University of Konstanz
Department of Mathematics and Statistics
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Nonlinear partial differential equations

Exercises and Questions, Part 3

Question 1. Let $G \subset \mathbb{R}^n$ be a domain. A function $u\colon G \to \mathbb{R}$ is called convex if

$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(x) \ (t \in [0,1],\ x, y \in U)$$

holds for all open convex subsets $U \subset G$ with $\overline{U} \subset G$ and $\overline{U}$ compact. The function u is called semi-convex if there exists a constant $C > 0$ such that $x \mapsto u(x) + \frac{C}{2}|x|^2$ is convex.

Let $u \in C^2(G)$ be semi-convex and $\theta \in C^2(\mathbb{R})$ with $\theta'(s) > 0$ $(s \in \mathbb{R})$. Show that $\theta \circ u$ is semi-convex. Can we replace $C^2(G)$ by $W_p^2(G)$ with $p \in [1, \infty]$? What do you know about convex functions in $W_\infty^1(G)$?

Question 2. Let $H \subset \mathbb{R}^{n+1}$ be a domain, U be a dense subset of $H \times \mathbb{R}^{n+1} \times \mathbb{R}_{\text{sym}}^{(n+1)\times(n+1)}$, and $E\colon U \to \mathbb{R}$ be a function. Then the equation $E = 0$ is called geometric in U if for all $\lambda > 0$ and $\mu \in \mathbb{R}$ there exist constants $C_1(\lambda, \mu), C_2(\lambda, \mu) > 0$ such that

$$C_1(\lambda, \mu) E(y, q, Y) \leq E(y, \lambda q, \lambda Y + \mu q \otimes q) \leq C_2(\lambda, \mu) E(y, q, Y)$$

holds whenever all terms are defined.

a) Show that if $E = 0$ is geometric in U then $E_* = 0$ and $E^* = 0$ are both geometric in $\overline{U}$.

b) Let $G \subset \mathbb{R}^n$ be a domain, $F\colon (0, T) \times G \times \mathbb{R}^n \times \mathbb{R}_{\text{sym}}^{n \times n} \to \mathbb{R}$ be a function. Set $D := \binom{\nabla}{\partial_t}$, $y := (t, x)$, and $E(y, Du(y), D^2u(y)) := \partial_t u + F(t, x, \nabla u(t, x), \nabla^2 u(t, x))$. Show that $E = 0$ is geometric if and only if for all $\lambda > 0$ and $\mu \in \mathbb{R}$

$$F(t, x, \lambda p, \lambda X + \mu p \otimes p) = \lambda F(t, x, p, X)$$

holds whenever all terms are defined.

Question 3. Show that the MCF equation

$$\partial_t u(t, x) - |\nabla u(t, x)| \operatorname{div}\Big(\frac{\nabla u(t, x)}{|\nabla u(t, x)|}\Big) = 0$$

is geometric.

Please prepare your answers for presentation and discussion on June 2nd (15.15-16.45, F 420).