# Test Sets for Positivity of Invariant Forms and Applications to Sums of Squares Representations

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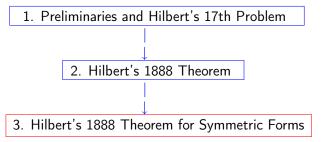
<sup>&</sup>lt;sup>1</sup>Dissertation of Ph.D. student Ms. Charu Goel

1. Preliminaries and Hilbert's 17th Problem

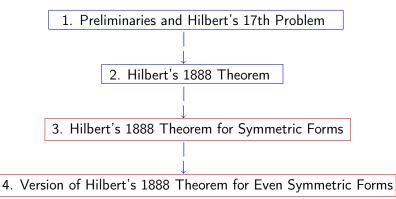
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Example: The Motzkin polynomial

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But what if rational functions are not allowed in the sos representation and we want only sos of polynomials?

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- ▶ (Q): For what pairs (n, 2d) we have  $\mathcal{P}_{n,2d} \subseteq \Sigma_{n,2d}$ ?

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Trivially,  $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow f \in \mathcal{P}_{n+j,2d} \setminus \sum_{n+j,2d} \forall \ j \geq 0$ . We claim:  $f \in \mathcal{P}_{n,2d} \setminus \sum_{n,2d} \Rightarrow x_1^{2i} f \in \mathcal{P}_{n,\ 2d+2i} \setminus \sum_{n,\ 2d+2i} \forall \ i \geq 0$ . Indeed, assume for a contradiction that  $x_1^2 f(x_1,\ldots,x_n) = \sum_{j=1}^k h_j^2(x_1,\ldots,x_n)$ . The L.H.S vanishes at  $x_1 = 0$ , so does the R.H.S. It follows that  $h_j(x_1,\ldots,x_n)$  vanishes at  $x_1 = 0$  and so  $x_1 \mid h_j \ \forall \ j$ , so  $x_1^2 \mid h_j^2 \ \forall \ j$ . So, R.H.S is divisible by  $x_1^2$ . Dividing both sides by  $x_1^2$  we get a sos representation of f, a contradiction.

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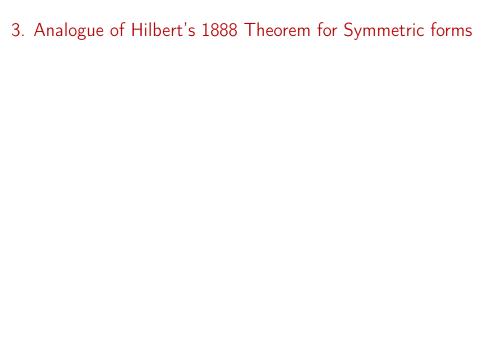
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► Choi and Lam, 1976  $Q(x, y, z, w) := w^4 + x^2y^2 + y^2z^2 + z^2x^2 - 4xyzw \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4},$  $S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$ 



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- ▶ Proposition [BCR]: Let R be a real closed field and p an irreducible polynomial in  $R[x_1, ..., x_n]$ . TFAE:
  - 1.  $(p) = \mathcal{I}(Z(p))$ , where  $\mathcal{I}(A) = \{g \in R[\underline{x}] \mid g(\underline{a}) = 0 \ \forall \ \underline{a} \in A\}$  is the ideal of vanishing polynomials on  $A \subseteq R^n$  and  $Z(p) = \{\underline{x} \in R^n \mid p(\underline{x}) = 0\}$  is the zero set of p.
  - 2. The sign of the polynomial p changes on  $R^n$  (i.e. p(x)p(y) < 0 for some  $x, y \in R^n$ ).

▶ Corollary 3.2 (G.): Let  $f \in \mathcal{P}_{n,2d} \setminus \Sigma_{n,2d}$  and p an irreducible indefinite form of degree r in  $\mathbb{R}[x_1, \ldots, x_n]$ . Then  $p^2 f \in \mathcal{P}_{n,2d+2r} \setminus \Sigma_{n,2d+2r}$ .

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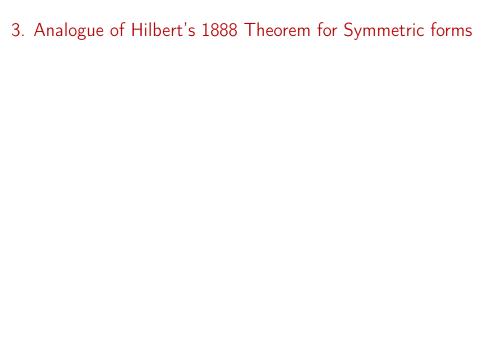
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  - ► Choi-Lam, 1976:  $f_{4,4} := \sum^6 x^2 y^2 + \sum^{12} x^2 yz 2xyzw \in \mathcal{SP}_{4,4} \setminus \mathcal{S}\Sigma_{4,4}.$  ["the construction of  $f_{n,4} \in \mathcal{SP}_{n,4} \setminus \mathcal{S}\Sigma_{n,4}$  (for  $n \geq 4$ ) requires considerable effort, so we shall not go into the full details here. Suffice it to record the special form  $f_{4,4}$ "]
  - ▶ We will construct explicit forms  $f \in SP_{n,4} \setminus S\Sigma_{n,4}$  for  $n \ge 5$



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- ▶ Corollary : (i) For a symmetric real polynomial f of degree 2d in n variables  $\exists \underline{x} \in \mathbb{R}^n$  s.t.  $f(\underline{x}) = 0 \Leftrightarrow \exists \underline{x} \in \Lambda_{n,k}$  s.t.  $f(\underline{x}) = 0$ .

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## 3.1. Symmetric psd not sos n-ary quartics for n > 5

▶ Consider the following symmetric quartic in  $n \ge 4$  variables,

$$L_n(x_1,...,x_n) := m(n-m) \sum_{i < j} (x_i - x_j)^4 - \Big(\sum_{i < j} (x_i - x_j)^2\Big)^2,$$
  
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$$= k(n - k)(r - s)^{4}[m(n - m) - k(n - k)]$$

$$= k(n - k)(r - s)^{4}[(m - k)(n - m - k)] > 0.$$

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  - ▶ construct explicit forms  $f \in S\mathcal{P}_{n,2d}^e \setminus S\Sigma_{n,2d}^e$  for the pairs  $(n,2d)=(3,12), (n,8)_{n\geq 5}$
  - ▶ deduce that for  $(n, 2d) = (n, 6)_{n \ge 3}$ ,  $(n, 8)_{n \ge 4}$ ,  $(3, 2d)_{d \ge 5}$ ,  $(n, 2d)_{n > 4, d > 7}$ , the answer to  $\mathcal{Q}(S^e)$  is negative.

▶ Lemma 4.1 (G.): If 2t = 4, 6, and  $n \ge 3$ , then

$$h_t(x_1,\ldots,x_n) := \sum_{i=1}^n x_i^{2t} - 10 \sum_{i \neq j} x_i^{2t-2} x_j^2$$

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### 4.2. Answer to $\mathcal{Q}(S^e)$ : for what (n,2d) $S\mathcal{P}_{n,2d}^e \subseteq S\Sigma_{n,2d}^e$ ?

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  - 1. (n,2d) = (n,8) for  $n \ge 5$ , and
  - 2. (n,2d) for  $n \ge 4, d = 5,6$ .

then the complete answer to  $Q(S^e)$  will be:

$$SP_{n,2d}^e \subseteq S\Sigma_{n,2d}^e$$
 if and only if  $n = 2, d = 1, (n,2d) = (n,4)_{n \ge 3}, (3,8)$ .

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- ▶ Proposition (Reduction to Basic Cases:) If we can find psd not sos even symmetric *n*—ary 2*d*—ic forms for the following pairs:
  - 1. (n, 2d) = (n, 8) for  $n \ge 5$ , and
  - 2. (n, 2d) for  $n \ge 4, d = 5, 6$ .

then the complete answer to  $Q(S^e)$  will be:

$$SP_{n,2d}^e \subseteq S\Sigma_{n,2d}^e$$
 if and only if  $n = 2, d = 1, (n,2d) = (n,4)_{n \ge 3}, (3,8)$ .

- ▶ Psd not sos even symmetric n—ary octics for  $n \ge 5$ 
  - ► Theorem (G.): The form

$$B(x_1,\ldots,x_5) := L_5(x_1^2,\ldots,x_5^2) \in S\mathcal{P}_{5,8}^e \setminus S\Sigma_{5,8}^e,$$

(recall that 
$$L_{2m+1} = m(m+1) \sum_{i < j} (x_i - x_j)^4 - \left(\sum_{i < j} (x_i - x_j)^2\right)^2$$
 is a symmetric psd not sos  $(2m+1)$ —ary quartic form).

### 4.2.1. Psd not sos even symmetric n-ary octics for $n \ge 6$

- ▶ Theorem (G.): For  $m \ge 3$ ,
  - 1.  $M_{2m+1}:=L_{2m+1}(x_1^2,\ldots,x_{2m+1}^2)\in S\mathcal{P}_{2m+1,8}^e\setminus S\Sigma_{2m+1,8}^e,$  and
  - 2.  $D_{2m} := C_{2m}(x_1^2, \dots, x_{2m}^2) \in S\mathcal{P}_{2m,8}^e \setminus S\Sigma_{2m,8}^e$ ,

Theorem (G.): 1.  $SP_{n,2d}^e = S\Sigma_{n,2d}^e$  for  $n = 2, d = 1, (n,2d) = (n,4)_{n \ge 3}, (3,8)$ .

2.  $SP_{n,2d}^e \supseteq S\Sigma_{n,2d}^e$  for  $(n,2d) = (n,6)_{n\geq 3}, (3,2d)_{d\geq 5}, (n,8)_{n\geq 4}$  and  $(n,2d)_{n\geq 4,d\geq 7}.$ 

i.e.

deg \ var	2	3	4	5	6	
2	<b>√</b>	<b>√</b>	<b>√</b>	<b>√</b>	<b>√</b>	
4	<b>√</b>	<b>√</b>	<b>√</b>	<b>√</b>	<b>√</b>	
6	<b>√</b>	×	×	×	×	
8	<b>√</b>	<b>√</b>	×	×	×	
10	<b>√</b>	×	?	?	?	?
12	<b>√</b>	×	?	?	?	?
14	<b>√</b>	×	×	×	×	
:	:	:	:	:	:	*