Pseudodifferential operators and maximal L^p-regularity

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A first example

We want to solve the heat equation

$$\partial_t u(t,x) - \Delta u(t,x) = f(t,x) \quad \text{in } [0,\infty) \times \mathbb{R}^n,$$
$$u(0,x) = 0 \quad \text{in } \mathbb{R}^n$$

and its parameter-dependent version

$$\lambda u(x) - \Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n.$$

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Main idea: apply Fourier transform

$$\mathscr{F} : u(x) \mapsto \hat{u}(\xi), \quad \partial_{x_j} u(x) \mapsto i\xi_j \hat{u}(\xi)$$

and obtain the solution

$$u = \mathscr{F}^{-1} \frac{1}{\lambda + |\xi|^2} \mathscr{F} f.$$

➡ pseudodifferential operator (PsDO)!

Solution spaces: L^p-theory

Elliptic case:

$$\lambda u(x) - \Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n.$$

Question: Is the operator

$$\lambda - \Delta : W_p^2(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

an isomorphism of Banach spaces?

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Parabolic case:

$$\partial_t u(t,x) - \Delta u(t,x) = f(t,x) \quad \text{in } [0,\infty) \times \mathbb{R}^n,$$
$$u(0,x) = 0 \quad \text{in } \mathbb{R}^n$$

Question: Is the operator

$$\partial_t - \Delta : W_p^1((0,\infty); L^p(\mathbb{R}^n)) \cap L^p((0,\infty); W_p^2(\mathbb{R}^n)) \to L^p((0,\infty) \times \mathbb{R}^n)$$

an isomorphism of Banach spaces?

Boundary value problems

Solution to boundary value problems

Consider the Dirichlet boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{aligned} (\lambda - \Delta)u &= f \quad \text{in } \Omega, \\ \gamma_0 u &= g \quad \text{on } \partial\Omega \end{aligned}$$

with $\gamma_0 u := u|_{\partial\Omega}$ (trace of u).

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with $\gamma_0 u := u|_{\partial\Omega}$ (trace of u).

Question: Is the operator

$$(\lambda - \Delta, \gamma_0)\colon W_p^2(\Omega) \to L^p(\Omega) \times W_p^{2-1/p}(\partial\Omega)$$

an isomorphism of Banach spaces?

- How can we compute u?
- How can we compute the normal trace $h := (\partial_\nu u)|_{\partial\Omega}$?

How to solve boundary value problems

We want to solve the boundary value problem

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How to solve boundary value problems

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Standard approach:

- Reduce to the case $f = 0$ by subtracting a whole space-solution,
- freeze the coefficients, take local coordinates ("model problem"),
- take partial Fourier transform in tangential directions,
- solve the ordinary differential equation in normal direction,
- take inverse partial Fourier transform.

Reduction to the boundary

Model problem in the half-space $\mathbb{R}^n_+$:

$$\begin{aligned}(\lambda - \Delta)u &= 0 \quad \text{in } \mathbb{R}^n_+,\\ u &= g \quad \text{on } \mathbb{R}^{n-1}.\end{aligned}$$

Partial Fourier transform $\mathscr{F}'$ $(x' \rightsquigarrow \xi')$ in the tangential variables $x' = (x_1, \dots, x_{n-1})$ gives the ODE

$$\begin{aligned}(\lambda + |\xi'|^2 - \partial_{x_n}^2)w(x_n) &= 0 \quad (x_n > 0),\\ w(0) &= (\mathscr{F}'g)(\xi').\end{aligned}$$

Reduction to the boundary

Model problem in the half-space $\mathbb{R}^n_+$:

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Partial Fourier transform $\mathscr{F}'$ $(x' \rightsquigarrow \xi')$ in the tangential variables $x' = (x_1, \ldots, x_{n-1})$ gives the ODE

$$\begin{aligned}(\lambda + |\xi'|^2 - \partial^2_{x_n})w(x_n) &= 0 \quad (x_n > 0),\\ w(0) &= (\mathscr{F}'g)(\xi').\end{aligned}$$

Stable (!) solution of is given by

$$\begin{aligned}w(x_n) = w(\xi', \lambda, x_n) &= \exp(-\sqrt{\lambda + |\xi'|^2}\, x_n)(\mathscr{F}'g)(\xi')\\ &= -\int_0^\infty \partial_{y_n}\big[\exp(-\sqrt{\lambda + |\xi'|^2}(x_n + y_n))(\mathscr{F}'\tilde{g})(\xi', y_n)\big]\, dy_n.\end{aligned}$$

The Dirichlet-Neumann operator

The normal trace of w is given by the symbol

$$(\partial_\nu w)(\xi', \lambda, 0) = \sqrt{\lambda + |\xi'|^2}(\mathscr{F}'g)(\xi').$$

We obtain the Dirichlet-Neumann operator

$$g \mapsto h = (\mathscr{F}')^{-1}\sqrt{\lambda + |\xi'|^2}(\mathscr{F}'g)(\xi').$$

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(Dirichlet-Neumann operator, Lopatinskii matrix)

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➡ pseudodifferential operator (PsDO) on the boundary $\partial\Omega$!

(Dirichlet-Neumann operator, Lopatinskii matrix)

More general:

- higher order operators ($\rightsquigarrow$ systems of PsDOs),
- additional equations on the boundary (e.g., free boundary value problems),

Summary: Why PsDOs?

1. Solution of $(\lambda - \Delta)u = f$ in the whole space:

$$u = \mathscr{F}^{-1} \frac{1}{\lambda + |\xi|^2} \mathscr{F} f.$$

2. Solution of $(\lambda - \Delta)u = 0,\ \gamma_0 u = g$ in the half-space:

$$u = (\mathscr{F}')^{-1} \int_0^\infty \partial_n \Big[\exp(-\sqrt{\lambda + |\xi'|^2}(x_n + y_n))(\mathscr{F}' \tilde{g})(\xi', y_n) \Big] dy_n.$$

3. Dirichlet-Neumann operator $g := u|_{\partial\Omega} \mapsto h := (\partial_\nu u)|_{\partial\Omega}$:

$$h = (\mathscr{F}')^{-1} \sqrt{\lambda + |\xi'|^2} \mathscr{F}' g.$$

Reduction to the boundary: an example

Spin-coating process (Geissert-Hieber-Saal-Sawada-D. 2011):

The spin-coating model leads to the following generalized Dirichlet-Neumann operator (Lopatinskii matrix):

$$L(\xi',\tau) = \begin{pmatrix} i\xi_1 & i\xi_2 & -\omega & 0 & 0 \\ 0 & 0 & 1 & \frac{|\xi'|}{\omega(\omega+|\xi'|)} & \lambda \\ \omega & 0 & -i\xi_1 & -\frac{i\xi_1(\omega-|\xi'|)}{\omega(\omega+|\xi'|)} & 0 \\ 0 & \omega & -i\xi_2 & -\frac{i\xi_2(\omega-|\xi'|)}{\omega(\omega+|\xi'|)} & 0 \\ 0 & 0 & -2\omega & -1 & \sigma|\xi'|^2 \end{pmatrix}$$

with $\omega := \sqrt{\lambda + |\xi'|^2}$.

Contents

Maximal L^p-regularity

One approach to solve nonlinear PDEs is to show maximal L^p-regularity for the linearized problems. Consider

$$\begin{aligned} \partial_t v + Av &= f \quad (t \in [0, T]), \\ v(0) &= 0. \end{aligned} \tag{1}$$

where A is a closed operator in the space X with domain $D(A)$.

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where A is a closed operator in the space X with domain $D(A)$.

Definition

The linear operator A has maximal L^p-regularity if for every $f \in L^p([0, T], X)$ equation (1) has a unique solution

$$v \in W_p^1([0, T], X) \cap L^p([0, T], D(A))$$

depending continuously on f.

$\mathcal{R}$-boundedness and maximal L^p-regularity

How to prove maximal L^p-regularity?

Theorem (Weis 2001)

Let X be a UMD space, $1 < p < \infty$, and A be a sectorial operator. Then A has maximal L^p-regularity (= well-posedness in L^p-Sobolev spaces) if and only if A is $\mathcal{R}$-sectorial, i.e. the resolvent is $\mathcal{R}$-bounded:

$$\mathcal{R}(\{\lambda(\lambda - A)^{-1} : \operatorname{Re}\lambda \geq 0\}) < \infty.$$

- UMD space: valid for many spaces, e.g. reflexive L^p-spaces,
- $\mathcal{R}$-bounded: similar to bounded in operator norm (but stronger),
- in Hilbert spaces: $\mathcal{R}$-bounded = bounded.

Mikhlin's theorem

Let $M = \mathrm{op}(m) := \mathscr{F}^{-1} m \mathscr{F}$. How to prove boundedness of M in $L^p(\mathbb{R}^n)$?

a) $p = 2$: $M \in L(L^2(\mathbb{R}^n))$ if and only if $m \in L^\infty(\mathbb{R}^n)$ by Plancherel's theorem.

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a) $p = 2$: $M \in L(L^2(\mathbb{R}^n))$ if and only if $m \in L^\infty(\mathbb{R}^n)$ by Plancherel's theorem.

b) $p \neq 2$:

Theorem (Mikhlin 1962 – Lizorkin 1963)

Let $1 < p < \infty$ and $m \in C^n(\mathbb{R}^n \setminus \{0\})$ with

$$\sup\left\{ |\xi^\beta \partial_\xi^\beta m(\xi)| : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0,1\}^n \right\} < \infty.$$

Then the associated Fourier multiplier $\mathscr{F}^{-1} m \mathscr{F}$ is a bounded operator in $L^p(\mathbb{R}^n)$.

$\mathcal{R}$-boundedness of Fourier multipliers

The following result is the "$\mathcal{R}$-bounded version" of Mikhlin's theorem:

Theorem (Girardi-Weis 2003)

Let $1 < p < \infty$, X be a UMD Banach spaces with property (α), Λ a set, and let $\{m_\lambda : \lambda \in \Lambda\}$ with $m_\lambda \in C^n(\mathbb{R}^n \setminus \{0\}, L(X))$ with

$$\mathcal{R}\Big(\big\{\xi^\beta \partial_\xi^\beta m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0,1\}^n,\ \lambda \in \Lambda\big\}\Big) < \infty.$$

Then the set of asscociated Fourier multipliers $\{\mathscr{F}^{-1} m_\lambda \mathscr{F} : \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded in $L(L^p(\mathbb{R}^n; X))$.

- property (α): valid for many spaces, e.g. L^p-spaces.

Pseudodifferential operators: scalar symbols

Let $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ and $\mu \in \mathbb{R}$.

Definition

$S^\mu(\mathbb{R}^n)$ is the space of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for which

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{\mu-|\alpha|} \quad (x,\xi \in \mathbb{R}^n,\, \alpha,\beta \in \mathbb{N}_0^n)$$

(space of all symbols of order μ).

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(space of all symbols of order μ).

Set $\Psi^\mu(\mathbb{R}^n) := \{\mathrm{op}(a) : a \in S^\mu(\mathbb{R}^n)\}$ with

$$[\mathrm{op}(a)u](x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} a(x,\xi)(\mathscr{F}u)(\xi)d\xi.$$

op(a) is called a pseudodifferential operator (PsDO) with symbol a.

Vector-valued PsDOs: $\mathcal{R}$-bounded symbols

Let X be a Banach space and $\mu \in \mathbb{R}$.

First we consider symbols which do not depend on x:

Definition

Let $a \in C^\infty(\mathbb{R}^n, L(X))$, $\xi \mapsto a(\xi)$. Then $a \in S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$ if for all $\alpha \in \mathbb{N}_0^n$

$$p_{\mathcal{R}}^{(\alpha)}(a) := \mathcal{R}\Big(\big\{\langle\xi\rangle^{-\mu+|\alpha|}\partial_\xi^\alpha a(\xi) : \xi \in \mathbb{R}^n\big\}\Big) < \infty.$$

⟹ Fréchet space $S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$ of $\mathcal{R}$-bounded symbols.

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➡ Fréchet space $S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$ of $\mathcal{R}$-bounded symbols.

- Let $S^{-\infty} := \bigcap_{\mu\in\mathbb{R}} S^\mu$ (smoothing symbols).
- There is a calculus for $\mathcal{R}$-bounded symbols.

Vector-valued PsDOs: constant coefficients

Some symbols are automatically $\mathcal{R}$-bounded:

Theorem (Krainer-D. 2007)

a) Constant smoothing symbols:

$$S^{-\infty}(\mathbb{R}^n, L(X)) \subset S^{-\infty}_{\mathcal{R}}(\mathbb{R}^n, L(X)).$$

b) Constant scalar symbols:

$$S^{\mu}(\mathbb{R}^n) \subset S^{\mu}_{\mathcal{R}}(\mathbb{R}^n, L(X))$$

via $a(\xi) \mapsto a(\xi)\,\mathrm{id}_X$.

Vector-valued PsDOs: constant coefficients

Some symbols are automatically $\mathcal{R}$-bounded:

Theorem (Krainer-D. 2007)

a) Constant smoothing symbols:

$$S^{-\infty}(\mathbb{R}^n, L(X)) \subset S_{\mathcal{R}}^{-\infty}(\mathbb{R}^n, L(X)).$$

b) Constant scalar symbols:

$$S^{\mu}(\mathbb{R}^n) \subset S_{\mathcal{R}}^{\mu}(\mathbb{R}^n, L(X))$$

via $a(\xi) \mapsto a(\xi)\,\mathrm{id}_X$.

Ideas of proof:

a) $S^{-\infty}(\mathbb{R}^n, L(X)) \cong \mathscr{S}(\mathbb{R}^n)\hat{\otimes}_{\pi} L(X)$

(completed projective tensor product).

b) Kahane's inequality.

Vector-valued PsDOs: classical symbols

$a \in S^\mu_{\mathrm{cl}} \;:\Leftrightarrow\; a \sim \sum_{j=0}^\infty a_j$ with a_j homogeneous of order $\mu - j$ (classical symbols)

Theorem (Krainer-D. 2007)

a) Every $a \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n, L(X))$ belongs to $S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$.
b) Every

$$a \in S^0_{\mathrm{cl}}(\mathbb{R}^n_x, S^\mu_{\mathrm{cl}}(\mathbb{R}^n_\xi, L(X)))$$

belongs to $S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$.
c) Let X be a UMD Banach spaces with property (α), and let

$$\{a_\lambda : \lambda \in \Lambda\} \subset S^0_{\mathrm{cl}}(\mathbb{R}^n_x, S^\mu_{\mathrm{cl}}(\mathbb{R}^n_\xi, L(X)))$$

be a bounded family of symbols. Then

$$\{\mathrm{op}(a_\lambda) : \lambda \in \Lambda\} \subset L(W^s_p(\mathbb{R}^n, X), W^{s-\mu}_p(\mathbb{R}^n, X))$$

is $\mathcal{R}$-bounded.

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$$u = (\mathscr{F}')^{-1} \int_0^\infty \partial_n \big[\exp(-\sqrt{\lambda + |\xi'|^2}(x_n + y_n))(\mathscr{F}'\tilde{g})(\xi', y_n) \big] \, dy_n.$$

3. Dirichlet-Neumann operator $g := u|_{\partial\Omega} \mapsto h := (\partial_\nu u)|_{\partial\Omega}$:

$$h = -(\mathscr{F}')^{-1} \sqrt{\lambda + |\xi'|^2} \mathscr{F}' g.$$

The Stokes equation

The pseudodifferential approach to the Stokes system:

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= 0 \quad \text{in } (0,\infty)\times\Omega,\\ \operatorname{div} u &= 0 \quad \text{in } (0,\infty)\times\Omega,\\ u &= 0 \quad \text{on } (0,\infty)\times\partial\Omega,\\ u\big|_{t=0} &= u_0 \quad \text{in } \Omega. \end{aligned}$$

The Stokes equation

The pseudodifferential approach to the Stokes system:

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Applying div to the first line, we obtain an elliptic boundary value problem for the pressure:

$$\begin{aligned} -\Delta p &= 0 \quad \text{in } \Omega, \\ \partial_{\nu_\Omega} p &= \gamma_0 \nu_\Omega \cdot \Delta u \quad \text{on } \partial\Omega \end{aligned}$$

where ν_Ω is the outer normal to $\partial\Omega$ and $\gamma_0 u := u|_{\partial\Omega}$.

Let $p = G[\gamma_0 \nu_\Omega \cdot \Delta u]$ with the solution operator G .

The Stokes system

Inserting the solution operator into the first equation, we obtain

$$\begin{aligned} (\partial_t - \Delta + \nabla G \gamma_0 \nu_\Omega \cdot \Delta) u &= 0 \quad \text{in } (0,\infty) \times \Omega, \\ u &= 0 \quad \text{on } (0,\infty) \times \partial\Omega, \\ u\big|_{t=0} &= u_0 \quad \text{in } \Omega. \end{aligned}$$

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Remarks:

- This is a nonlocal boundary value problem for u,
- the equality $\operatorname{div} u = 0$ is satisfied automatically,
- the nonlocal term $\nabla G\gamma_0\nu_\Omega \cdot \Delta$ is not of lower order.

(Grubb-Kokholm 1993), (Grubb-Solonnikov 1991)

The structure of the solution operator

We have to solve the Neumann-Laplace equation for the pressure:

$$\begin{aligned} -\Delta p &= 0 \quad \text{in } \Omega, \\ \partial_{\nu_\Omega} p &= \gamma_0 \nu_\Omega \cdot \Delta u =: \gamma_0 g \quad \text{on } \partial\Omega. \end{aligned}$$

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Locally, the solution operator can be written in the form

$$(Gg)(x) = (\mathscr{F}')^{-1}\Big[\int_0^\infty k(x',\xi',x_n,y_n)(\mathscr{F}'g)(\xi',y_n)dy_n\Big]$$

with some kernel function k containing the fundamental solution of the ODE .

Singular Green operators

We have to study the resolvent of the non-local operator

$$A := -\Delta + \nabla G \gamma_0 \nu_\Omega \cdot \Delta \quad \text{ in } L^p(\mathbb{R}^n_+)$$

(model problem). The last term is an example of a singular Green operator.

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Definition

A singular Green operator (of type 0) has the form

$$(Gf)(x) = [\mathrm{op}_G(k)f](x) = (\mathscr{F}')^{-1}\Big[\int_0^\infty k(x',\xi',x_n,y_n)(\mathscr{F}'f)(\xi',y_n)dy_n\Big]$$

with a kernel k belonging to the symbol class S^d_G defined as all C^∞-functions for which

$$\Big\|x_n^\ell D_{x_n}^{\ell'} y_n^m D_{y_n}^{m'} D_{x'}^{\alpha'} D_{\xi'}^{\beta'} k(x',\xi',x_n,y_n)\Big\|_{L^2((0,\infty)_{x_n}\times(0,\infty)_{y_n})} \le C\langle\xi'\rangle^{d-\ell+\ell'-m+m'-|\beta'|}.$$

(see (Grubb 1996), (Schrohe 2001), also parameter-dependent versions)

The Boutet-de Monvel calculus

Singular Green operators appear in the calculus of PsDO boundary value problems (Boutet-de Monvel calculus). Here the operators (and their inverses!) have the form

$$\begin{pmatrix} P+G & K \\ T & S \end{pmatrix}$$

with

- P: PsDO in Ω,
- G: singular Green operator in Ω,
- T: boundary operators (trace) $\Omega \rightsquigarrow \partial\Omega$,
- K: Poisson operators $\partial\Omega \rightsquigarrow \Omega$,
- S: PsDO on $\partial\Omega$.

The Boutet-de Monvel calculus

Theorem

The resolvent $R(\lambda) := (\lambda + A)^{-1}$ *of the non-local operator* A *is again of the form*

$$R(\lambda) = P(\lambda) + \tilde{G}(\lambda)$$

with P *being a PsDO and* $\tilde{G}$ *being a singular Green operator .*

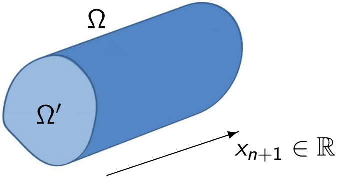
Proof: This follows directly from the (parameter-dependent) Boutet-de Monvel calculus.

Using the boundedness of PsDOs and of singular Green operators, one can show unique solvability in L^p-Sobolev spaces for the Stokes equation.

(Grubb-Kokholm 1993), (Grubb-Solonnikov 1991)

The Stokes equation in a cylindrical domain

Let $\Omega = \Omega' \times \mathbb{R} \subset \mathbb{R}^{n+1}$ be a cylindrical domain with $\Omega' \subset \mathbb{R}^n$ being a bounded smooth domain.



Consider the Stokes equation in Ω:

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= 0 \quad \text{in } (0,\infty) \times \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } (0,\infty) \times \Omega, \\ u &= 0 \quad \text{on } (0,\infty) \times \partial\Omega, \\ u\big|_{t=0} &= u_0 \quad \text{in } \Omega. \end{aligned}$$

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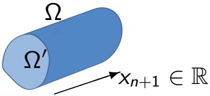

Idea to treat the Stokes system in the cylindrical domain $\Omega = \Omega' \times \mathbb{R}$:
use Fourier transform $\mathscr{F}_{x_{n+1} \to \xi_{n+1}}$ in unbounded direction!

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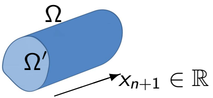

Idea to treat the Stokes system in the cylindrical domain $\Omega = \Omega' \times \mathbb{R}$:
use Fourier transform $\mathscr{F}_{x_{n+1} \to \xi_{n+1}}$ in unbounded direction!

➡ obtain a parameter-dependent problem with an additional parameter $\tau = \xi_{n+1}$ in a bounded domain:

$$\begin{aligned} \lambda u - \Delta u + \tau^2 u + \nabla p &= f \quad \text{in } \Omega', \\ \operatorname{div}' \bar{u} + i\tau u_{n+1} &= 0 \quad \text{in } \Omega', \\ u &= 0 \quad \text{on } \partial\Omega'. \end{aligned}$$

The parameters are $\lambda (\leftrightarrow \partial_t)$ and $\tau (\leftrightarrow x_{n+1})$.

The Stokes system in a cylindrical domain

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Let $x = (\bar{x}, x_{n+1}) \in \Omega$, $u = (\bar{u}, u_{n+1})$ etc.

Then we obtain with $\tau := \xi_{n+1}$

$$(-\Delta + \tau^2)p = 0 \quad \text{in } \Omega',$$
$$\partial_{\nu_{\Omega'}} p = \gamma_0 \nu_{\Omega'} \cdot \Delta_n \bar{u} \quad \text{on } \partial\Omega'$$

with a parameter-dependent solution operator $p = G_\tau[\gamma_0 \nu_{\Omega'} \cdot \Delta \bar{u}]$.

The Stokes system in a cylindrical domain

Inserting this into the first equation, we obtain a nonlocal boundary value problem for $\bar{u}$ with two parameters:

$$(\lambda - \Delta + \tau^2 + \nabla G_\tau \gamma_0 \nu_{\Omega'} \cdot \Delta)\bar{u} = 0 \quad + \text{b.c.}$$

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Aim: Prove maximal L^p-regularity for

$$Au := \mathscr{F}^{-1}_{\tau \to x_{n+1}} \underbrace{(-\Delta + \tau^2 + \nabla G_\tau \gamma_0 \nu_{\Omega'} \cdot \Delta)}_{A_\tau} \mathscr{F}_{x_{n+1} \to \tau} \quad + \text{b.c.}$$

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- use Girardi-Weis theorem $\rightsquigarrow$ show $\mathcal{R}$-boundedness of the resolvent $\lambda(\lambda - A_\tau)^{-1}$
- use Boutet-de Monvel calculus $\rightsquigarrow$ the resolvent is the sum of a PsDO and a singular Green operator

➡ it remains to show that singular Green operators are $\mathcal{R}$-bounded!

$\mathcal{R}$-boundedness of singular Green operators

We have to prove the $\mathcal{R}$-boundedness of the operator family

$$[\mathrm{op}_G(k)f](x) = \mathscr{F}'^{-1}_{\xi'\to x'}\Big[\int_0^\infty k(x',\xi',\lambda,\tau,x_n,y_n)(\mathscr{F}'_{x'\to\xi'}f)(y_n)dy_n\Big].$$

Theorem (Seiler-D. 2011)

Let $\mathcal{K}$ be a bounded subset in the symbol space of singular Green operators of order $d \le 0$ (and regularity $\nu \ge 1/2$) on the half-space $\mathbb{R}^n_+$. Let $1 < p < \infty$. Then

$$\{\,\mathrm{op}_G(k) : k \in \mathcal{K}\} \subset L(L^p(\mathbb{R}^n_+))$$

is $\mathcal{R}$-bounded .

Ingredients of the proof:

- Girardi-Weis theorem,
- $\mathcal{R}$-boundedness of integral operators in $L^p(\mathbb{R}_+)$,
- interpolation of Sobolev spaces.

Maximal L^p-regularity for the Stokes system

Corollary

The Stokes equation in a cylindrical domain $\Omega = \Omega' \times \mathbb{R}$ with Dirichlet boundary conditions has maximal L^p-regularity for every $1 < p < \infty$.

(see also (Farwig-Ri 2007))

Maximal L^p-regularity for the Stokes system

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Proof:

- By results of Grubb-Solonnikov, the reduced Stokes system is a parabolic non-local boundary problem in the Boutet-de Monvel calculus .
- ➡ The resolvent is the sum of a pseudodifferential operator and a singular Green operator .
- These operators are $\mathcal{R}$-bounded (see above).
- By the theorem of Weis, $\mathcal{R}$-sectoriality is equivalent to maximal regularity .

Extensions and remarks

- nonsmooth Boutet-de Monvel calculus
  (Abels 2005)
- Finite cylinder: $\Omega = \Omega' \times [0, 2\pi]^m$

  ➡ $\mathcal{R}$-boundedness of Fourier series

  (Arendt-Bu 2002, Nau-D. 2011)
- L^p in time and L^q in space

  ➡ Triebel-Lizorkin spaces as trace spaces

  (Hieber-Prüss-D. 2007, Kaip 2011)
- anisotropic symbol structure

  ➡ Newton polygon method

  (Volevich-D. 2008, Saal-Seiler-D. 2008)

Thank you for your attention!