V \in L(\mathcal{F}_T)$. By definition of C_t^V, one has $U_t^V(X) = U_t^V(C_t^V(X))$. Hence, it follows from (2) that $U_s^V(X) = U_s^V(C_t^V(X))$, and therefore, $C_s^V(X) = C_s^V(C_t^V(X))$. This shows (2) $\Rightarrow$ (4). If (4) holds and $U_t^V(X) \geq U_t^V(Y)$ for some $X, Y, V \in L(\mathcal{F}_T)$, then $C_t^V(X) \geq C_t^V(Y)$ and therefore, $C_s^V(X) = C_s^V(C_t^V(X)) \geq C_s^V(C_t^V(Y)) = C_s^V(Y)$. This shows $U_s^V(X) \geq U_s^V(Y)$ and hence, (4) $\Rightarrow$ (2).
(1) $\Leftrightarrow$ (3) follows like (2) $\Leftrightarrow$ (4), and the proof is complete. □

Lemma 3.2 allows us to prove the following uniqueness result for time-consistent utility functions.

Proposition 3.3 *Let* $(U_t)_{t=0}^T$ *and* $(\tilde{U}_t)_{t=0}^T$ *be two dynamic utility functions and* $V \in L(\mathcal{F}_T)$. *Denote by* $(C_t^V)_{t=0}^T$ *and* $(\tilde{C}_t^V)_{t=0}^T$ *the certainty equivalents corresponding to* $(U_t)_{t=0}^T$, $(\tilde{U}_t)_{t=0}^T$ *and* V. *Assume that* $C_0^V = \tilde{C}_0^V$. *Then* $C_t^V = \tilde{C}_t^V$ *for all* $1 \le t \le T$.

Proof. Fix $1 \le t \le T$ and $X \in L(\mathcal{F}_T)$. Denote $A = \left\{C_t^V(X) > \tilde{C}_t^V(X)\right\} \in \mathcal{F}_t$ and observe that

$$C_0^V\left(1_A C_t^V(X)\right) = C_0^V(1_A X) = \tilde{C}_0^V(1_A X) = \tilde{C}_0^V\left(1_A \tilde{C}_t^V(X)\right).$$

This implies that A is empty. Otherwise, it would follow from (SM) that $C_0^V\left(1_A C_t^V(X)\right) > \tilde{C}_0^V\left(1_A \tilde{C}_t^V(X)\right)$. Analogously, it follows that the set $\left\{C_t^V(X) < \tilde{C}_t^V(X)\right\} \in \mathcal{F}_t$ is empty, and the proposition is proved. □

We call a sequence of functions $f_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)$, $t = 0, \dots, T$, recursive if $f_s \circ f_t = f_s$ for all $s < t$. The next theorem shows that a dynamic utility function leads to recursive indifference prices if an only if the corresponding certainty equivalents have the translation property (T). The precise statement is as follows. The proof is given in the Appendix.

Theorem 3.4 *Let* $(U_t)_{t=0}^T$ *be a time-consistent dynamic utility function.*

(1) *If* C_t^0 *has the translation property* (T) *for every* $t = 1, \dots, T$, *then* $(b_t^V)_{t=0}^T$ *is recursive for all* $V \in L(\mathcal{F}_T)$.

(2) *If* U_1 *is loss sensitive and* $(b_t^V)_{t=0}^T$ *is recursive for all* $V \in L(\mathcal{F}_T)$, *then* C_t^V *satisfies* (T) *for each* $t = 1, \dots, T$ *and all* $V \in L(\mathcal{F}_T)$.

Remarks 3.5

1. In part (2) of Theorem 3.4 we do not obtain that C_0^V has the translation property. For instance, if $T = 1$, then for every dynamic utility function $(U_t)_{t=0}^1$ and $V \in L(\mathcal{F}_1)$, the corresponding indifference bid prices trivially satisfy $b_0^V(b_1^V(X)) = b_0^V(X)$ for $X \in L(\mathcal{F}_1)$. Hence, $(b_t^V)_{t=0}^1$ is recursive even if the certainty equivalent C_0^V does not have the translation property (T).

2. Part (2) of Theorem 3.4 does not hold if the filtration is not strongly refining. If, for example, $T = 2$, $\mathcal{F}_0 = \mathcal{F}_1$ and $U_0 = U_1$, then for every $V \in L(\mathcal{F}_2)$, $b_0^V = b_1^V$ and consequently, $b_0^V(b_1^V(X)) = b_0^V(X)$ for all $X \in L(\mathcal{F}_2)$. Since one also has $b_1^V(b_2^V(X)) = b_1^V(X)$, $(b_t^V)_{t=0}^2$ is recursive. But again, the corresponding certainty equivalents $C_0^V = C_1^V$ do not necessarily satisfy (T).

Examples 3.6

1. Dynamic utility functions with the translation property

Let $(U_t)_{t=0}^T$ be a time-consistent sequence of utility functions with the translation property (T). Then the normalized utility functions $(U_t^0)_{t=0}^T$ also satisfy (T), and it follows from (5) of Proposition 2.5 that $U_t^0 = C_t^0$. Hence, by part (1) of Theorem 3.4, $(U_t)_{t=0}^T$ induces recursive indifference prices.

It is shown in Föllmer and Schied [10] that every concave function $U : L(\mathcal{F}_T) \to \mathbb{R}$ with the properties (M) and (T) has a representation of the form

$$U(X) = \min_{\mathbb{Q} \in \mathcal{Q}} \{\mathrm{E}_{\mathbb{Q}}[X] + c(\mathbb{Q})\} \tag{3.2}$$

for a non-empty set of probability measures $\mathcal{Q}$ on Ω and a function $c : \mathcal{Q} \to \mathbb{R}$.

If $(U_t)_{t=0}^T$ is a dynamic utility function such that all U_t satisfy (T) and (CC), then it follows from Theorem 3.23 of Cheridito et al. [4] that it can be represented as

$$U_t(X) = \min_{\mathbb{Q} \in \mathcal{Q}_t} \{\mathrm{E}_{\mathbb{Q}}[X \mid \mathcal{F}_t] + c_t(\mathbb{Q})\} , \quad X \in L(\mathcal{F}_T) , t = 0, \ldots, T ,$$

for non-empty subsets $\mathcal{Q}_t$ of strictly positive probability measures on Ω and a sequence of functions c_t from $\mathcal{Q}_t$ to $L(\mathcal{F}_t)$. Necessary and sufficient conditions for time-consistency in terms of the sequence $(\mathcal{Q}_t, c_t)_{t=0}^T$ can be found in Cheridito et al. [4, 5].

A specific example of a time-consistent dynamic utility function with the property (T) is the dynamic entropic utility function

$$U_t(X) := -\frac{1}{\gamma} \log \mathbb{E}_{\mathbb{P}}[e^{-\gamma X} \mid \mathcal{F}_t] , \quad X \in L(\mathcal{F}_T) , t = 0, \ldots, T , \tag{3.3}$$

for a constant $\gamma > 0$. It gives the same preferences as conditional expected exponential utility $\mathbb{E}_{\mathbb{P}}[e^{-\gamma X} \mid \mathcal{F}_t]$. A generalized version of (3.3) is discussed in Example 5.6 of Cheridito et al. [4].

2. Dynamic expected utility

Let $\mathbb{Q}$ be a fixed strictly positive probability measure on Ω and u a strictly increasing, continuous function from $\mathbb{R}$ to $\mathbb{R}$. Define

$$U_t(X) := \mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t] , \quad X \in L(\mathcal{F}_T) , t = 0, \ldots, T .$$

It follows from the tower property of the conditional expectation that $(U_t)_{t=0}^T$ is time-consistent. If $\lim_{x \to -\infty} u(x) = -\infty$, then all U_t are loss sensitive. Hence, it follows from Theorem 3.4 that the indifference bid prices $(b_t^V)_{t=0}^T$ induced by $(U_t)_{t=0}^T$ are recursive for all $V \in L(\mathcal{F}_T)$ if and only if the certainty equivalents

$$C_t^0(X) = u^{-1}\left(\mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t]\right) , \quad X \in L(\mathcal{F}_T) , t = 1, \ldots, T ,$$

have the translation property (T). It is well-known that this is exactly the case when u has constant absolute risk aversion, that is, either

$$\begin{aligned} u(x) &= a + bx \quad \text{for } a \in \mathbb{R} \text{ and } b > 0 , \\ u(x) &= a - be^{-\gamma x} \quad \text{for } a \in \mathbb{R} \text{ and } b, \gamma > 0 , \quad \text{or} \\ u(x) &= a + be^{\gamma x} \quad \text{for } a \in \mathbb{R} \text{ and } b, \gamma > 0 , \end{aligned}$$

where the last case is ruled out by the condition $\lim_{x \to -\infty} u(x) = -\infty$. A related result was obtained by Gerber [12] in Example 5.e of Section 5.4, where the iterativity of the premium principle of zero utility is discussed.

3. Dynamic multi-prior expected utility

Let $(U_t)_{t=0}^T$ be a dynamic utility function given by

$$U_t(X) = \min_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t] \,, \quad X \in L(\mathcal{F}_T)\,,\, t = 0, \dots, T\,,$$

for a non-empty closed convex set $\mathcal{Q}$ of strictly positive probability measures and a strictly increasing, concave function u from $\mathbb{R}$ to $\mathbb{R}$. Necessary and sufficient conditions on the set $\mathcal{Q}$ for $(U_t)_{t=0}^T$ to be time-consistent have been studied in Epstein and Schneider [9], Artzner et al. [2], Delbaen [8], Riedel [22], Roorda et al. [23]. Let us assume that these conditions are satisfied and $\lim_{x \to -\infty} u(x) = -\infty$. Then $(U_t)_{t=0}^T$ is a time-consistent sequence of loss sensitive utility functions. By Theorem 3.4, the corresponding indifference prices are recursive if and only if the certainty equivalents

$$C_t^0(X) = u^{-1}\left(\min_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t]\right) \,, \quad X \in L(\mathcal{F}_T)\,,\, t = 1, \dots, T\,,$$

have the translation property (T). This is equivalent to the fact that for all $t \in \{1, \dots, T\}$ and $m \in L(\mathcal{F}_t)$,

$$\min_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[u(X) \mid \mathcal{F}_t] \quad \text{and} \quad \min_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[u(X+m) \mid \mathcal{F}_t]$$

induce the same conditional preference on $L(\mathcal{F}_T)$. This is the case if and only if for all $m \in \mathbb{R}$, the shifted function $u^m(.) := u(m + .)$ is an affine transformation of u, which is true if and only if u has constant absolute risk aversion.

4 The case of a financial market

Until now the only investment opportunity available to our agent has been to invest in the money market account. In the following we introduce a more general financial market and adapt the definition of indifference prices accordingly. We show that if the agent has a time-consistent utility function whose certainty equivalents have the translation property (T), then indifference prices remain recursive.

To allow for frictions and constraints, we model trading gains attainable over the time interval $[t-1, t]$ by a general set $\mathcal{M}_t$ of $\mathcal{F}_t$-measurable random variables containing 0 and satisfying the following property:

$$1_A X + 1_{A^c} Y \in \mathcal{M}_t \quad \text{for all } X, Y \in \mathcal{M}_t \quad \text{and } A \in \mathcal{F}_{t-1}. \tag{4.1}$$

We set $\hat{U}_T := U_T$,

$$\hat{U}_t(X) := \sup_{Z \in \mathcal{M}_{t+1} + \dots + \mathcal{M}_T} U_t\left(X + Z\right) \quad \text{for} \quad X \in L(\mathcal{F}_T) \text{ and } t \leq T-1$$

and say condition (A) holds if for every $t \le T-1$ and all $X \in L(\mathcal{F}_T)$, there exist unique $Z_{t+1} \in \mathcal{M}_{t+1}, \dots, Z_T \in \mathcal{M}_T$ depending $||.||_\infty$-continuously on X such that $\hat{U}_t(X) = U_t(X + Z_{t+1} + \dots + Z_T)$. The following lemma shows that for a dynamic utility function $(U_t)_{t=0}^T$ and a financial market $(\mathcal{M}_t)_{t=1}^T$ satisfying condition (A), $(\hat{U}_t)_{t=0}^T$ is again a dynamic utility function.

Lemma 4.1 *Let $(U_t)_{t=0}^T$ be a dynamic utility function and $(\mathcal{M}_t)_{t=1}^T$ a financial market such that condition* (A) *holds. Then $\hat{U}_t$ has the properties* (LP), (SM), (C) *for all $t = 0, \dots, T$.*

Proof. Since $\hat{U}_T = U_T$, it is enough to show the claim for $t \le T-1$. $\hat{U}_t$ inherits the local property (LP) from U_t. To show that $\hat{U}_t$ satisfies (SM), consider $X \in L(\mathcal{F}_T)$, $\varepsilon > 0$, $\omega \in \Omega$ and let A_t^k be the $\mathcal{F}_t$-atom containing ω. By condition (A), there exists $Z \in \mathcal{M}_{t+1} + \dots + \mathcal{M}_T$ such that $\hat{U}_t(X) = U_t(X+Z)$, and one obtains

$$\hat{U}_t(X + \varepsilon 1_{\{\omega\}}) \ge U_t(X + \varepsilon 1_{\{\omega\}} + Z) > U_t(X+Z) = \hat{U}_t(X) \quad \text{on } A_t^k .$$

To prove (C), choose a sequence $(X^n)_{n\in\mathbb{N}}$ in $L(\mathcal{F}_T)$ converging to some $X \in L(\mathcal{F}_T)$ with respect to $||.||_\infty$. Then, by condition (A), there exist $Z^n \in \mathcal{M}_{t+1} + \dots + \mathcal{M}_T$ converging to $Z \in \mathcal{M}_{t+1} + \dots + \mathcal{M}_T$ with respect to $||.||_\infty$ such that $\hat{U}_t(X^n) = U_t(X^n + Z^n)$ and $\hat{U}_t(X) = U_t(X+Z)$. Hence, if follows from the continuity of U_t that $\hat{U}_t(X^n) \to \hat{U}_t(X)$. □

We now adapt the definition of indifference prices to the case where there exists a financial market.

Definition 4.2 *Let $(U_t)_{t=0}^T$ be a time-consistent dynamic utility function, $V \in L(\mathcal{F}_T)$ and $(\mathcal{M}_t)_{t=1}^T$ a financial market satisfying condition* (A)*. Let $Z_t \in \mathcal{M}_t$, $1 \le t \le T$, such that $\hat{U}_0(V) = U_0(V + Z_1 + \dots + Z_T)$. The utility indifference bid price $\hat{b}_t^V(X)$ of $X \in L(\mathcal{F}_t)$ with respect to $(U_t)_{t=0}^T$, V and $(\mathcal{M}_t)_{t=1}^T$ is the unique $m \in L(\mathcal{F}_t)$ such that*

$$\hat{U}_t(V + Z_1 + \dots + Z_t + X - m) = \hat{U}_t(V + Z_1 + \dots + Z_t). \tag{4.2}$$

In (4.2) we assume that the agent has endowment V at time 0 and starts trading according to $Z_1, \dots, Z_T$ to maximize utility at time T. But then at time t s/he is offered the payoff X and has to decide whether to buy it at price m and adapt the trading strategy after time t or reject and continue trading as planned.

The following corollary extends part (1) of Theorem 3.4 to the case of a financial market.

Corollary 4.3 *Let $(U_t)_{t=0}^T$ be a time-consistent dynamic utility function and $(\mathcal{M}_t)_{t=1}^T$ a financial market such that condition* (A) *holds. Assume C_t^0 has the translation property* (T) *for every $t = 1, \dots, T$. Then for all $V \in L(\mathcal{F}_T)$ and $t = 0, \dots, T$,*

$$\hat{U}_t(V + X - \hat{b}_t^V(X)) = \hat{U}_t(V), \tag{4.3}$$

and $(\hat{b}_t^V)_{t=0}^T$ is recursive.

Proof. By Lemma 4.1, $(\hat{U}_t)_{t=0}^T$ is a dynamic utility function, and $\hat{C}_t^0$ inherits the translation property (T) from C_t^0 for all $t = 1, \dots, T$. As a consequence one has (4.3) for every $t = 1, \dots, T$, and for $t = 0$, (4.3) is equal to the defining equation (4.2). To complete the proof, it is enough to show that $(\hat{U}_t)_{t=0}^T$ is time-consistent. It then follows from Theorem 3.4 and (4.3) that $(\hat{b}_t^V)_{t=0}^T$ is recursive. To show the time-consistency of $(\hat{U}_t)_{t=0}^T$, let $s < t$ and $X, Y \in L(\mathcal{F}_T)$ such that $\hat{U}_t(X) \geq \hat{U}_t(Y)$. Then there exists a $Z^X \in \mathcal{M}_{t+1} + \dots + \mathcal{M}_T$ such that $U_t(X + Z^X) \geq U_t(Y + Z^2)$ for every $Z^2 \in \mathcal{M}_{t+1} + \dots + \mathcal{M}_T$. Since the preference order induced by U_t is translation-invariant, one has $U_t(X + Z^1 + Z^X) \geq U_t(Y + Z^1 + Z^2)$ for all $Z^1 \in \mathcal{M}_{s+1} + \dots + \mathcal{M}_t$, and therefore, by time-consistency, $U_s(X + Z^1 + Z^X) \geq U_s(Y + Z^1 + Z^2)$. This shows that $\hat{U}_s(X) \geq \hat{U}_s(Y)$, and the corollary is proved. □

A Appendix: Proof of Theorem 3.4

In the whole appendix, $(U_t)_{t=0}^T$ is a dynamic utility function. C_t^V and b_t^V denote the certainty equivalent and indifference bid price corresponding to U_t and endowment $V \in L(\mathcal{F}_T)$.

Proof of Theorem 3.4.1.
Let $V \in L(\mathcal{F}_T)$. It follows from the time-consistency of $(U_t)_{t=0}^T$ and Lemma 3.2 that $(C_t^V)_{t=0}^T$ is recursive. By (7) of Proposition 2.5, C_t^V satisfies (T) for all $t = 1, \dots, T$. Hence, it follows from (5) of Proposition 2.7 that $C_t^V = b_t^V$ for all $t = 1, \dots, T$. This shows that $(b_t^V)_{t=1}^T$ is recursive. It remains to prove that

$$b_0^V(b_1^V(X)) = b_0^V(X) \quad \text{for all } X \in L(\mathcal{F}_T)\,. \tag{A.1}$$

From $C_1^V = b_1^V$ and $C_0^V = C_0^V \circ C_1^V$ one obtains

$$C_0^V(X - b_0^V(b_1^V(X))) = C_0^V(C_1^V(X - b_0^V(C_1^V(X)))) = C_0^V(C_1^V(X) - b_0^V(C_1^V(X))) = 0\,,$$

which shows (A.1) and concludes the proof. □

Our proof of Theorem 3.4.2 is based on a characterization of time-consistency of dynamic utility functions in terms of indifference sets. This extends results on the decomposition property of acceptance sets of monetary risk measures in Delbaen [8] and Cheridito et al. [4].

For $0 \leq s \leq t \leq T$ and $V \in L(\mathcal{F}_T)$, we introduce the indifference set

$$\mathcal{I}_{s,t}^V := \left\{X \in L(\mathcal{F}_t) : U_s^V(X) = 0\right\}\,.$$

It follows from the definitions of U_s^V and C_s^0 that

$$\mathcal{I}_{s,t}^V = \{X \in L(\mathcal{F}_t) : U_s(V + X) = U_s(V)\} = \left\{X \in L(\mathcal{F}_t) : C_s^0(V + X) = C_s^0(V)\right\}\,.$$

Thus, $\mathcal{I}_{s,t}^V$ consists of all payoffs $X \in L(\mathcal{F}_t)$ that leave an agent with endowment V indifferent at time s. Also, it is clear that

$$\mathcal{I}_{s,t}^V = \left\{X \in L(\mathcal{F}_t) : C_s^V(X) = 0\right\} = \left\{X \in L(\mathcal{F}_t) : b_s^V(X) = 0\right\}\,.$$

The local property (LP) of U_s^V translates into the following property for $\mathcal{I}_{s,t}^V$:

(LP') **Local property:** $1_A X + 1_{A^c} Y \in \mathcal{I}_{s,t}^V$ for all $X, Y \in \mathcal{I}_{s,t}^V$ and $A \in \mathcal{F}_s$.

Proposition A.1 *For all $0 \le s < t \le T$, the following are equivalent:*

(1) $C_s^0(X) = C_s^0(C_t^0(X))$ *for all* $X \in L(\mathcal{F}_T)$

(2) $C_s^V(X) = C_s^V(C_t^V(X))$ *for all* $V, X \in L(\mathcal{F}_T)$

(3) $\mathcal{I}_{s,T}^V = \bigcup_{Y \in \mathcal{I}_{s,t}^V} \left(Y + \mathcal{I}_{t,T}^{V+Y}\right)$ *for all* $V \in L(\mathcal{F}_s)$

(4) $\mathcal{I}_{s,T}^V = \bigcup_{Y \in \mathcal{I}_{s,t}^V} \left(Y + \mathcal{I}_{t,T}^{V+Y}\right)$ *for all* $V \in L(\mathcal{F}_T)$

Proof. We show (1) ⇒ (2) ⇒ (4) ⇒ (3) ⇒ (1). (1) ⇒ (2) is part of Lemma 3.2.
(2) ⇒ (4): Fix $V \in L(\mathcal{F}_T)$. We first prove $\mathcal{I}_{s,T}^V \supset \bigcup_{Y \in \mathcal{I}_{s,t}^V} \left(Y + \mathcal{I}_{t,T}^{V+Y}\right)$. Let $Y \in \mathcal{I}_{s,t}^V$ and $Z \in \mathcal{I}_{t,T}^{V+Y}$. Since

$$C_t^{V+Y}(Z) = 0 = C_t^{V+Y}(0)\,,$$

it follows from (6) of Proposition 2.5 that

$$C_t^V(Y+Z) = C_t^V(Y)\,.$$

So (2) implies

$$C_s^V(Y+Z) = C_s^V(C_t^V(Y+Z)) = C_s^V(C_t^V(Y)) = C_s^V(Y) = 0\,,$$

which shows that $Y + Z \in \mathcal{I}_{s,T}^V$.

To show $\mathcal{I}_{s,T}^V \subset \bigcup_{Y \in \mathcal{I}_{s,t}^V} \left(Y + \mathcal{I}_{t,T}^{V+Y}\right)$, we choose $X \in \mathcal{I}_{s,T}^V$ and decompose it into $X = Y + Z$ for $Y = C_t^V(X)$ and $Z = X - C_t^V(X)$. Since

$$C_s^V(C_t^V(X)) = C_s^V(X) = 0\,,$$

$Y = C_t^V(X)$ is in $\mathcal{I}_{s,t}^V$. Moreover, by (6) of Proposition 2.5, it follows from

$$C_t^V(X) = C_t^V(C_t^V(X))$$

that

$$C_t^{V+C_t^V(X)}(X - C_t^V(X)) = C_t^{V+C_t^V(X)}(C_t^V(X) - C_t^V(X)) = 0\,,$$

which shows that $Z = X - C_t^V(X)$ is in $\mathcal{I}_{t,T}^{V+C_t^V(X)} = \mathcal{I}_{t,T}^{V+Y}$.
(4) ⇒ (3) is trivial.
(3) ⇒ (1):
Let $X \in L(\mathcal{F}_T)$ and write

$$X - C_s^0(C_t^0(X)) = Y + Z$$

for

$$Y = C_t^0(X) - C_s^0(C_t^0(X)) \quad \text{and} \quad Z = X - C_t^0(X)\,.$$

By (6) of Proposition 2.5, it follows from

$$C_s^0(C_t^0(X)) = C_s^0(C_s^0(C_t^0(X))) \quad \text{and} \quad C_t^0(X) = C_t^0(C_t^0(X))$$

that

$$C_s^{C_s^0(C_t^0(X))}(Y) = C_s^{C_s^0(C_t^0(X))}(0) = 0 \quad \text{and} \quad C_t^{C_t^0(X)}(Z) = C_t^{C_t^0(X)}(0) = 0\,,$$

and therefore,

$$Y \in \mathcal{I}_{s,t}^{C_s^0(C_t^0(X))} \quad \text{and} \quad Z \in \mathcal{I}_{t,T}^{C_t^0(X)}\,.$$

(3) implies $Y + Z = X - C_s^0(C_t^0(X)) \in \mathcal{I}_{s,T}^{C_s^0(C_t^0(X))}$, which shows that

$$C_s^{C_s^0(C_t^0(X))}(X - C_s^0(C_t^0(X))) = 0 = C_s^{C_s^0(C_t^0(X))}(0)\,,$$

and therefore, by (6) of Proposition 2.5, $C_s^0(X) = C_s^0(C_s^0(C_t^0(X))) = C_s^0(C_t^0(X))$. □

Corollary A.2 *Fix $V \in L(\mathcal{F}_T)$ and $0 \le s < t \le T$. Then the following are equivalent:*

(1) *For all $X, Y \in L(\mathcal{F}_T)$: $b_t^V(X) \ge b_t^V(Y)$ implies $b_s^V(X) \ge b_s^V(Y)$*

(2) *$b_s^V(X) = b_s^V(b_t^V(X))$ for all $X \in L(\mathcal{F}_T)$*

(3) *$\mathcal{I}_{s,T}^V = \mathcal{I}_{s,t}^V + \mathcal{I}_{t,T}^V$*

Proof. For fixed $V \in L(\mathcal{F}_T)$ and $r \in \{s,t\}$, $\tilde{U}_r := b_r^V$ is a utility function at time r which is equal to its own certainty equivalent. Therefore, the equivalence of (1) and (2) follows directly from the equivalence of (1) and (3) in Lemma 3.2. Furthermore, one has

$$\tilde{\mathcal{I}}_{r,T}^0 := \left\{ X \in L(\mathcal{F}_T) : \tilde{U}_r(X) = 0 \right\} = \mathcal{I}_{r,T}^V\,, \tag{A.2}$$

and since b_r^V satisfies (T),

$$\tilde{\mathcal{I}}_{r,T}^W := \left\{ X \in L(\mathcal{F}_T) : \tilde{U}_r(X + W) = \tilde{U}_r(W) \right\} = \tilde{\mathcal{I}}_{r,T}^0 \quad \text{for all } W \in L(\mathcal{F}_r)\,. \tag{A.3}$$

It follows from the equivalence of (1) and (3) in Proposition A.1 that (2) is equivalent to

$$\tilde{\mathcal{I}}_{s,T}^W = \bigcup_{Y \in \tilde{\mathcal{I}}_{s,t}^W} (Y + \tilde{\mathcal{I}}_{t,T}^{W+Y}) \quad \text{for all } W \in L(\mathcal{F}_s)\,.$$

By (A.3), this reduces to

$$\tilde{\mathcal{I}}_{s,T}^0 = \tilde{\mathcal{I}}_{s,t}^0 + \tilde{\mathcal{I}}_{t,T}^0\,,$$

which by (A.2), is equivalent to (3). □

Corollary A.3 *Assume that $(U_t)_{t=0}^T$ is time-consistent and $(b_t^V)_{t=0}^T$ recursive for all $V \in L(\mathcal{F}_T)$. Then*

$$\mathcal{I}_{t,T}^V = \mathcal{I}_{t,T}^{V+Y}$$

for all $V \in L(\mathcal{F}_T)$, $t = 1, \ldots, T$, and $Y \in \mathcal{I}_{t-1,t}^V$.

Proof. Fix $V \in L(\mathcal{F}_T)$ and $Y \in \mathcal{I}_{t-1,t}^V$.

i) We first show $\mathcal{I}_{t,T}^{V+Y} \subset \mathcal{I}_{t,T}^V$:

Assume to the contrary that there exists a Z in $\mathcal{I}_{t,T}^{V+Y} \setminus \mathcal{I}_{t,T}^V$. Then, at least one of the sets $A := \left\{b_t^V(Z) > 0\right\}$ or $B := \left\{b_t^V(Z) < 0\right\}$ is non-empty. Let us assume A is. The case where B is non-empty works completely analogously. Since $\mathcal{I}_{t,T}^{V+Y}$ has the local property (LP'), $1_A Z$ is still in $\mathcal{I}_{t,T}^{V+Y}$. On the other hand, one has

$$b_t^V(1_A Z) = 1_A b_t^V(Z) > 0 \quad \text{on } A\,. \tag{A.4}$$

By Proposition A.1, $Y + 1_A Z \in \mathcal{I}_{t-1,T}^V$, and by Corollary A.2, $\mathcal{I}_{t-1,T}^V = \mathcal{I}_{t-1,t}^V + \mathcal{I}_{t,T}^V$. But

$$Y + 1_A Z = (Y + b_t^V(1_A Z)) + (1_A Z - b_t^V(1_A Z))$$

is the unique decomposition of $Y + 1_A Z$ into the sum of two random variables such that the first one is in $L(\mathcal{F}_t)$ and the second one in $\mathcal{I}_{t,T}^V$. Therefore, $Y + b_t^V(1_A Z)$ must belong to $\mathcal{I}_{t-1,T}^V$. But together with (A.4), this contradicts the strict monotonicity (SM) of U_{t-1}^V. This shows $\mathcal{I}_{t,T}^{V+Y} \subset \mathcal{I}_{t,T}^V$.

ii) $\mathcal{I}_{t,T}^V \subset \mathcal{I}_{t,T}^{V+Y}$:

Since Y is in $\mathcal{I}_{t-1,t}^V$, we have

$$C_{t-1}^V(0) = 0 = C_{t-1}^V(Y)\,,$$

which by (6) of Proposition 2.5 is equivalent to

$$C_{t-1}^{V+Y}(-Y) = C_{t-1}^{V+Y}(0) = 0\,.$$

This shows that $-Y$ is in $\mathcal{I}_{t-1,t}^{V+Y}$. So it follows from i) that

$$\mathcal{I}_{t,T}^V = \mathcal{I}_{t,T}^{V+Y-Y} \subset \mathcal{I}_{t,T}^{V+Y}\,.$$

□

Proof of Theorem 3.4.2.

We fix $V, X \in L(\mathcal{F}_T)$, $t \in \{1, \ldots, T\}$ and show that

$$C_t^V(X+m) = C_t^V(X) + m \tag{A.5}$$

for all $m \in L(\mathcal{F}_t)$.

i) In a first step we show (A.5) for $m \in \mathcal{I}_{t-1,t}^{V+X}$:
By (6) of Proposition 2.5, it follows from

$$C_t^V(C_t^V(X)) = C_t^V(X)$$

that

$$C_t^{V+X}(C_t^V(X) - X) = C_t^{V+X}(0) = 0\,.$$

Hence, $C_t^V(X) - X \in \mathcal{I}_{t,T}^{V+X}$. By Corollary A.3, $C_t^V(X) - X \in \mathcal{I}_{t,T}^{V+X+m}$, or equivalently,

$$C_t^{V+X+m}(C_t^V(X) - X) = 0 = C_t^{V+X+m}(0)\,,$$

which by (6) of Proposition 2.5 shows that

$$C_t^V(X) + m = C_t^V(C_t^V(X) + m) = C_t^V(X + m)\,.$$

ii) We now show (A.5) for all $m \in L_+(\mathcal{F}_t) := \{Z \in L(\mathcal{F}_t) : Z \geq 0\}$:
Every $m \in L_+(\mathcal{F}_t)$ can be written as $m = \sum_{k=1}^{N_t} m_k 1_{A_t^k}$, for the atoms $A_t^1, \ldots, A_t^{N_t}$ of $\mathcal{F}_t$ and non-negative real numbers $m_1, \ldots, m_{N_t}$. For every k, we set $\bar{A}_t^k := \mathcal{F}_{t-1}(A_t^k) \setminus A_t^k$, where $\mathcal{F}_{t-1}(A_t^k)$ denotes the $\mathcal{F}_{t-1}$-atom that contains A_t^k. By assumption, the filtration is strongly refining. Therefore, $\bar{A}_t^k$ is non-empty. U_t^{V+X} satisfies (M), (C) and (LS) (that it satisfies (LS) follows from the fact that U_1 does). Therefore, there exist $n_k \in \mathbb{R}$ such that $U_{t-1}^{V+X}(m_k 1_{A_t^k} + n_k 1_{\bar{A}_t^k}) = 0$. Hence,

$$Y_k := m_k 1_{A_t^k} + n_k 1_{\bar{A}_t^k} \in \mathcal{I}_{t-1,t}^{V+X}\,.$$

Now, since C_t^V satisfies (LP), one obtains from i) that

$$\begin{aligned} C_t^V(X+m) &= C_t^V\left(X + \sum_{k=1}^{N_t} m_k 1_{A_t^k}\right) = \sum_{k=1}^{N_t} 1_{A_t^k} C_t^V(X + Y_k) \\ &= \sum_{k=1}^{N_t} 1_{A_t^k}(C_t^V(X) + Y_k) = C_t^V(X) + \sum_{k=1}^{N_t} m_k 1_{A_t^k} = C_t^V(X) + m\,. \end{aligned}$$

iii) Finally, we show (A.5) for general $m \in L(\mathcal{F}_t)$:
For $m \leq 0$, it follows from ii) that

$$C_t^V(X) = C_t^V(X + m - m) = C_t^V(X + m) - m\,,$$

and therefore,

$$C_t^V(X + m) = C_t^V(X) + m\,. \tag{A.6}$$

For general $m \in L(\mathcal{F}_t)$, denote $m^+ := \max(m, 0)$ and $m^- := -\min(m, 0)$. Then, by ii) and (A.6),

$$\begin{aligned} C_t^V(X+m) &= C_t^V(X + m^+ - m^-) = C_t^V(X - m^-) + m^+ \\ &= C_t^V(X) + m^+ - m^- = C_t^V(X) + m\,. \end{aligned}$$

□

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