Discrete Fourier multipliers and cylindrical boundary-value problems

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We consider operator-valued boundary-value problems in $(0, 2\pi)^n$ with periodic or, more generally, ν-periodic boundary conditions. Using the concept of discrete vector-valued Fourier multipliers, we give equivalent conditions for the unique solvability of the boundary-value problem. As an application, we study vector-valued parabolic initial boundary-value problems in cylindrical domains $(0, 2\pi)^n \times V$ with ν-periodic boundary conditions in the cylindrical directions. We show that, under suitable assumptions on the coefficients, we obtain maximal L^q-regularity for such problems. For symmetric operators such as the Laplacian, related results for mixed Dirichlet–Neumann boundary conditions on $(0, 2\pi)^n \times V$ are deduced.

1. Introduction

In this paper we first study boundary-value problems with operator-valued coefficients of the form

$$P(D)u + Q(D)Au = f \quad \text{in } (0, 2\pi)^n, \tag{1.1}$$

$$D^\beta u|_{x_j=2\pi} - e^{2\pi\nu_j} D^\beta u|_{x_j=0} = 0 \quad (j = 1, \ldots, n, \ |\beta| < m_1). \tag{1.2}$$

Here, $P(D)$ is a partial differential operator of order m_1 acting on $u = u(x)$ with $x \in (0, 2\pi)^n$, $Q(D)$ is a partial differential operator of order $m_2 \leqslant m_1$, A is a closed linear operator acting in a Banach space X, and $\nu := (\nu_1, \ldots, \nu_n)^{\mathrm{T}} \in \mathbb{C}^n$. We refer to the boundary conditions as ν-periodic. Note that for $\nu_j = 0$ we have periodic boundary conditions in direction j, whereas for $\nu_j = \mathrm{i}/2$ we have antiperiodic boundary conditions in this direction. In general, we have different boundary conditions (i.e. different ν_j) in different directions.

As a motivation for studying problem (1.1), (1.2), we mention two classes of problems. First, the boundary-value problem (1.1), (1.2) includes equations of the form

$$u_t(t) + Au(t) = f(t) \quad (t \in (0, 2\pi)) \tag{1.3}$$

and

$$u_{tt}(t) - aAu_t(t) - \alpha Au(t) = f(t) \quad (t \in (0, 2\pi)) \tag{1.4}$$

with periodic or ν-periodic boundary conditions. Equations of the form (1.3), (1.4) were considered in [1] and [19], respectively. These equations fit into our context

by taking $n = 1$, $P(D) = \partial_t$ and $Q(D) = 1$ for (1.3) and by taking $P(D) = \partial_t^2$, $Q(D) = -a\partial_t - \alpha$ for (1.4).

As a second motivation for studying (1.1), (1.2), we consider a boundary-value problem of cylindrical type where the domain is of the form $\Omega = (0, 2\pi)^n \times V$, with $V \subset \mathbb{R}^{n_V}$ being a sufficiently smooth domain with compact boundary. The operator is assumed to split in the sense that

$$\mathcal{A}(x, D) = P(x^1, D_1) + Q(x^1, D_1)A_V(x^2, D_2), \qquad (1.5)$$

where the differential operators $P(x^1, D_1)$ and $Q(x^1, D_1)$ act on $x^1 \in (0, 2\pi)^n$ only and the differential operator $A_V(x^2, D_2)$ acts on $x^2 \in V$ only. The boundary conditions are assumed to be ν-periodic in the x^1-direction, whereas in V the operator $A_V(x^2, D_2)$ of order $2m_V$ may be supplemented with general boundary conditions $B_1(x^2, D_2), \ldots, B_{m_V}(x^2, D_2)$. The simplest example of such an operator is the Laplacian in a finite cylinder $(0, 2\pi)^n \times V$ with ν-periodic boundary conditions in the cylindrical directions and Dirichlet boundary conditions on $(0, 2\pi)^n \times \partial V$.

Our first main result (theorem 3.6) gives, under appropriate assumptions on P, Q and A, equivalent conditions for the unique solvability of (1.1), (1.2) in L^p-Sobolev spaces. This result generalizes results from [1] and [19] on (1.3) and (1.4), respectively.

In particular, in connection with operators of the form (1.5) in cylindrical domains, one is also interested in parabolic theory. Therefore, in §4 we study problems of the form

$$
\left.
\begin{aligned}
u_t + \mathcal{A}(x, D)u &= f & (t \in [0, T], \ x \in (0, 2\pi)^n \times V), \\
B_j(x, D)u &= 0 & (t \in [0, T], \ x \in (0, 2\pi)^n \times \partial V, \\
& & j = 1, \ldots, m_V), \\
(D^\beta u)|_{x_j=2\pi} - \mathrm{e}^{2\pi\nu_j}(D^\beta u)|_{x_j=0} &= 0 & (j = 1, \ldots, n; \ |\beta| < m_1), \\
u(0, x) &= u_0(x) & (x \in (0, 2\pi)^n \times V).
\end{aligned}
\right\}
$$
$$(1.6)$$

Here, $\mathcal{A}(x, D)$ is defined as in (1.5). If $(A_V, B_1, \ldots, B_{m_V})$ is a parabolic boundary-value problem in the sense of parameter-ellipticity (see [10, §8]), we obtain, under suitable assumptions on P and Q, maximal L^q-regularity for (1.6) (see theorems 4.3 and 4.6). The proof of maximal regularity is based on the $\mathcal{R}$-boundedness of the resolvent related to (1.6).

Periodic boundary values appear, for instance, in the study of the formation of keratin networks, which are a component of the cytoskeleton of biological cells. In [3] the evolution of a pool of soluble polymers fuelling network growth is modelled by the Laplace operator with periodic boundary conditions.

Apart from being of interest in itself, the consideration of ν-periodic boundary conditions also allows us to address boundary conditions of mixed type. As the simplest example, when $a = 0$ we can analyse (1.4) with Dirichlet–Neumann-type boundary conditions

$$u(0) = 0, \qquad u_t(\pi) = 0.$$

The connection to periodic and antiperiodic boundary conditions is given by suitable extensions of the solution. This was also considered in [1], where, starting from

periodic boundary conditions, the pure Dirichlet and the pure Neumann case could be treated.

The main tool to address problems (1.1), (1.2) and (1.6) is the theory of discrete vector-valued Fourier multipliers. Taking the Fourier series in the cylindrical directions, we are faced with the question of under which conditions an operator-valued Fourier series defines a bounded operator in L^p. This question was answered by Arendt and Bu in [1] for the one-dimensional case $n = 1$, where a discrete operator-valued Fourier multiplier result for UMD spaces and applications to periodic Cauchy problems of first and second order in Lebesgue and Hölder spaces can be found. For general n, the main result on vector-valued Fourier multipliers is contained in [7]. A shorter proof of this result by means of induction, based on the result for $n = 1$ in [1], is given in [6]. As pointed out by Arendt and Bu in [1] and Bu and Kim in [7], the results can also be deduced from [24, theorems 3.7 and 3.8].

A generalization of the results in [1] to periodic first-order integro-differential equations in Lebesgue, Besov and Hölder spaces is given in [18]. Here the concept of 1-regularity in the context of sequences is introduced (see remark 2.11).

In [19] one finds a comprehensive treatment of periodic second-order differential equations of type (1.4) in Lebesgue and Hölder spaces. In particular, the special case of a Cauchy problem of second order, i.e. $\alpha = 0$, $a = 1$, where A is the generator of a strongly continuous cosine function, is investigated. In [20] more general equations are treated in the aforementioned spaces as well as in Triebel–Lizorkin spaces. Moreover, applications to nonlinear equations are presented.

Maximal regularity of second-order initial-value problems of the type

$$u_{tt}(t) + Bu_t(t) + Au(t) = f(t) \quad (t \in [0, T)),$$
$$u(0) = u_t(0) = 0$$

is treated in [8] and [9]. In particular, p-independence of maximal regularity for second-order problems of this type is shown. The same equation involving dynamic boundary conditions is studied in [27]. The non-autonomous second-order problem, involving t-dependent operators $B(t)$ and $A(t)$, is treated in [5]. We also refer the reader to [26] for the treatment of higher-order Cauchy problems.

In [2] various properties such as, for example, the Fredholmness of the operator $\partial_t - A(\cdot)$ associated with the non-autonomous periodic first-order Cauchy problem in the L^p-context are investigated. Results on this operator based on Floquet theory are obtained in [14]. We remark that in Floquet theory ν-periodic (instead of periodic) boundary conditions appear in a natural way.

For the treatment of boundary-value problems in $(0, 1)$ with operator-valued coefficients subject to numerous types of homogeneous and inhomogeneous boundary conditions, we refer the reader to [11–13] and the references therein. Their approaches rely mainly on semigroup theory and do not allow for an easy generalization to $(0, 1)^n$. In [13], however, applications to boundary-value problems in the cylindrical space domain $(0, 1) \times V$ can be found.

The use of operator-valued multipliers to treat cylindrical-in-space boundary-value problems was first carried out in [15, 16] in a Besov space setting. In these papers, Guidotti constructs semi-classical fundamental solutions for a class of elliptic operators on infinite cylindrical domains $\mathbb{R}^n \times V$. This proves to be a strong tool for the treatment of related elliptic and parabolic (see [15] and [16]), as well

as hyperbolic (see [16]), problems. Operators in cylindrical domains with a similar splitting property to those in the present paper were, in the case of an infinite cylinder, also considered in [22].

2. Discrete Fourier multipliers and $\mathcal{R}$-boundedness

In the subsequent lines, let X and Y be Banach spaces, let $1 \leqslant p < \infty$, let $n \in \mathbb{N}$ and let $\mathcal{Q}_n := (0, 2\pi)^n$. By $\mathcal{L}(X, Y)$ we denote the space of all bounded linear operators from X to Y, and we set $\mathcal{L}(X) := \mathcal{L}(X, X)$. By $L^p(\mathcal{Q}_n, X)$ we denote the standard Bochner space of X-valued L^p-functions defined on $\mathcal{Q}_n$ (see, for example, [4]). For $f \in L^p(\mathcal{Q}_n, X)$ and $\boldsymbol{k} \in \mathbb{Z}^n$, the $\boldsymbol{k}$th Fourier coefficient of f is given by

$$\hat{f}(\boldsymbol{k}) := \frac{1}{(2\pi)^n} \int_{\mathcal{Q}_n} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot x} f(x) \, \mathrm{d}x. \tag{2.1}$$

By Fejer's theorem we see that $f(x) = 0$ almost everywhere if $\hat{f}(\boldsymbol{k}) = 0$ for all $\boldsymbol{k} \in \mathbb{Z}^n$, and that $f(x) = \hat{f}(\boldsymbol{0})$ almost everywhere if $\hat{f}(\boldsymbol{k}) = 0$ for all $\boldsymbol{k} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. Moreover, for $f, g \in L^p(\mathcal{Q}_n, X)$ and a closed operator A in X it holds that $f(x) \in D(A)$ and $Af(x) = g(x)$ almost everywhere if and only if $\hat{f}(\boldsymbol{k}) \in D(A)$ and $A\hat{f}(\boldsymbol{k}) = \hat{g}(\boldsymbol{k})$ for all $\boldsymbol{k} \in \mathbb{Z}^n$. We will make frequent use of these observations without further comment.

DEFINITION 2.1. A function $M \colon \mathbb{Z}^n \to \mathcal{L}(X, Y)$ is called a (discrete) L^p-multiplier if for each $f \in L^p(\mathcal{Q}_n, X)$ there exists a $g \in L^p(\mathcal{Q}_n, Y)$ such that

$$\hat{g}(\boldsymbol{k}) = M(\boldsymbol{k})\hat{f}(\boldsymbol{k}) \quad (\boldsymbol{k} \in \mathbb{Z}^n).$$

In this case there exists a unique operator $T_M \in \mathcal{L}(L^p(\mathcal{Q}_n, X), L^p(\mathcal{Q}_n, Y))$ associated with M such that

$$(T_M f)\hat{}(\boldsymbol{k}) = M(\boldsymbol{k})\hat{f}(\boldsymbol{k}) \quad (\boldsymbol{k} \in \mathbb{Z}^n). \tag{2.2}$$

The property of being a Fourier multiplier is closely related to the concept of $\mathcal{R}$-boundedness. Here, we give only the definition and some properties that will be used later on; as references for $\mathcal{R}$-boundedness we mention [21] and [10].

DEFINITION 2.2. A family $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded if there exist a $C > 0$ and a $p \in [1, \infty)$ such that, for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and all independent symmetric $\{-1, 1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{A}, P)$ (e.g. the Rademacher sequence) for $j = 1, \ldots, N$, we have that

$$\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_{L^p(\Omega, Y)} \leqslant C_p \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_{L^p(\Omega, X)}. \tag{2.3}$$

The smallest $C_p > 0$ such that (2.3) is satisfied is called the $\mathcal{R}_p$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}_p(\mathcal{T})$.

By Kahane's inequality, (2.3) holds for all $p \in [1, \infty)$ if it holds for one $p \in [1, \infty)$. Therefore, we will drop the p-dependence of $\mathcal{R}_p(\mathcal{T})$ in the notation and write $\mathcal{R}(\mathcal{T})$.

LEMMA 2.3.

(a) *Let Z be a third Banach space and let $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X, Y)$ and $\mathcal{U} \subset \mathcal{L}(Y, Z)$ be $\mathcal{R}$-bounded. Then, $\mathcal{T} + \mathcal{S}$, $\mathcal{T} \cup \mathcal{S}$ and $\mathcal{U}\mathcal{T}$ are $\mathcal{R}$-bounded as well and we have that*

$$\mathcal{R}(\mathcal{T} + \mathcal{S}), \mathcal{R}(\mathcal{T} \cup \mathcal{S}) \leqslant \mathcal{R}(\mathcal{S}) + \mathcal{R}(\mathcal{T}), \qquad \mathcal{R}(\mathcal{U}\mathcal{T}) \leqslant \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

Furthermore, if $\bar{\mathcal{T}}$ denotes the closure of $\mathcal{T}$ with respect to the strong operator topology, then we have that $\mathcal{R}(\bar{\mathcal{T}}) = \mathcal{R}(\mathcal{T})$.

(b) Contraction principle of Kahane. *Let $p \in [1, \infty)$. Then, for all $N \in \mathbb{N}$, $x_j \in X$, ε_j as above, and for all $a_j, b_j \in \mathbb{C}$ with $|a_j| \leqslant |b_j|$ for $j = 1, \ldots, N$, we have that*

$$\left\| \sum_{j=1}^{N} a_j \varepsilon_j x_j \right\|_{L^p(\Omega, X)} \leqslant 2 \left\| \sum_{j=1}^{N} b_j \varepsilon_j x_j \right\|_{L^p(\Omega, X)}. \tag{2.4}$$

For $M: \mathbb{Z}^n \to \mathcal{L}(X, Y)$ and $1 \leqslant j \leqslant n$ we inductively define the differences (discrete derivatives)

$$\Delta_j^\ell M(\boldsymbol{k}) := \Delta_j^{\ell-1} M(\boldsymbol{k}) - \Delta_j^{\ell-1} M(\boldsymbol{k} - \boldsymbol{e}_j) \quad (\ell \in \mathbb{N}, \ \boldsymbol{k} \in \mathbb{Z}^n),$$

where $\boldsymbol{e}_j$ denotes the jth unit vector in $\mathbb{R}^n$, and where we have set $\Delta_j^0 M(\boldsymbol{k}) := M(\boldsymbol{k})(\boldsymbol{k} \in \mathbb{Z}^n)$. As $\Delta_i^{\gamma_i}$ and $\Delta_j^{\gamma_j}$ commute for $1 \leqslant i, j \leqslant n$, for a multi-index $\gamma \in \mathbb{N}_0^n$ the expression

$$\Delta^\gamma M(\boldsymbol{k}) := (\Delta_1^{\gamma_1} \cdots \Delta_n^{\gamma_n} M)(\boldsymbol{k}) \quad (\boldsymbol{k} \in \mathbb{Z}^n)$$

is well defined. Given $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, we write $\alpha \leqslant \gamma \leqslant \beta$ if $\alpha_j \leqslant \gamma_j \leqslant \beta_j$ for all $1 \leqslant j \leqslant n$. We also set $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\boldsymbol{0} := (0, \ldots, 0)$ and $\boldsymbol{1} := (1, \ldots, 1)$. We agree to write $\boldsymbol{0} < \gamma$ if $\boldsymbol{0} \leqslant \gamma$ and $0 < \gamma_j$ for at least one $1 \leqslant j \leqslant n$.

We recall that a Banach space X is called a UMD space, or a Banach space of class $\mathcal{HT}$, if for some $q \in (1, \infty)$ (equivalently, if for all $q \in (1, \infty)$) the Hilbert transform defines a bounded operator in $L^q(\mathbb{R}, X)$. Thus, Hilbert spaces are UMD spaces. For $1 < p < \infty$ and an arbitrary domain $G \subset \mathbb{R}^n$ the spaces $X := L^p(G, E)$ are UMD spaces, provided the Banach space E has the UMD property. In particular, $L^p(G)$ is a UMD space. A Banach space X is said to have the property (α) if there exists a $C > 0$ such that for all $N \in \mathbb{N}$, all $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leqslant 1$, all $x_{ij} \in X$ and all independent symmetric $\{+1, -1\}$-valued random variables $\varepsilon_i^{(1)}$ on a probability space $(\Omega_1, \mathcal{A}_1, P_1)$ and $\varepsilon_j^{(2)}$ on a probability space $(\Omega_2, \mathcal{A}_2, P_2)$ for $i, j = 1, \ldots, N$, we have that

$$\left\| \sum_{i,j=1}^{N} \alpha_{ij} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{ij} \right\|_{L^2(\Omega_1 \times \Omega_2, X)} \leqslant C \left\| \sum_{i,j=1}^{N} \varepsilon_i^{(1)} \varepsilon_j^{(2)} x_{ij} \right\|_{L^2(\Omega_1 \times \Omega_2, X)}.$$

Again, the spaces $X := L^p(G, E)$ enjoy the property (α), provided that E has this property. Here we do not have to exclude $p = 1$. Since $\mathbb{C}$ is known to have property (α), this extends to the space $L^p(G)$. The following result from Bu and Kim [7] provides a sufficient condition for discrete Fourier multipliers by $\mathcal{R}$-boundedness. Here and henceforth we are confined to the range $1 < p < \infty$.

THEOREM 2.4 (Bu and Kim [7]).

(a) *Let $1 < p < \infty$, let X, Y be UMD spaces and let $\mathcal{T} \subset \mathcal{L}(X, Y)$ be $\mathcal{R}$-bounded. If $M \colon \mathbb{Z}^n \to \mathcal{L}(X, Y)$ satisfies*

$$\{|\boldsymbol{k}|^{|\gamma|}\Delta^\gamma M(\boldsymbol{k}) \colon \boldsymbol{k} \in \mathbb{Z}^n \setminus [-1,1]^n, \ \boldsymbol{0} < \gamma \leqslant \boldsymbol{1}\} \cup \{M(\boldsymbol{k}) \colon \boldsymbol{k} \in \mathbb{Z}^n\} \subset \mathcal{T}, \ (2.5)$$

then M defines a Fourier multiplier.

(b) *If X, Y additionally enjoy property (α), then*

$$\{\boldsymbol{k}^\gamma \Delta^\gamma M(\boldsymbol{k}) \colon \boldsymbol{k} \in \mathbb{Z}^n \setminus [-1,1]^n, \ \boldsymbol{0} < \gamma \leqslant \boldsymbol{1}\} \cup \{M(\boldsymbol{k}) \colon \boldsymbol{k} \in \mathbb{Z}^n\} \subset \mathcal{T} \quad (2.6)$$

is sufficient. In this case the set

$$\{T_M \colon \ M \text{ satisfies condition (2.6)}\} \subset \mathcal{L}(L^p(\mathcal{Q}_n, X), L^p(\mathcal{Q}_n, Y))$$

is again $\mathcal{R}$-bounded.

REMARK 2.5. In [7], theorem 2.4 is stated with discrete derivatives $\tilde{\Delta}$ defined in such a way that $\Delta^\gamma M(\boldsymbol{k} + \gamma) = \tilde{\Delta}^\gamma M(\boldsymbol{k})$. However, as for fixed $\boldsymbol{0} \leqslant \gamma \leqslant \boldsymbol{1}$ there exist $c, C > 0$ such that $c|\boldsymbol{k} - \gamma| \leqslant |\boldsymbol{k}| \leqslant C|\boldsymbol{k} - \gamma|$ for $\boldsymbol{k} \in \mathbb{Z}^n \setminus [-1,1]^n$, lemma 2.3 shows our formulation to be equivalent to the one in [7]. Throughout this article, we make frequent use of this estimate without any further comment. Furthermore, Bu and Kim [7] chose the slightly stronger conditions

$$\{|\boldsymbol{k}|^{|\gamma|}\Delta^\gamma M(\boldsymbol{k}) \colon \boldsymbol{k} \in \mathbb{Z}^n, \ \boldsymbol{0} \leqslant \gamma \leqslant \boldsymbol{1}\} \subset \mathcal{T} \tag{2.7}$$

and

$$\{\boldsymbol{k}^\gamma \Delta^\gamma M(\boldsymbol{k}) \colon \boldsymbol{k} \in \mathbb{Z}^n, \ \boldsymbol{0} \leqslant \gamma \leqslant \boldsymbol{1}\} \subset \mathcal{T} \tag{2.8}$$

in their article. However, the proof is the same and conditions (2.5) and (2.6) are more convenient to verify.

The following lemma states some properties for discrete derivatives, where

$$(S_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathbb{Z}^n} \quad \text{and} \quad (T_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathbb{Z}^n}$$

denote arbitrary commuting sequences in $\mathcal{L}(X)$. An introduction to difference operators including the subsequent Leibniz rule can be found in [23]. For $\alpha \in \mathbb{N}_0^n \setminus \{\boldsymbol{0}\}$, let

$$\mathcal{Z}_\alpha := \left\{\mathcal{W} = (\omega^1, \ldots, \omega^r); \ 1 \leqslant r \leqslant |\alpha|, \ \boldsymbol{0} \leqslant \omega^j \leqslant \alpha, \ \omega^j \neq \boldsymbol{0}, \ \sum_{j=1}^r \omega^j = \alpha\right\}$$

denote the set of all additive decompositions of α into $r = r_{\mathcal{W}}$ multi-indices, and set $\mathcal{Z}_{\boldsymbol{0}} := \{\emptyset\}$ and $r_\emptyset := 0$. For $\mathcal{W} \in \mathcal{Z}_\alpha$ we set $\omega_j^* := \sum_{l=j+1}^r \omega^l$. In the following, $c_{\alpha,\beta}$ and $c_{\mathcal{W}}$ denote integer constants depending on α and β and $\mathcal{W}$, respectively.

LEMMA 2.6.

(a) Leibniz rule. *For $\alpha \in \mathbb{N}_0^n$ and $\boldsymbol{k} \in \mathbb{Z}^n$ we have that*

$$\Delta^\alpha(ST)(\boldsymbol{k}) = \sum_{\boldsymbol{0} \leqslant \beta \leqslant \alpha} c_{\alpha,\beta}(\Delta^{\alpha-\beta}S)(\boldsymbol{k} - \beta)(\Delta^\beta T)(\boldsymbol{k}).$$

(b) *Let* $(S^{-1})(\boldsymbol{k}) := (S_{\boldsymbol{k}})^{-1}$ *exist for all* $\boldsymbol{k} \in \mathbb{Z}^n$. *Then, for* $\alpha \in \mathbb{N}_0^n$ *and* $\boldsymbol{k} \in \mathbb{Z}^n$ *we have that*

$$\Delta^{\alpha}(S^{-1})(\boldsymbol{k}) = \sum_{\mathcal{W} \in \mathcal{Z}_{\alpha}} c_{\mathcal{W}}(S^{-1})(\boldsymbol{k} - \alpha) \prod_{j=1}^{r_{\mathcal{W}}} ((\Delta^{\omega^j} S) S^{-1})(\boldsymbol{k} - \omega_j^*).$$

Proof. We show both assertions by induction on $|\alpha|$, the case $|\alpha| = 0$ being obvious. For part (a) see also [23, lemma 3.3.6].

(a) By definition, we have that

$$(\Delta^{\boldsymbol{e}_j}(ST))(\boldsymbol{k}) = (ST)(\boldsymbol{k}) - (ST)(\boldsymbol{k} - \boldsymbol{e}_j) = S(\boldsymbol{k} - \boldsymbol{e}_j)(\Delta^{\boldsymbol{e}_j}T)(\boldsymbol{k}) + (\Delta^{\boldsymbol{e}_j}S)(\boldsymbol{k})T(\boldsymbol{k}),$$

and for $\alpha' := \alpha - \boldsymbol{e}_j$, where $\alpha_j \neq 0$, we obtain that

$$\begin{aligned}(\Delta^{\alpha}(ST))(\boldsymbol{k}) &= \Delta^{\boldsymbol{e}_j} \sum_{\beta \leqslant \alpha'} c_{\alpha'\beta}(\Delta^{\alpha'-\beta}S)(\boldsymbol{k} - \beta)(\Delta^{\beta}T)(\boldsymbol{k}) \\ &= \sum_{\beta \leqslant \alpha} c_{\alpha\beta}(\Delta^{\alpha-\beta}S)(\boldsymbol{k} - \beta)(\Delta^{\beta}T)(\boldsymbol{k}).\end{aligned}$$

(b) For $|\alpha| \geqslant 1$, we apply (a) to SS^{-1} and get that

$$\begin{aligned}0 &= (\Delta^{\alpha}(SS^{-1}))(\boldsymbol{k}) \\ &= S(\boldsymbol{k} - \alpha)(\Delta^{\alpha}S^{-1})(\boldsymbol{k}) + \sum_{\beta < \alpha} c_{\alpha\beta}(\Delta^{\alpha-\beta}S)(\boldsymbol{k} - \beta)(\Delta^{\beta}S^{-1})(\boldsymbol{k}).\end{aligned}$$

Hence,

$$\begin{aligned}(\Delta^{\alpha}S^{-1})(\boldsymbol{k}) &= -S^{-1}(\boldsymbol{k} - \alpha) \sum_{\beta < \alpha} c_{\alpha\beta}(\Delta^{\alpha-\beta}S)(\boldsymbol{k} - \beta)(\Delta^{\beta}S^{-1})(\boldsymbol{k}) \\ &= -\sum_{\beta < \alpha} \sum_{\mathcal{W} \in \mathcal{Z}_{\beta}} c_{\mathcal{W}} S^{-1}(\boldsymbol{k} - \alpha)((\Delta^{\alpha-\beta}S)S^{-1})(\boldsymbol{k} - \beta) \\ &\qquad\qquad\qquad \times \prod_{j=1}^{r_{\mathcal{W}}} ((\Delta^{\omega^j} S) S^{-1})(\boldsymbol{k} - \omega_j^*) \\ &= \sum_{\mathcal{W} \in \mathcal{Z}_{\alpha}} c_{\mathcal{W}} S^{-1}(\boldsymbol{k} - \alpha)((\Delta^{\omega^1} S) S^{-1})(\boldsymbol{k} - \omega_1^*) \\ &\qquad\qquad\qquad \times \prod_{j=2}^{r_{\mathcal{W}}} ((\Delta^{\omega^j} S) S^{-1})(\boldsymbol{k} - \omega_j^*).\end{aligned}$$

$$\square$$

DEFINITION 2.7. Consider a polynomial $P \colon \mathbb{R}^n \to \mathbb{C}$; $\xi \mapsto P(\xi)$ and let $P^{\#}$ denote its principal part.

  (a) P is called elliptic if $P^{\#}(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$.

  (b) Let $\phi \in (0, \pi)$ and let $\Sigma_{\phi} := \{\lambda \in \mathbb{C} \setminus \{0\} \colon |\arg(\lambda)| < \phi\}$ be the open sector with opening angle 2ϕ. Then, P is called parameter-elliptic in $\bar{\Sigma}_{\pi-\phi}$ if $\lambda + P^{\#}(\xi) \neq 0$ for $(\lambda, \xi) \in \bar{\Sigma}_{\pi-\phi} \times \mathbb{R}^n \setminus \{(0, \boldsymbol{0})\}$. In this case,

$$\varphi_P := \inf\{\phi \in (0, \pi) \colon P \text{ is parameter-elliptic in } \bar{\Sigma}_{\pi-\phi}\}$$

    is called the angle of parameter-ellipticity of P.

REMARK 2.8.

(a) By quasi-homogeneity of $(\lambda, \xi) \mapsto \lambda + P^{\#}(\xi)$, we easily see that P is parameter-elliptic in $\bar{\Sigma}_{\pi - \phi}$ if and only if, for all polynomials N with $\deg N \leqslant \deg P$, there exist $C > 0$ and a bounded subset $G \subset \mathbb{R}^n$ such that the estimate $|\xi|^m |N(\xi)| \leqslant C |\lambda + P(\xi)|$ holds for all $\lambda \in \bar{\Sigma}_{\pi - \phi}$, all $0 \leqslant m \leqslant \deg P - \deg N$ and all $\xi \in \mathbb{R}^n \setminus G$ (see, for example, [17, theorem 3.3.6]).

(b) In the same way, P is elliptic if and only if the assertion in (a) is valid for $\lambda = 0$.

(c) By induction, one can see that for $|\alpha| \leqslant \deg P$ the discrete polynomial $\Delta^{\alpha} P(\boldsymbol{k})$ defines a polynomial of degree not greater than $\deg P - |\alpha|$. If P is elliptic, by (b) this implies that $|\boldsymbol{k}|^{|\alpha|} |\Delta^{\alpha} P(\boldsymbol{k})| \leqslant C |P(\boldsymbol{k})| (\boldsymbol{k} \in \mathbb{Z}^n \setminus G)$ for some finite set $G \subset \mathbb{Z}^n$.

In what follows the assumption that $(\lambda + \mu A)^{-1}$ exists for $\lambda, \mu \in \mathbb{C}$ is meant to imply both that $(\lambda + \mu A)^{-1} \in \mathcal{L}(X)$ and that $(\lambda + \mu A)^{-1}(X) = D(A)$. Hence, $\mu \neq 0$ and $\lambda \in \rho(-\mu A)$.

PROPOSITION 2.9. *Let A be a closed linear operator in a UMD space X. Consider polynomials $P, Q \colon \mathbb{Z}^n \to \mathbb{C}$ such that*

(i) *P and Q are elliptic,*

(ii) *$(P(\boldsymbol{k}) + Q(\boldsymbol{k})A)^{-1}$ exists for all $\boldsymbol{k} \in \mathbb{Z}^n$,*

(iii) *$\{P(\boldsymbol{k})(P(\boldsymbol{k}) + Q(\boldsymbol{k})A)^{-1} \colon \boldsymbol{k} \in \mathbb{Z}^n\}$ is $\mathcal{R}$-bounded.*

Then, for every polynomial N with $\deg N \leqslant \deg P$, the map

$$M \colon \mathbb{Z}^n \to \mathcal{L}(X); \quad \boldsymbol{k} \mapsto N(\boldsymbol{k})(P(\boldsymbol{k}) + Q(\boldsymbol{k})A)^{-1}$$

defines an L^p-multiplier for $1 < p < \infty$.

Proof. Lemma 2.6 yields that

$$|\boldsymbol{k}|^{|\gamma|} \Delta^{\gamma} M(\boldsymbol{k})$$
$$= \sum_{\beta \leqslant \gamma} \sum_{\mathcal{W} \in \mathcal{Z}_{\beta}} c_{\mathcal{W}} |\boldsymbol{k}|^{|\gamma - \beta|} (\Delta^{\gamma - \beta} N)(\boldsymbol{k} - \beta)(P(\boldsymbol{k} - \beta) + Q(\boldsymbol{k} - \beta)A)^{-1}$$
$$\times \prod_{j=1}^{r_{\mathcal{W}}} |\boldsymbol{k}|^{|\omega^j|} (\Delta^{\omega^j} P(\boldsymbol{k} - \omega_j^*) + \Delta^{\omega^j} Q(\boldsymbol{k} - \omega_j^*)A)(P(\boldsymbol{k} - \omega_j^*) + Q(\boldsymbol{k} - \omega_j^*)A)^{-1}.$$

By remark 2.8, we know that $\deg(\Delta^{\gamma - \beta} N) \leqslant \deg N - |\gamma - \beta|$. This and the ellipticity of P imply that $|\boldsymbol{k}|^{|\gamma - \beta|} |\Delta^{\gamma - \beta} N(\boldsymbol{k})| \leqslant C |P(\boldsymbol{k})|$ for $\boldsymbol{k} \in \mathbb{Z}^n \setminus G$ with a finite set $G \subset \mathbb{Z}^n$. By Kahane's contraction principle, we obtain the $\mathcal{R}$-boundedness of

$$\{|\boldsymbol{k}|^{|\gamma - \beta|} \Delta^{\gamma - \beta} N(\boldsymbol{k} - \beta)(P(\boldsymbol{k} - \beta) + Q(\boldsymbol{k} - \beta)A)^{-1} \colon \boldsymbol{k} \in \mathbb{Z}^n \setminus G\}.$$

Since

$$Q(\boldsymbol{k})A(P(\boldsymbol{k}) + Q(\boldsymbol{k})A)^{-1} = \mathrm{id}_X - P(\boldsymbol{k})(P(\boldsymbol{k}) + Q(\boldsymbol{k})A)^{-1},$$

in the same way the $\mathcal{R}$-boundedness of

$$\{|\boldsymbol{k}|^{|\omega^j|}\Delta^{\omega^j}Q(\boldsymbol{k}-\omega_j^*)A(P(\boldsymbol{k}-\omega_j^*)+Q(\boldsymbol{k}-\omega_j^*)A)^{-1}\colon \boldsymbol{k}\in\mathbb{Z}^n\setminus G\}$$

follows from the ellipticity of Q. Now, the assertion follows from lemma 2.3 and theorem 2.4. $\qquad\square$

Proposition 2.9 is closely related to the concept of 1-regularity of complex-valued sequences, introduced in [18] for the one-dimensional case $n=1$. In fact, if $Q(\boldsymbol{k})\neq 0$ for all $\boldsymbol{k}\in\mathbb{Z}^n$, we may write

$$M(\boldsymbol{k})=\frac{N(\boldsymbol{k})}{Q(\boldsymbol{k})}\left(\frac{P(\boldsymbol{k})}{Q(\boldsymbol{k})}+A\right)^{-1}.$$

Hence, for $n=1$ we enter the framework of [20, proposition 5.3], i.e. $M(k)=a_k(b_k-A)^{-1}$, with $(a_k)_{k\in\mathbb{Z}},(b_k)_{k\in\mathbb{Z}}\subset\mathbb{C}$. We give a generalization of this concept to arbitrary n and briefly indicate the connection to the results above.

DEFINITION 2.10. We call a pair of sequences $(a_{\boldsymbol{k}},b_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^n}\subset\mathbb{C}^2$ 1-regular if for all $\boldsymbol{0}\leqslant\gamma\leqslant\boldsymbol{1}$ there exist a finite set $K\subset\mathbb{Z}^n$ and a constant $C>0$ such that

$$|\boldsymbol{k}^\gamma|\max\{|(\Delta^\gamma a)_{\boldsymbol{k}}|,|(\Delta^\gamma b)_{\boldsymbol{k}}|\}\leqslant C|b_{\boldsymbol{k}}|\quad(\boldsymbol{k}\in\mathbb{Z}^n\setminus K). \tag{2.9}$$

We say that the pair $(a_{\boldsymbol{k}},b_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^n}$ is strictly 1-regular if $|\boldsymbol{k}^\gamma|$ can be replaced by $|\boldsymbol{k}|^{|\gamma|}$ in (2.9). A sequence $(a_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^n}$ is called (strictly) 1-regular if $(a_{\boldsymbol{k}},a_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^n}$ has this property.

REMARK 2.11.

(a) In the case $n=1$, a sequence $(a_k)_{k\in\mathbb{Z}}\subset\mathbb{C}\setminus\{0\}$ is 1-regular in $\mathbb{Z}$ in the sense of definition 2.10 if and only if the sequence $(k(a_{k+1}-a_k)/a_k)_{k\in\mathbb{Z}}$ is bounded. Hence, our definition extends the one from [18] for a sequence $(a_k)_{k\in\mathbb{Z}}$.

(b) With $\gamma=0$, the definition especially requests that $|a_{\boldsymbol{k}}|\leqslant C|b_{\boldsymbol{k}}|$ for $\boldsymbol{k}\in\mathbb{Z}^n\setminus K$.

(c) Strict 1-regularity implies 1-regularity. If $n=1$, both concepts are equivalent.

(d) Under the assumptions of proposition 2.9, if $Q(\boldsymbol{k})\neq 0$ for $\boldsymbol{k}\in\mathbb{Z}^n$, then the pair $(a_{\boldsymbol{k}},b_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^n}$, with $a_{\boldsymbol{k}}:=N(\boldsymbol{k})/Q(\boldsymbol{k})$, $b_{\boldsymbol{k}}:=P(\boldsymbol{k})/Q(\boldsymbol{k})$, is strictly 1-regular.

(e) Again from lemma 2.6, we deduce the following variant of proposition 2.9. Let $b_{\boldsymbol{k}}\in\rho(A)$ for all $\boldsymbol{k}\in\mathbb{Z}^n$, let $\mathcal{R}(\{b_{\boldsymbol{k}}(b_{\boldsymbol{k}}-A)^{-1}\colon\boldsymbol{k}\in\mathbb{Z}^n\setminus G\})<\infty$ for some finite subset $G\subset\mathbb{Z}^n$, and let $(a_{\boldsymbol{k}},b_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^n}$ be strictly 1-regular. Then, $M(\boldsymbol{k}):=a_{\boldsymbol{k}}(b_{\boldsymbol{k}}-A)^{-1}$ defines a Fourier multiplier.

3. ν-periodic boundary-value problems

DEFINITION 3.1. Let X be a Banach space, let $m\in\mathbb{N}_0$, $n\in\mathbb{N}$ and $\nu\in\mathbb{C}^n$. We set $D^\alpha:=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, with $D_j=-\mathrm{i}\partial/\partial j$, and denote by $W_{\nu,\mathrm{per}}^{m,p}(\mathcal{Q}_n,X)$ the space of all $u\in W^{m,p}(\mathcal{Q}_n,X)$ such that for all $j\in\{1,\dots,n\}$ and all $|\alpha|<m$ it holds that

$$(D^\alpha u)|_{x_j=2\pi}=\mathrm{e}^{2\pi\nu_j}(D^\alpha u)|_{x_j=0}.$$

For the sake of convenience we set $W_{\mathrm{per}}^{m,p}(\mathcal{Q}_n,X):=W_{0,\mathrm{per}}^{m,p}(\mathcal{Q}_n,X)$.

We give some useful characterizations of the space $W_{\nu,\mathrm{per}}^{m,p}(\mathcal{Q}_n, X)$ where we omit the rather simple proof.

LEMMA 3.2. *The following assertions are equivalent:*

(i) $u \in W_{\nu,\mathrm{per}}^{m,p}(\mathcal{Q}_n, X)$;

(ii) $u \in W^{m,p}(\mathcal{Q}_n, X)$ *and for all* $|\alpha| \leqslant m$ *it holds that*

$$(\mathrm{e}^{-\nu\cdot}D^\alpha u)^{\widehat{\,}}(\boldsymbol{k}) = (\boldsymbol{k} - \mathrm{i}\nu)^\alpha (\mathrm{e}^{-\nu\cdot}u)^{\widehat{\,}}(\boldsymbol{k})$$

for all $\boldsymbol{k} \in \mathbb{Z}^n$;

(iii) *there exists* $v \in W_{\mathrm{per}}^{m,p}(\mathcal{Q}_n, X)$ *such that* $u = \mathrm{e}^{\nu\cdot}v$.

The following lemma characterizes multipliers such that the associated operators map $L^p(\mathcal{Q}_n, X)$ into $W_{\mathrm{per}}^{\alpha,p}(\mathcal{Q}_n, X)$. The proof follows that for the case $n = 1$ of [1, lemma 2.2].

LEMMA 3.3. *Let* $1 \leqslant p < \infty$, $m \in \mathbb{N}$ *and* $M: \mathbb{Z}^n \to \mathcal{L}(X)$. *Then, the following assertions are equivalent:*

(i) M *is an* L^p*-multiplier such that the associated operator* $T_M \in \mathcal{L}(L^p(\mathcal{Q}_n, X))$ *maps* $L^p(\mathcal{Q}_n, X)$ *into* $W_{\mathrm{per}}^{m,p}(\mathcal{Q}_n, X)$;

(ii) $M_\alpha: \mathbb{Z}^n \to \mathcal{L}(X)$; $\boldsymbol{k} \mapsto \boldsymbol{k}^\alpha M(\boldsymbol{k})$ *is an* L^p*-multiplier for all* $|\alpha| = m$.

Let X be a UMD space and let A be a closed linear operator in X. With $n \in \mathbb{N}$ and $\nu \in \mathbb{C}^n$ we consider the boundary-value problem in $\mathcal{Q}_n$ given by

$$\left.\begin{array}{l} \mathcal{A}(D)u = f \quad (x \in \mathcal{Q}_n), \\ (D^\beta u)|_{x_j=2\pi} - \mathrm{e}^{2\pi\nu_j}(D^\beta u)|_{x_j=0} = 0 \quad (j = 1, \ldots, n; \ |\beta| < m_1). \end{array}\right\} \tag{3.1}$$

In view of the boundary conditions, we refer to the boundary-value problem (3.1) as ν-periodic. Here,

$$\mathcal{A}(D) := P(D) + Q(D)A := \sum_{|\alpha|\leqslant m_1} p_\alpha D^\alpha + \sum_{|\alpha|\leqslant m_2} q_\alpha D^\alpha A,$$

with $m_1, m_2 \in \mathbb{N}$, $m_2 \leqslant m_1$ and $p_\alpha, q_\alpha \in \mathbb{C}$. In what follows, with $m := m_1$, we frequently write

$$\mathcal{A}(D) = \sum_{|\alpha|\leqslant m} (p_\alpha D^\alpha + q_\alpha D^\alpha A),$$

where additional coefficients q_α are understood to be equal to zero, that is, where $m_2 < |\alpha| \leqslant m_1$. We also consider the complex polynomials

$$P(z) := \sum_{|\alpha|\leqslant m_1} p_\alpha z^\alpha \quad \text{and} \quad Q(z) := \sum_{|\alpha|\leqslant m_2} q_\alpha z^\alpha \quad \text{for } z \in \mathbb{C}^n.$$

DEFINITION 3.4. A solution of the boundary-value problem (3.1) is understood as a function $u \in W_{\nu,\mathrm{per}}^{m_1,p}(\mathcal{Q}_n, X) \cap W^{m_2,p}(\mathcal{Q}_n, D(A))$ such that $\mathcal{A}(D)u(x) = f(x)$ for almost every $x \in \mathcal{Q}_n$.

REMARK 3.5. Since the trace operator with respect to one direction and the tangential derivation commute, the ν-periodic boundary conditions as imposed in (3.1) are equivalent to

$$(D_j^\ell u)|_{x_j=2\pi} - e^{2\pi\nu_j}(D_j^\ell u)|_{x_j=0} = 0 \quad (j = 1,\ldots,n,\ 0 \leqslant \ell < m_1).$$

Recall that the existence of $(\lambda + \mu A)^{-1}$ implies both $\mu \neq 0$ and $\lambda \in \rho(-\mu A)$.

THEOREM 3.6. *Let* $1 < p < \infty$, *and assume that* P *and* Q *are elliptic. The following assertions are then equivalent:*

(i) *For each* $f \in L^p(\mathcal{Q}_n, X)$ *there exists a unique solution of (3.1);*

(ii) $(P(\boldsymbol{k} - i\nu) + Q(\boldsymbol{k} - i\nu)A)^{-1}$ *exists for all* $\boldsymbol{k} \in \mathbb{Z}^n$, *and*

$$M_\alpha(\boldsymbol{k}) := \boldsymbol{k}^\alpha(P(\boldsymbol{k} - i\nu) + Q(\boldsymbol{k} - i\nu)A)^{-1}$$

defines a Fourier multiplier for every $|\alpha| = m_1$;

(iii) $(P(\boldsymbol{k} - i\nu) + Q(\boldsymbol{k} - i\nu)A)^{-1}$ *exists for all* $\boldsymbol{k} \in \mathbb{Z}^n$, *and for all* $|\alpha| = m_1$ *there exists a finite subset* $G \subset \mathbb{Z}^n$ *such that the sets* $\{M_\alpha(\boldsymbol{k});\ \boldsymbol{k} \in \mathbb{Z}^n \setminus G\}$ *are $\mathcal{R}$-bounded.*

Proof. (i) $\Rightarrow$ (ii). Let $f \in L^p(\mathcal{Q}_n, X)$ be arbitrary and let u be a solution of (3.1) with right-hand side $e^{\nu \cdot} f$. Then, $e^{-\nu \cdot}\mathcal{A}(D)u = f$.

To compute the Fourier coefficients, we first remark that

$$(e^{-\nu \cdot}P(D)u)\widehat{}(\boldsymbol{k}) = P(\boldsymbol{k} - i\nu)(e^{-\nu \cdot}u)\widehat{}(\boldsymbol{k}) \quad (\boldsymbol{k} \in \mathbb{Z}^n)$$

by lemma 3.2. Concerning $e^{-\nu \cdot}Q(D)Au$, note that by definition of a solution we have $Au \in W^{m_2,p}(\mathcal{Q}_n, X)$. Due to the closedness of A, we obtain that $D^\alpha Au = AD^\alpha u$ for $|\alpha| \leqslant m_2$, and, consequently, $Au \in W^{m_2,p}_{\nu,\mathrm{per}}(\mathcal{Q}_n, X)$. We can now apply lemma 3.2 to see that

$$(e^{-\nu \cdot}Q(D)Au)\widehat{}(\boldsymbol{k}) = Q(\boldsymbol{k} - i\nu)(e^{-\nu \cdot}Au)\widehat{}(\boldsymbol{k}) = Q(\boldsymbol{k} - i\nu)A(e^{-\nu \cdot}u)\widehat{}(\boldsymbol{k}).$$

Writing $\boldsymbol{k}_\nu := \boldsymbol{k} - i\nu$ for short, we obtain that

$$(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)(e^{-\nu \cdot}u)\widehat{}(\boldsymbol{k}) = \hat{f}(\boldsymbol{k}) \quad (\boldsymbol{k} \in \mathbb{Z}^n). \tag{3.2}$$

For arbitrary $y \in X$ and $\boldsymbol{k} \in \mathbb{Z}^n$, the choice $f := e^{i\boldsymbol{k}\cdot}y$ shows $(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)$ to be surjective. Let $z \in D(A)$ such that $(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)z = 0$. For fixed $\boldsymbol{k} \in \mathbb{Z}^n$ set $v := e^{i\boldsymbol{k}\cdot}z$ and $u := e^{\nu \cdot}v$. Then,

$$P(\boldsymbol{k}_\nu)(e^{-\nu \cdot}u)\widehat{}(\boldsymbol{k}) + Q(\boldsymbol{k}_\nu)A(e^{-\nu \cdot}u)\widehat{}(\boldsymbol{k}) = 0.$$

As $(e^{-\nu \cdot}u)\widehat{}(\boldsymbol{m}) = 0$ for all $\boldsymbol{m} \neq \boldsymbol{k}$, this gives $\mathcal{A}(D)u = 0$; hence, $v = u = 0$ and $z = 0$.

Altogether, we have shown bijectivity of $P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A$ for $\boldsymbol{k} \in \mathbb{Z}^n$. The closedness of A yields that $(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1} \in \mathcal{L}(X)$.

For $f \in L^p(\mathcal{Q}_n, X)$ let u be a solution of (3.1) with right-hand side $e^{\nu \cdot}f$ and $v := e^{-\nu \cdot}u$. Then, $v \in W^{m_1,p}_{\mathrm{per}}(\mathcal{Q}_n, X)$, and (3.2) implies that

$$\hat{v}(\boldsymbol{k}) = (P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1}\hat{f}(\boldsymbol{k}) \quad (\boldsymbol{k} \in \mathbb{Z}^n).$$

This shows

$$M_0 \colon \mathbb{Z}^n \to \mathcal{L}(L^p(\mathcal{Q}_n, X)); \ \boldsymbol{k} \mapsto (P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1}$$

to be a Fourier multiplier such that T_{M_0} maps $L^p(\mathcal{Q}_n, X)$ into $W_{\mathrm{per}}^{m_1,p}(\mathcal{Q}_n, X)$. Due to lemma 3.3, we have that M_α is a Fourier multiplier for all $|\alpha| = m_1$.

(ii) $\Rightarrow$ (iii). This follows as in [1, proposition 1.11].

(iii) $\Rightarrow$ (i). For $\boldsymbol{k} \neq \boldsymbol{0}$ it holds that

$$P(\boldsymbol{k}_\nu)(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1} = \frac{P(\boldsymbol{k}_\nu)}{\sum_{j=1}^n \boldsymbol{k}^{m_1 \boldsymbol{e}_j}} \left(\sum_{j=1}^n \boldsymbol{k}^{m_1 \boldsymbol{e}_j} (P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1} \right),$$

and as there exists $C > 0$ such that $|P(\boldsymbol{k}_\nu)| \leqslant C |\sum_{j=1}^n \boldsymbol{k}^{m_1 \boldsymbol{e}_j}|$ for $\boldsymbol{k} \in \mathbb{Z}^n \setminus G$ with suitably chosen finite $G \subset \mathbb{Z}^n$, lemma 2.3 shows that the set

$$\{P(\boldsymbol{k}_\nu)(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1} \colon \boldsymbol{k} \in \mathbb{Z}^n \setminus G\}$$

is $\mathcal{R}$-bounded as well. By proposition 2.9 it follows that M_α for $|\alpha| = m_1$ and $P(\cdot - \mathrm{i}\nu)M_0$ are Fourier multipliers. For arbitrary $f \in L^p(\mathcal{Q}_n, X)$, we therefore get that $v := T_{M_0}(\mathrm{e}^{-\nu\cdot}f) \in W_{\mathrm{per}}^{m_1,p}(\mathcal{Q}_n, X)$. As

$$Q(\boldsymbol{k}_\nu)A(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1} = \mathrm{id}_X - P(\boldsymbol{k}_\nu)(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1}, \qquad (3.3)$$

$Q(\cdot - \mathrm{i}\nu)AM_0$ is also a Fourier multiplier. Once more by the ellipticity of Q and lemma 2.3, the same holds for $\boldsymbol{k}^\alpha A(P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A)^{-1}$, $|\alpha| \leqslant m_2$.

Set $u := \mathrm{e}^{\nu\cdot}v = \mathrm{e}^{\nu\cdot}T_{M_0}\mathrm{e}^{-\nu\cdot}f$. Then, u solves (3.1) by construction, and lemma 3.3 yields $u \in W_{\nu,\mathrm{per}}^{m_1,p}(\mathcal{Q}_n, X)$ and $Au \in W_{\nu,\mathrm{per}}^{m_2,p}(\mathcal{Q}_n, X)$. Finally, the uniqueness of u follows immediately from the uniqueness of the representation as a Fourier series. $\qquad\square$

REMARK 3.7. We have seen in the proof that if one of the equivalent conditions in theorem 3.6 is satisfied, we have $Au \in W_{\nu,\mathrm{per}}^{m_2,p}(\mathcal{Q}_n, X)$. In particular, we get

$$(D^\beta Au)|_{x_j = 2\pi} - \mathrm{e}^{2\pi\nu_j}(D^\beta Au)|_{x_j = 0} = 0 \quad (j = 1, \ldots, n; \ |\beta| < m_2)$$

as additional boundary conditions in (3.1).

Theorem 3.6 enables us to treat Dirichlet–Neumann-type boundary conditions on $\tilde{\mathcal{Q}}_n := (0, \pi)^n$ for symmetric operators, provided that P and Q are of appropriate structure. More precisely, we call a differential operator

$$\mathcal{A}(D) = \sum_{|\alpha| \leqslant m} (p_\alpha D^\alpha + q_\alpha D^\alpha A)$$

symmetric if for all $|\alpha| \leqslant m$ either $p_\alpha = q_\alpha = 0$ or $\alpha \in 2\mathbb{N}_0^n$. In particular, m_1 is even. As examples, the operators $\mathcal{A}(D_t) := D_t^2 + A$ and $\mathcal{A}(D_1, D_2) := (D_1^2 + D_2^2)^2 + (D_1^4 + D_2^4)A$ are symmetric and satisfy the conditions on P and Q from theorem 3.6.

In each direction $j \in \{1, \ldots, n\}$, we will consider one of the following boundary conditions:

(i) $D_j^\ell u|_{x_j=0} = D_j^\ell u|_{x_j=\pi} = 0 \quad (\ell = 0, 2, \ldots, m_1 - 2)$,

(ii) $D_j^\ell u|_{x_j=0} = D_j^\ell u|_{x_j=\pi} = 0 \quad (\ell = 1, 3, \ldots, m_1 - 1)$,

(iii) $D_j^\ell u|_{x_j=0} = D_j^{\ell+1} u|_{x_j=\pi} = 0 \quad (\ell = 0, 2, \ldots, m_1 - 2)$,

(iv) $D_j^{\ell+1} u|_{x_j=0} = D_j^\ell u|_{x_j=\pi} = 0 \quad (\ell = 0, 2, \ldots, m_1 - 2)$.

Note that, for a second-order operator, (i) is of Dirichlet type, (ii) is of Neumann type, and (iii) and (iv) are of mixed type. For instance, in case (iii) we have that $u|_{x_j=0} = 0$ and $D_j u|_{x_j=\pi} = 0$. Therefore, we refer to these boundary conditions as conditions of Dirichlet–Neumann type. Note that the types may be different in different directions.

THEOREM 3.8. *Let $\mathcal{A}(D)$ be symmetric, with P and Q being elliptic, and let the boundary conditions be of Dirichlet–Neumann type, as explained above. Define $\nu \in \mathbb{C}^n$ by setting $\nu_j := 0$ in cases (i) and (ii) and $\nu_j := \mathrm{i}/2$ in cases (iii) and (iv). If, for this ν, one of the equivalent conditions of theorem 3.6 is satisfied, then for each $f \in L^p(\tilde{\mathcal{Q}}_n, X)$ there exists a unique solution $u \in W^{m,p}(\tilde{\mathcal{Q}}_n, X)$ of $\mathcal{A}(D)u = f$ satisfying the boundary conditions.*

Proof. Following an idea from [1], the solution is constructed by a suitable even or odd extension of the right-hand side from $(0, \pi)^n$ to $(-\pi, \pi)^n$. For simplicity of notation, we first consider the case $n = 2$ and boundary conditions of type (ii) in direction x_1 and of type (iii) in direction x_2. By definition, this leads to $\nu_1 = 0$ and $\nu_2 = \mathrm{i}/2$.

Let $f \in L^p(\tilde{\mathcal{Q}}_2, X)$ be arbitrary. Considering the even extension of f to the rectangle $(-\pi, \pi) \times (0, \pi)$ and then its odd extension to $(-\pi, \pi) \times (-\pi, \pi)$, we end up with a function F which satisfies $F(x_1, x_2) = F(-x_1, x_2)$ as well as $F(x_1, x_2) = -F(x_1, -x_2)$ almost everywhere in $(-\pi, \pi)^2$.

We can now apply theorem 3.6 with $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$, as above. (Here and in the following, the result of theorem 3.6 has to be shifted from the interval $(0, 2\pi)^n$ to the interval $(-\pi, \pi)^n$.) This yields a unique solution U of

$$\left.\begin{aligned} \mathcal{A}(D)U &= F && \text{in } (-\pi, \pi) \times (-\pi, \pi), \\ D_1^\ell U|_{x_1=-\pi} &= D_1^\ell U|_{x_1=\pi} && (\ell = 0, \ldots, m_1 - 1), \\ -D_2^\ell U|_{x_2=-\pi} &= D_2^\ell U|_{x_2=\pi} && (\ell = 0, \ldots, m_1 - 1). \end{aligned}\right\} \tag{3.4}$$

Symmetry of $\mathcal{A}(D)$ now shows that $V_1(x_1, x_2) := U(-x_1, x_2)$ and $V_2(x_1, x_2) := -U(x_1, -x_2)$ $(x \in (-\pi, \pi)^2)$ are also solutions of (3.4). By uniqueness, $V_1 = U = V_2$ follows.

Hence, $U_{x_2} := U(\cdot, x_2) \in W^{m,p}((-\pi, \pi), X) \subset C^{m-1}((-\pi, \pi), X)$ for almost every (a.e.) $x_2 \in (-\pi, \pi)$ is even. Together with the symmetry of U_{x_2} due to (3.4), this yields that

$$U_{x_2}^{(\ell)}(0) = U_{x_2}^{(\ell)}(\pi) = 0 \quad (\ell = 1, 3, \ldots, m_1 - 1).$$

Accordingly, for a.e. $x_1 \in (-\pi, \pi)$ we have that U_{x_1} is odd, and antisymmetry due to (3.4) gives

$$U_{x_1}^{(\ell)}(0) = U_{x_1}^{(\ell+1)}(\pi) = 0 \quad (\ell = 0, 2, \ldots, m_1 - 2).$$

Therefore, $u := U|_{(0,\pi)^2}$ solves $\mathcal{A}(D)u = f$ with boundary conditions (ii) for $j = 1$ and (iii) for $j = 2$.

For arbitrary $n \in \mathbb{N}$ and arbitrary boundary conditions of Dirichlet–Neumann type, the construction of the solution follows on the same lines. Here we choose even extensions in the cases (ii) and (iv) and odd extensions in the cases (i) and (iii).

On the other hand, let u be a solution of $\mathcal{A}(D)u = f$ satisfying boundary conditions of Dirichlet–Neumann type. We extend u and f to U and F on $(-\pi, \pi)^n$, as described above. Then, $U \in W^{m,p}((-\pi, \pi)^n, X)$, $Q(D)AU \in L^p((-\pi, \pi)^n, X)$ and due to symmetry of $\mathcal{A}(D)$ we see that, apart from a shift, U solves (3.1) with right-hand side F and ν defined as above. Thus, the uniqueness of U yields the uniqueness of u and the proof is complete. $\qquad\square$

REMARK 3.9. In the case $n = 1$ the ellipticity of P no longer forces P to be of even order. Hence, the same results can be achieved if $\mathcal{A}(D)$ is antisymmetric in the obvious sense, e.g. $\mathcal{A}(D_t) := D_t^3 + D_t + D_t A$.

4. Maximal regularity of cylindrical boundary-value problems with ν-periodic boundary conditions

Let F be a UMD space and let $\Omega := \mathcal{Q}_n \times V \subset \mathbb{R}^{n+n_V}$, with $V \subset \mathbb{R}^{n_V}$. For $x \in \Omega$ we write $x = (x^1, x^2) \in \mathcal{Q}_n \times V$ whenever we want to refer to the cylindrical geometry of Ω. Accordingly, we write $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}_0^n \times \mathbb{N}_0^{n_V}$ for a multi-index $\alpha \in \mathbb{N}_0^{n+n_V}$ and $D^\alpha = D^{(\alpha^1, \alpha^2)} =: D_1^{\alpha^1} D_2^{\alpha^2}$.

In the following we investigate the vector-valued parabolic initial boundary-value problem

$$\left.\begin{aligned} u_t + \mathcal{A}_\delta(x, D)u &= f & (t \in J,\ x \in \mathcal{Q}_n \times V), \\ B_j(x, D)u &= 0 & (t \in J,\ x \in \mathcal{Q}_n \times \partial V, \\ & & j = 1, \ldots, m_V), \\ (D^\beta u)|_{x_j = 2\pi} - \mathrm{e}^{2\pi\nu_j}(D^\beta u)|_{x_j = 0} &= 0 & (j = 1, \ldots, n;\ |\beta| < m_1), \\ u(0, x) &= u_0(x) & (x \in \mathcal{Q}_n \times V). \end{aligned}\right\} \tag{4.1}$$

Here $J := [0, T)$, $0 < T \leqslant \infty$, denotes a time interval, and the differential operator $\mathcal{A}_\delta(x, D)$ has the form

$$\begin{aligned} \mathcal{A}_\delta(x, D) &= P(x^1, D_1) + Q_\delta(D_1)A_V(x^2, D_2) \\ &:= P(x^1, D_1) + (Q(D_1) + \delta)A_V(x^2, D_2) \\ &:= \sum_{|\alpha^1| \leqslant m_1} p_{\alpha^1}(x^1)D_1^{\alpha^1} + \sum_{|\alpha^1| \leqslant m_2} q_{\alpha^1}D_1^{\alpha^1}A_V(x^2, D_2) + \delta A_V(x^2, D_2), \end{aligned}$$

where $\delta \geqslant 0$ is to be specified. The operator $A_V(x^2, D_2)$ is assumed to be of order $2m_V$ and is supplemented with boundary conditions

$$B_j(x, D) = B_j(x^2, D_2) \quad (j = 1, \ldots, m_V),$$

with operators $B_j(x^2, D_2)$ of order $m_j < 2m_V$ acting on the boundary of V. We want to restrict ourselves to $\nu = 0$ or to purely imaginary components of ν, since in that case $\mathrm{i}\nu \in \mathbb{R}^n$. In view of the boundary conditions, it is sufficient to consider $\nu \in \mathrm{i}(-1, 1)^n$. Note that periodic, as well as antiperiodic, boundary conditions are still captured.

This class of equations fits into the framework of § 3 if we define the operator $A = A_V$ in § 3 as the L^p-realization of the boundary-value problem

$$(A_V(x^2, D_2), B_1(x^2, D_2), \ldots, B_{m_V}(x^2, D_2)).$$

More precisely, for $1 < p < \infty$ we define the operator A_V in $L^p(V, F)$ by

$$D(A_V) := \{u \in W^{2m,p}(V, F) : B_j(x^2, D_2)u = 0 \ (j = 1, \ldots, m_V)\},$$
$$A_V u := A_V(x, D)u := A_V(x^2, D_2)u \ (u \in D(A_V)).$$

Throughout this section, we assume that the boundary-value problem $(A_V, B_1, \ldots, B_{m_V})$ satisfies standard smoothness and parabolicity assumptions as given in [10, theorem 8.2], for example. In particular, V is assumed to be a domain with compact C^{2m_V}-boundary, and $(A_V, B_1, \ldots, B_{m_V})$ is assumed to be parameter-elliptic with angle $\varphi_{A_V} \in [0, \pi)$. For the notion of parameter-ellipticity of a boundary-value problem, we refer the reader to [10, § 8.1].

Recall that a densely defined operator A is called $\mathcal{R}$-sectorial if there exists a $\theta \in (0, \pi)$ such that

$$\mathcal{R}(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\theta}\}) < \infty. \tag{4.2}$$

For an $\mathcal{R}$-sectorial operator, $\phi_A^{\mathcal{R}} := \inf\{\theta \in (0, \pi) : (4.2) \text{ holds}\}$ is called the $\mathcal{R}$-angle of A (see [10, p. 42]). We mention that we do not impose injectivity for $\mathcal{R}$-sectoriality of an operator A. In the study of time-dependent problems, $\mathcal{R}$-sectoriality of an operator is closely related to maximal regularity. Recall that a closed and densely defined operator in a Banach space X has maximal L^q-regularity if for each $f \in L^q((0, \infty), X)$ there exists a unique solution $w : (0, \infty) \to D(A)$ of the Cauchy problem

$$w_t + Aw = f \quad \text{in } (0, \infty),$$
$$w(0) = 0$$

satisfying the estimate

$$\|w_t\|_{L^q((0,\infty),X)} + \|Aw\|_{L^q((0,\infty),X)} \leqslant C\|f\|_{L^q((0,\infty),X)},$$

with a constant C independent of f. By a well-known result due to Weis [25, theorem 4.2], $\mathcal{R}$-sectoriality in a UMD space with $\mathcal{R}$-angle less than $\pi/2$ is equivalent to maximal L^q-regularity for all $1 < q < \infty$. In [10] it was shown that standard parameter-elliptic problems lead to $\mathcal{R}$-sectorial operators.

PROPOSITION 4.1 (Denk *et al.* [10, theorem 8.2]). *Under the assumptions above, for each $\phi > \varphi_{A_V}$ there exists a $\delta_V = \delta_V(\phi) \geqslant 0$ such that $A_V + \delta_V$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi_{A_V+\delta_V}^{\mathcal{R}} \leqslant \phi$. Moreover,*

$$\mathcal{R}(\{\lambda^{1-|\alpha^2|/2m_V} D^{\alpha^2} (\lambda + A_V + \delta_V)^{-1}; \ \lambda \in \Sigma_{\pi-\phi}, \ 0 \leqslant |\alpha^2| \leqslant 2m_V\}) < \infty. \tag{4.3}$$

We show that, under suitable assumptions on P and Q, $\mathcal{R}$-sectoriality of A_V implies $\mathcal{R}$-sectoriality of the operator related to the cylindrical problem (4.1). For this, consider the resolvent problem corresponding to (4.1), which is given by

$$\left.\begin{aligned}
\lambda u + \mathcal{A}_\delta(x, D)u &= f \quad (x \in \mathcal{Q}_n \times V), \\
B_j(x, D)u &= 0 \quad (x \in \mathcal{Q}_n \times \partial V, \; j = 1, \dots, m_V), \\
(D^\beta u)|_{x_j=2\pi} - \mathrm{e}^{2\pi\nu_j}(D^\beta u)|_{x_j=0} &= 0 \quad (j = 1, \dots, n, \; |\beta| < m_1).
\end{aligned}\right\} \quad (4.4)$$

For the sake of readability, we assume that $m_1 = 2m_V$. The $L^p(\Omega, F)$-realization of the boundary-value problem (4.4) is defined as

$$D(\mathbb{A}_\delta) := \{u \in W^{m_1,p}(\Omega, F) \cap W^{m_1,p}_{\nu,\mathrm{per}}(\mathcal{Q}_n, L^p(V, F)):$$
$$B_j(x, D)u = 0 \; (j = 1, \dots, m_V), \; A_V(x, D)u \in W^{m_2,p}(\mathcal{Q}_n, L^p(V, F))\},$$
$$\mathbb{A}_\delta u := \mathcal{A}_\delta(x, D)u \quad (u \in D(\mathbb{A}_\delta)).$$

REMARK 4.2.

(a) Since $m_2 \leqslant m_1$, it holds that

$$D(\mathbb{A}_\delta) = W^{m_1,p}(\Omega, F) \cap W^{m_1,p}_{\nu,\mathrm{per}}(\mathcal{Q}_n, L^p(V, F)) \cap W^{m_2,p}(\mathcal{Q}_n, D(A_V)).$$

(b) The following techniques also apply to equations with mixed orders $m_1 \neq 2m_V$. In that case, in the definition of $D(\mathbb{A}_\delta)$, the space $W^{m_1,p}(\Omega, F)$ has to be replaced by

$$\left\{u \in L^p(\Omega, F): D^\alpha u \in L^p(\Omega, F) \text{ for } \frac{|\alpha^1|}{m_1} + \frac{|\alpha^2|}{2m_V} \leqslant 1\right\}.$$

4.1. Constant coefficients

As is assumed for $Q(D_1)$, within this section we first assume that $P(x^1, D_1) = P(D_1)$ also has constant coefficients, and set

$$\mathcal{A}_{\delta,0} := \mathcal{A}_{\delta,0}(x^2, D) := P(D_1) + Q_\delta(D_1)(A_V + \delta_V).$$

With $\mathbb{A}_{\delta,0}u := \mathcal{A}_{\delta,0}u$ for $u \in D(\mathbb{A}_{\delta,0}) := D(\mathbb{A}_\delta)$, we formally get that $(\lambda + \mathbb{A}_{\delta,0})^{-1} = \mathrm{e}^{\nu\cdot}T_{M_\lambda}\mathrm{e}^{-\nu\cdot}$ where T_{M_λ} denotes the associated operator to

$$M_\lambda(\boldsymbol{k}) := (\lambda + P(\boldsymbol{k} - \mathrm{i}\nu) + Q_\delta(\boldsymbol{k} - \mathrm{i}\nu)(A_V + \delta_V))^{-1}.$$

More generally, the Leibniz rule shows that

$$D^\alpha(\lambda + \mathbb{A}_{\delta,0})^{-1} = D^\alpha \mathrm{e}^{\nu\cdot}T_{M_\lambda}\mathrm{e}^{-\nu\cdot} = \sum_{\beta \leqslant \alpha} g_\beta(\nu)\mathrm{e}^{\nu\cdot}T_{M_\lambda^\beta}\mathrm{e}^{-\nu\cdot},$$

where g_β is a polynomial depending on β. Here, $T_{M_\lambda^\beta}$ denotes the associated operator to

$$M_\lambda^\beta(\boldsymbol{k}) := \boldsymbol{k}^{\beta_1}D^{\beta_2}(\lambda + P(\boldsymbol{k} - \mathrm{i}\nu) + Q_\delta(\boldsymbol{k} - \mathrm{i}\nu)(A_V + \delta_V))^{-1},$$

where $\beta = (\beta_1, \beta_2)^{\mathrm{T}} \leqslant \alpha$. In the case $\nu = 0$ we simply have that

$$D^\alpha(\lambda + \mathbb{A}_{\delta,0})^{-1} = T_{M_\lambda^\alpha}.$$

THEOREM 4.3. *Let $1 < p < \infty$, let F be a UMD space enjoying the property (α), let $\nu \in \mathrm{i}(-1,1)^n$ and let the boundary-value problem (A_V, B) satisfy the conditions of [10, theorem 8.2] with angle of parameter-ellipticity φ_{A_V}.*

For P and Q assume that

(i) *P is homogeneous and parameter-elliptic with angle $\varphi_P \in [0, \pi)$,*

(ii) *Q is homogeneous and parameter-elliptic with angle $\varphi_Q \in [0, \pi)$,*

(iii) *$\varphi_P + \varphi_Q + \varphi_{A_V} < \pi$.*

Set $\varphi_0 := \max\{\varphi_P, \varphi_Q + \varphi_{A_V}\}$.

Then, for each $\delta > 0$ the L^p-realization $\mathbb{A}_{\delta,0}$ of $\mathcal{A}_{\delta,0}$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^{\mathcal{R}}_{\mathbb{A}_{\delta,0}} \leqslant \varphi_0$. Moreover, for each $\phi > \varphi_0$ it holds that

$$\mathcal{R}(\{\lambda^{1-|\alpha|/m_1} D^{\alpha}(\lambda + \mathbb{A}_{\delta,0})^{-1} \colon \lambda \in \Sigma_{\pi-\phi}, \ \alpha \in \mathbb{N}_0^{n+n_v}, \ 0 \leqslant |\alpha| \leqslant m_1\}) < \infty. \quad (4.5)$$

In particular, if $\varphi_0 < \pi/2$, then $\mathbb{A}_{\delta,0}$ has maximal L^q-regularity for every $1 < q < \infty$, i.e. the initial boundary-value problem (4.1) is well posed in $L^q([0,T], L^p(\Omega, F))$.

If $\nu \neq 0$ or $Q \equiv c$, $c \neq 0$, the assertion remains valid for $\delta = 0$.

Proof. Let $\phi > \varphi_0$. Due to conditions (i)–(iii) on φ_P, φ_Q and φ_{A_V}, there exists $\vartheta > \varphi_{A_V}$ such that

$$\frac{\lambda + P(\xi)}{Q(\xi)} \in \Sigma_{\pi-\vartheta} \quad (\lambda \in \Sigma_{\pi-\phi}, \ \xi \in \mathbb{R}^n \setminus \{0\}).$$

First consider $\nu \neq 0$ and $\delta = 0$. Let $\alpha \in \mathbb{N}_0^{n+n_V}$, $0 \leqslant |\alpha| \leqslant m_1 = 2m_V$, $\mathbf{0} \leqslant \beta \leqslant \alpha$ and $\mathbf{0} \leqslant \gamma \leqslant \mathbf{1}$. For sake of convenience we drop the shift of A_V, i.e. we assume that $\delta_V = 0$. To prove (4.5) we apply lemma 2.6 in order to calculate $\mathbf{k}^{\gamma} \Delta^{\gamma} M^{\beta}_{\lambda+\delta}(\mathbf{k})$. In what follows we again write $\mathbf{k}_{\nu} := \mathbf{k} - \mathrm{i}\nu$ for short. Recall that $\mathrm{i}\nu \in (-1, 1)^n \setminus \{0\}$. As in the proof of proposition 2.9, it suffices to show that the operator families

$$\{\lambda^{1-|\alpha|/m_1} \mathbf{k}^{\omega} \Delta^{\omega} N(\mathbf{k}) D^{\beta_2}(\lambda + P(\mathbf{k}_{\nu}) + Q(\mathbf{k}_{\nu})A_V)^{-1} \colon \lambda \in \Sigma_{\pi-\phi}, \ \mathbf{k} \in \mathbb{Z}^n\} \quad (4.6)$$

with $N(\mathbf{k}) := \mathbf{k}^{\beta_1}$ and arbitrary $\omega \leqslant \gamma$,

$$\{\mathbf{k}^{\omega} \Delta^{\omega} P(\mathbf{k}_{\nu})(\lambda + P(\mathbf{k}_{\nu}) + Q(\mathbf{k}_{\nu})A_V)^{-1} \colon \lambda \in \Sigma_{\pi-\phi}, \ \mathbf{k} \in \mathbb{Z}^n\} \quad (4.7)$$

with $\mathbf{0} < \omega \leqslant \gamma$, and

$$\{\mathbf{k}^{\omega} \Delta^{\omega} Q(\mathbf{k}_{\nu})A_V(\lambda + P(\mathbf{k}_{\nu}) + Q(\mathbf{k}_{\nu})A_V)^{-1} \colon \lambda \in \Sigma_{\pi-\phi}, \ \mathbf{k} \in \mathbb{Z}^n\} \quad (4.8)$$

with $\mathbf{0} < \omega \leqslant \gamma$ are $\mathcal{R}$-bounded. Due to our assumptions and proposition 4.1, in particular (4.3), for $0 \leqslant |\beta_2| \leqslant m_1 = 2m_V$ the set

$$\left\{\left(\frac{\lambda + P(\mathbf{k}_{\nu})}{Q(\mathbf{k}_{\nu})}\right)^{1-|\beta_2|/m_1} D^{\beta_2}\left(\frac{\lambda + P(\mathbf{k}_{\nu})}{Q(\mathbf{k}_{\nu})} + A_V\right)^{-1} \colon \lambda \in \Sigma_{\pi-\phi}, \ \mathbf{k} \in \mathbb{Z}^n\right\}$$

is $\mathcal{R}$-bounded. For $\beta_2 = 0$ this yields the $\mathcal{R}$-boundedness of

$$\{(\lambda + P(\mathbf{k}_{\nu}))(\lambda + P(\mathbf{k}_{\nu}) + Q(\mathbf{k}_{\nu})A_V)^{-1} \colon \lambda \in \Sigma_{\pi-\phi}, \ \mathbf{k} \in \mathbb{Z}^n\} \quad (4.9)$$

and with it the $\mathcal{R}$-boundedness of

$$\{Q(\boldsymbol{k}_\nu)A_V(\lambda + P(\boldsymbol{k}_\nu) + Q(\boldsymbol{k}_\nu)A_V)^{-1}\colon \lambda \in \Sigma_{\pi-\phi}, \ \boldsymbol{k} \in \mathbb{Z}^n\}. \qquad (4.10)$$

Since ν is supposed to have at least one non-zero component, there exists $\varepsilon > 0$ such that $|\boldsymbol{k}_\nu| > \varepsilon$ holds true for all $\boldsymbol{k} \in \mathbb{Z}^n$. Moreover, there exists $C > 0$ such that

$$\frac{\lambda^{1-|\alpha|/m_1}|\boldsymbol{k}|^{|\omega|}|\Delta^\omega N(\boldsymbol{k})||Q(\boldsymbol{k}_\nu)|^{1-|\beta_2|/m_1}}{|\lambda + P(\boldsymbol{k}_\nu)|^{1-|\beta_2|/m_1}|Q(\boldsymbol{k}_\nu)|} \leqslant C \quad \text{and} \quad \frac{|\boldsymbol{k}|^{|\omega|}|\Delta^\omega P(\boldsymbol{k}_\nu)|}{|\lambda + P(\boldsymbol{k}_\nu)|} \leqslant C$$

for all $\boldsymbol{k} \in \mathbb{Z}^n$ and all $\lambda \in \Sigma_{\pi-\phi}$, due to the parameter-ellipticity of P and the ellipticity of Q. (In the case $\delta > 0$, the parameter-ellipticity of Q has to be used.)

Again we apply the contraction principle of Kahane to prove (4.6) and (4.7).

Similarly, the ellipticity of Q proves (4.8) as well as $D^\alpha A_V(\lambda + \mathbb{A}_{\delta,0})^{-1}f \in L^p(\Omega, F)$ for $|\alpha| \leqslant m_2$.

Now, consider the case $\nu = 0$. Note that the ideas of the first part of the proof carry over to this situation only if $\boldsymbol{k} \neq 0$. Two $\mathcal{R}$-boundedness statements have to be proven in order to apply the multiplier theorem. First, the $\mathcal{R}$-boundedness of

$$\{\lambda^{1-|\alpha|/m_1}M_\lambda^\alpha\colon \lambda \in \Sigma_{\pi-\phi}, \ \boldsymbol{k} \in \mathbb{Z}^n\}.$$

This follows immediately due to homogeneity arguments. Recall the structure of M_λ^α, in particular the fact that we no longer have to consider M_λ^β with $|\beta| < |\alpha|$, and that

$$\lambda^{1-|\alpha|/m_1}M_\lambda^\alpha(0) = \begin{cases} 0, & \alpha_1 \neq 0 \\ \lambda^{1-|\alpha_2|/m_1}D^{\alpha_2}(\lambda + \delta A_V)^{-1}, & \alpha_1 = 0. \end{cases}$$

Second, we have to prove the $\mathcal{R}$-boundedness of (4.6)–(4.8), this time, however, with $\boldsymbol{k} \in \mathbb{Z}^n \setminus \{0\}$ instead of $\boldsymbol{k} \in \mathbb{Z}^n$. Hence, $\boldsymbol{k} \neq 0$ and part one of the proof applies verbatim.

The last claim on maximal L^q-regularity now follows from [25, theorem 4.2]. $\square$

REMARK 4.4. We have seen in the proof that $A_V u \in W^{m_2,p}_{\nu,\mathrm{per}}(\mathcal{Q}_n, L^p(V, F))$, i.e. the solution u of (4.4) satisfies the further boundary condition

$$(D^\beta A_V u)|_{x_j=2\pi} - \mathrm{e}^{2\pi\nu_j}(D^\beta A_V u)|_{x_j=0} = 0 \quad (j = 1, \ldots, n; \ |\beta| < m_2)$$

(see remark 3.7). Additionally, we have seen in the proof that

$$\mathcal{R}(\{D^\alpha A_V(\lambda + \mathbb{A}_{\delta,0})^{-1}\colon \lambda \in \Sigma_{\pi-\phi}, \ 0 \leqslant |\alpha| \leqslant m_2\}) < \infty. \qquad (4.11)$$

Note that the shift $\delta > 0$ cannot be neglected in the case $Q \not\equiv c$, $c \in \mathbb{R}$, and $\nu = 0$. To see this, take a right-hand side $f \in L^p(\Omega, F)$, which is given as a constant extension of a function in $g \in L^p(V, F) \setminus D(A_V)$. If $\lambda u + \mathcal{A}(D)u = f$, then $\lambda\hat{u}(0) = \hat{f}(0) = g$ by the parameter-ellipticity of P and Q. Hence, $u \notin D(\mathbb{A}_0)$.

REMARK 4.5. Consider again the boundary-value problems in $(0,\pi)^n \times V$ with Dirichlet–Neumann-type boundary conditions and a symmetric setting with respect to $(0,\pi)^n$. As the extension and restriction operators defined above are bounded, we can immediately see from theorem 3.8 the related result for Dirichlet–Neumann-type boundary conditions. In particular, we obtain maximal regularity results also for boundary conditions of mixed type (iii) and (iv); see the text before theorem 3.8.

4.2. Non-constant coefficients of P

In this subsection, $P(x^1, D_1)$ is allowed to have non-constant coefficients, where we assume that

$$\left.\begin{aligned}
p_{\alpha^1} &\in C_{\mathrm{per}}(\mathcal{Q}_n) \quad \text{for } |\alpha^1| = m_1, \\
p_{\alpha^1} &\in L^{r_\eta}(\mathcal{Q}_n) \quad \text{for } |\alpha^1| = \eta < m_1, \ r_\eta \geqslant p, \ \frac{m_1 - \eta}{n - k} > \frac{1}{r_\eta}.
\end{aligned}\right\} \tag{4.12}$$

Here $C_{\mathrm{per}}(\mathcal{Q}_n) := \{f \in C([0, 2\pi]^n) \colon f|_{x_j=0} = f|_{x_j=2\pi} \ (j = 1, \ldots, n)\}$. However, in order to apply perturbation results similar to [10] or [22], we assume that $Q \equiv 1$, i.e. we consider

$$\mathcal{A}(x, D) := P(x^1, D_1) + A_V(x^2, D_2).$$

THEOREM 4.6. *Let $1 < p < \infty$, let F be a UMD space enjoying the property (α), let $\Omega := \mathcal{Q}_n \times V$ and let the boundary-value problem (A_V, B) satisfy the conditions of [10, theorem 8.2] with angle of parameter-ellipticity φ_{A_V}.*
 For P assume that

(i) *the coefficients satisfy (4.12) and*

(ii) *P is parameter-elliptic with angle $\varphi_P \in [0, \pi - \varphi_{A_V})$ uniformly in $x \in \bar{\mathcal{Q}}_n$.*

Set $\varphi_0 := \max\{\varphi_P, \varphi_{A_V}\}$. Then, for each $\phi > \varphi_0$ there exists $\mu = \mu(\phi) \geqslant 0$ such that the L^p-realization $\mathbb{A} + \mu$ of $\mathcal{A} + \mu$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi_{\mathbb{A}+\mu}^{\mathcal{R}} \leqslant \phi$. Moreover, we have that

$$\mathcal{R}(\{\lambda^{1-|\alpha|/m_1} D^\alpha (\lambda + \mathbb{A} + \mu)^{-1} \colon \lambda \in \Sigma_{\pi-\phi}, \ \alpha \in \mathbb{N}_0^{n+n_v}, \ 0 \leqslant |\alpha| \leqslant m_1\}) < \infty. \tag{4.13}$$

In particular, if $\varphi_0 < \pi/2$, then there exists $\mu > 0$ such that $\mathbb{A} + \mu$ has maximal L^q-regularity for every $1 < q < \infty$.

Proof. As a first step, we consider $P(x, D)$ to be a homogeneous differential operator with slightly varying coefficients. That is, we consider

$$\mathcal{A}^{va}(x, D) := P_0(D_1) + R(x^1, D_1) + A_V(x^2, D_2),$$

where

$$P_0(D_1) := \sum_{|\alpha^1|=2m} p_{\alpha^1} D_1^{\alpha^1}$$

is assumed to have constant coefficients and

$$R(x^1, D_1) := \sum_{|\alpha^1|=2m} r_{\alpha^1}(x^1) D_1^{\alpha^1}$$

satisfies

$$\sum_{|\alpha^1|=2m} \|r_{\alpha^1}\|_\infty \leqslant \eta$$

with $\eta > 0$ sufficiently small. The claim then follows due to perturbation results for $\mathcal{R}$-sectorial operators (see [10, 22]) from theorem 4.3.

As a second step, we choose a finite but sufficiently fine open covering of $\mathcal{Q}_n$. In view of the periodicity of the top-order coefficients, we may assume every open set of the covering that intersects with $\mathbb{R}^n \setminus \mathcal{Q}_n$ to be cut at the boundary of $\mathcal{Q}_n$ and continued within $\mathcal{Q}_n$ on the opposite side. By means of reflection and cut-off techniques, this enables us to define local operators with slightly varying coefficients. With the help of a partition of the unity and perturbation results for lower-order terms subject to condition (4.12), just as in [22], the claim follows. $\qquad\square$

References

1   W. Arendt and S. Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.* **240** (2002), 311–343.

2   W. Arendt and P. J. Rabier. Linear evolution operators on spaces of periodic functions. *Commun. Pure Appl. Analysis* **8** (2009), 5–36.

3   W. Arendt, M. Beil, F. Fleischer, S. Lück, S. Portet and V. Schmidt. The Laplacian in a stochastic model for spatio-temporal reaction systems. *Ulmer Seminare* **13** (2008), 133–144.

4   W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, vol. 96, 2nd edn (Birkhäuser, 2011).

5   C. J. K. Batty, R. Chill and S. Srivastava. Maximal regularity for second-order non-autonomous Cauchy problems. *Studia Math.* **189** (2008), 205–223.

6   S. Bu. On operator-valued Fourier multipliers. *Sci. China* A **49** (2006), 574–576.

7   S. Bu and J.-M. Kim. Operator-valued Fourier multiplier theorems on L_p-spaces on $\mathbb{T}^d$. *Arch. Math.* **82** (2004), 404–414.

8   R. Chill and S. Srivastava. L^p-maximal regularity for second-order Cauchy problems. *Math. Z.* **251** (2005), 751–781.

9   R. Chill and S. Srivastava. L^p-maximal regularity for second-order Cauchy problems is independent of p. *Boll. UMI* **1** (2008), 147–157.

10   R. Denk, M. Hieber and J. Prüss. $\mathcal{R}$-*boundedness, Fourier multipliers and problems of elliptic and parabolic type*, vol. 166 (Providence, RI: American Mathematical Society, 2003).

11   A. Favini and Y. Yakubov. Irregular boundary-value problems for second-order elliptic differential-operator equations in UMD Banach spaces. *Math. Annalen* **348** (2010), 601–632.

12   A. Favini, R. Labbas, S. Maingot, H. Tanabe and A. Yagi. A simplified approach in the study of elliptic differential equations in UMD spaces and new applications. *Funkcial. Ekvac.* **51** (2008), 165–187.

13   A. Favini, V. Shakhmurov and Y. Yakubov. Regular boundary-value problems for complete second-order elliptic differential-operator equations in UMD Banach spaces. *Semigroup Forum* **79** (2009), 22–54.

14   T. Gauss. *Floquet theory for a class of periodic evolution equations in an L^p-setting* (Karlsruhe: KIT Scientific Publishing, 2001).

15   P. Guidotti. Elliptic and parabolic problems in unbounded domains. *Math. Nachr.* **272** (2004), 32–45.

16   P. Guidotti. Semiclassical fundamental solutions. *Abstr. Appl. Analysis* **1** (2005), 45–57.

17   L. Hörmander. *Linear partial differential operators*, Grundlehren der Mathematischen Wissenschaften, vol. 116 (Springer, 1976).

18   V. Keyantuo and C. Lizama. Fourier multipliers and integro-differential equations in Banach spaces. *J. Lond. Math. Soc. (2)* **69** (2004), 737–750.

19   V. Keyantuo and C. Lizama. Periodic solutions of second-order differential equations in Banach spaces. *Math. Z.* **253** (2006), 489–514.

20   V. Keyantuo, C. Lizama and V. Poblete. Periodic solutions of integro-differential equations in vector-valued function spaces. *J. Diff. Eqns* **246** (2009), 1007–1037.

21   P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, Lecture Notes in Mathematics, vol. 1855, pp. 65–311 (Springer, 2004).

22   T. Nau and J. Saal. $\mathcal{R}$-sectoriality of cylindrical boundary value problems. In *Parabolic problems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 80, pp. 479–505 (Birkhäuser, 2011).

23   M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries: background analysis and advanced topics*, Pseudo-Differential Operators: Theory and Applications, vol. 2 (Birkhäuser, 2010).

24   Ž. Štrkalj and L. Weis. On operator-valued Fourier multiplier theorems. *Trans. Am. Math. Soc.* **359** (2007), 3529–3547.

25   L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Annalen* **319** (2001), 735–758.

26   T.-J. Xiao and J. Liang. *The Cauchy problem for higher-order abstract differential equations*, Lecture Notes in Mathematics, vol. 1701 (Springer, 1998).

27   T.-J. Xiao and J. Liang. Second-order parabolic equations in Banach spaces with dynamic boundary conditions. *Trans. Am. Math. Soc.* **356** (2004), 4787–4809.