New Pathways between Group Theory and Model Theory
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Hahn Groups and Hahn Fields.

The aim of this talk is to survey some of our work on those Hausdorff - Hahn constructions, illustrating their role in ordered structures, model theory, valuation theory, transcendental number theory, models of arithmetic and very recently in Conway's field of surreal numbers.

Part I: Lexicographic powers of ordered sets:

Hausdorff's universal, saturated η_α *- chains.*

• Holland, W.- Kuhlmann, S.- McCleary, S.: *Lexicographic Exponentiation of chains*, JSL **70**, 389-409 **(2005)**

• Kuhlmann, F.-V. – Kuhlmann, S. – Shelah, S.: *Functorial equations for lexicographic products*, Proc. AMS **131** 2969-2976 **(2003)**

Part II: Hahn Groups and Fields:

Hahn's and Kaplansky's embedding theorems, constructions of universal saturated objects.

• D'Aquino, P. - Kuhlmann, S.: *A note on* $\aleph_\alpha$*-saturated o-minimal expansions of real closed fields*, to appear in Algebra and Logic **(2016)**

• D'Aquino, P. – Kuhlmann, S. – Lange, K.: *A valuation theoretic characterization of recursively saturated real closed fields*, JSL **80**, 194-206 **(2015)**

Part III: The κ-bounded EL- Hahn field EL(Γ, σ): *Constructing models of real exponentiation starting with a chain Γ equipped with a right shift automorphism σ.*

- Kuhlmann- Shelah: *κ–bounded Exponential Logarithmic Power Series Fields*, APAL 136, 284-296, **(2005)**.
- Kuhlmann : *Ordered Exponential Fields*, Fields Institute Monograph Series vol. 12, AMS **(2000)**

Part IV: Hardy Type Derivations on EL- series and applications to Ax-Schanuel's conjecture: *Developing a general method to construct derivations on EL(Γ, σ). Combinatorial criteria on supports.*

- Kuhlmann- Matusinski- Shkop: *A Note on Schanuel's Conjectures for Exponential Logarithmic Power Series Fields*, Archiv der Mathematik, 100, 431-436 **(2013)**
- Kuhlmann- Matusinski: *Hardy type derivations in generalized series fields*, J. of Algebra, 351, 185-203, **(2012)**
- Kuhlmann- Matusinski: *Hardy type derivations on fields of exponential logarithmic series*, J. of Algebra, 345, 171-189 **(2011)**

Part V: Integer parts of Hahn fields and models of arithmetic:

Study of discretely ordered subrings of real closed fields and their relation to models of fragments of Peano arithmetic:

- Fornasiero- Kuhlmann, F.-V.- Kuhlmann, S.: *Towers of complements and truncation closed embeddings of valued fields* J. of Algebra, **323**, 574-600 **(2010)**
- Biljakovic- Kotchetov-Kuhlmann: *Primes and Irreducibles in Truncation Integer Parts of Real Closed Fields*, LNL **26**, ASL, 42-65 **(2006)**

Part VI: The exponential rank of Gonshors exponentiation on the surreals.

Finding a complete system of representatives for the log-exp equivalence classes; the class of kappa numbers.

- Kuhlmann- Matusinski: *The exponential-logarithmic equivalence classes of surreal numbers*, ORDER **32**, 53-68 **(2014)**

Lexicographic powers of ordered sets.

Let Δ and Γ be (*linearly i.e. totally*) ordered sets. Fix a distinguished element $0 \in \Delta$. The lexicographic power Δ^Γ is the following set:

$$\begin{aligned}\Delta^\Gamma &:= \{s : \Gamma \to \Delta\, ;\ \text{support}\, s \ \text{ is well-ordered}\} \\ &= \{s \in \prod_{\gamma\in\Gamma} \Delta\, ;\ \text{support}\, s \ \text{ is well-ordered}\}\ ,\end{aligned}$$

ordered lexicographically from the left,

that is "order by first differences".

Here $\text{support}\, s := \{\gamma \in \Gamma\, ;\ s_\gamma \neq 0\}$.

F. Hausdorff and others studied their order types, generalizing Cantor's ordinal arithmetic and construction of saturated and universal models (the so-called η_α*−sets) for the theory of dense linear ordering without endpoints.*

• If α are β ordinals, then the lexicographic power α^{β^*} has order type the ordinal α^{β}.

• Hausdorff's η_α−set is constructed with the lexicographic power $2^{\aleph_\alpha}$.

• Other Examples: $\mathbb{Z}^{\mathbb{N}}$ has the order type of the set of irrationals, $\mathbb{N}^{\mathbb{N}}$ that of the set non-negative real $\mathbb{R}^+$, $2^{\mathbb{N}}$ that of the Cantor set.

Many fascinating problems (studied with W.C. Holland and S. McCleary) such as: dependence on the choice of the distinguished element 0*, isomorphism of powers with same base but different exponents or vice-versa, etc....*

Lexicographically ordered abelian goups.

If Δ is an ordered abelian group, e.g. $\Delta = \mathbb{R}$ for simplicity, we can endow the lexicographic power $\mathbb{R}^\Gamma$, which we then denote by $\mathbf{H}_\Gamma\mathbb{R}$, with an ordered abelian group structure.

Indeed, using now for $s \in \mathbb{R}^\Gamma$ the notation

$$s = \sum_\gamma s_\gamma \mathbf{1}_\gamma$$

define pointwise addition:

$$s + r = \sum_\gamma s_\gamma \mathbf{1}_\gamma + \sum_\gamma r_\gamma \mathbf{1}_\gamma := \sum_\gamma (s_\gamma + r_\gamma)\mathbf{1}_\gamma \, .$$

Obviously, the support of $s+r$ is still well-ordered, so $s + r$ is well-defined.

H. Hahn and others introduced and studied these so-called Hahn-groups. They are used for constructing saturated and universal models for the theory of divisible ordered abelian groups:

Theorem [Hahn Embedding's Theorem]:
Every ordered abelian group G is isomorphic to a subgroup of a Hahn group $\mathbf{H}_\Gamma\mathbb{R}$ for a suitable chain Γ.

More precisely, the invariant Γ is uniquely determined by G, it is the so-called archimedean "rank" of G. So Hahn's theorem generalizes O.L. Hölder's Theorem to the non-archimedean case.

Theorem [N.L. Alling - S.K.]:
Let Γ be an $\eta_\alpha-$set, then the Hahn group $\mathbf{H}_\Gamma\mathbb{R}$ is an $\aleph_\alpha-$saturated divisible ordered abelian group.

Let us continue enriching our lexicographic powers...

Lexicographically ordered fields.

If Δ is an ordered field, e.g. $\Delta = \mathbb{R}$ for simplicity, *and* Γ an ordered abelian group call it G, we can endow the lexicographic power $\mathbb{R}^G$, which we then denote by $\mathbb{R}((G))$, with a field structure.

Indeed, using now for $s \in \mathbb{R}^G$ the notation

$$s = \sum_g s_g t^g$$

define multiplication via convolution:

$$s.r = \sum_g \Big(\sum_{g'+g''=g} s_{g'} r_{g''} \Big) \; t^g \; .$$

Is $s.r$ well-defined? That is, is it true that (i) for every $g \in G$,

$$\sum_{g'+g''=g} s_{g'} r_{g''}$$

is a finite sum? and (ii) $s.r$ has well-ordered support? The answer is yes. Why?

Summability

We need the following key notion. Let I be an infinite index set, $F := \{s_i \in \mathbb{R}((G))\ ;\ i \in I\}$ a family of series, set

$$\text{Support } F := \bigcup_{i \in I} \text{support } s_i \ .$$

F is said to be **summable** if:

(i) For any $g \in$ Support F, the set

$$S_g := \{i \in I \mid g \in \text{support } s_i\} \subseteq I$$

is finite.

(ii) Support F is well-ordered.

Write $s_i = \sum_g (s_i)_g t^g$ for each $s_i \in \mathcal{F}$. If $\mathcal{F}$ is summable. Then

$$\sum_{i \in I} s_i := \sum_{g \in \text{Support}\mathcal{F}} (\sum_{i \in S_g} (s_i)_g)\ t^g$$

is a well-defined element of $\mathbb{R}((G))$ that we call the **sum** of $\mathcal{F}$.

Returning to multiplication: $s.r$ is well-defined because one can verify that the family

$$\{t^{g'}.r\ ;\ g' \in \text{support } s\}$$

is summable.

W.Krull, I. Kaplansky and others studied the field $\mathbb{R}((\Gamma))$ *while developing valuation theory, again we have universality and saturation:*

Non-archimedean Real Closed Fields

In what follows, G is non-trivial, i.e. $G \neq 0$.
If G is divisible, then $\mathbb{R}((G))$ is a non-standard model of the o-minimal $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$.

Theorem [Kaplansky Embedding's Theorem]:
Every real closed field R is isomorphic to a subfield of a field of generalized series $\mathbb{R}((G))$ for a suitable G.

More precisely, G is uniquely determined by R, it is the so-called value group of R.

Theorem [N.L. Alling - S.K.]:
Let G be an $\aleph_\alpha$−saturated divisible ordered abelian group. Then the field of generalized series $\mathbb{R}((G))$ is an $\aleph_\alpha$−saturated real closed field.

We have studied dense linear orderings without endpoints, divisible ordered abelian groups, real closed fields, all are so-called o-minimal structures. Let us continue enriching our fields of power series with further o-minimal structure....

Exponentiation The subring $\mathbb{R}((G^{\geq 0}))$ of $\mathbb{R}((G))$ (consisting of series with support contained in the non-negative cone $G^{\geq 0}$ of G) is a valuation ring, with a unique maximal ideal $\mathbb{R}((G^+))$ consisting of infinitesimal series, i.e. series with strictly positive support.

Theorem [Neumann's Lemma] Let $\epsilon \in \mathbb{R}((G^+))$ and $c_i \in \mathbb{R}, i \in \mathbb{N}$. Then $\{c_i\epsilon^i \ ; \ i \in \mathbb{N}\}$ is summable. In particular one can define $f(\epsilon)$ for any real analytic function. Define the exponential function on $\mathbb{R}((G^+))$ and the logarithm on the multiplicative group of 1- units $1+\mathbb{R}((G^+))$:

$$\exp(\epsilon) := \sum \frac{\epsilon^i}{i!}$$

$$\log(1+\epsilon) := \sum (-1)^{i-1} \ \frac{\epsilon^i}{i}$$

How to define a total surjective exponential function on $\mathbb{R}((G))$, *i.e. an ordering preserving isomorphism from the ordered additive group* $(\mathbb{R}((G)), +)$ *onto the ordered multiplicative group* $(\mathbb{R}((G))^{>0}, \times)$*?*

Lexicographic Decomposition

We have the following direct sum (respectively, multiplicative direct sum) decompositions:

$$\begin{array}{rcccccc} \mathbb{R}((G)) & = & \mathbb{R}((G^-)) & \oplus & \mathbb{R} & \oplus & \mathbb{R}((G^+)), \\ \mathbb{R}((G))^{>0} & = & t^G & \times & \mathbb{R}^+ & \times & (\,1+\mathbb{R}((G^+))\,). \end{array}$$

Indeed given $s \in \mathbb{R}((G))$ write

- $s = s_{<0} + s_0 + s_{>0}$ and
- for $s > 0$ and $g :=\ \min\ \text{support}\ s$, write

$$s = t^g \cdot c \cdot (1 + \epsilon)$$

with $c \in \mathbb{R}$, $c > 0$, $\epsilon \in \mathbb{R}((G^+))$.

Left Exponentiation? It is thus necessary and sufficient to construct a **left exponential**, that is, an ordering preserving isomorphism from the ordered additive group $\mathbb{R}((G^-))$ onto the ordered multiplicative group t^G :

Theorem[K-K–Shelah]

$\mathbb{R}((G))$ does not admit left-exponentiation (unless G is a proper class).

The "field" **No** *of surreal numbers was invented by J. Conway, studied by H. Gonshor, D. Knuth, M. Kruskal, N.L. Alling, P. Ehrlich and others. It admits left-exponentiation, however it is not a "field" since it is a proper class!*

Construction of non-archimedean models of real exponentiation

Fix a regular uncountable cardinal κ.

• The κ**-bounded Hahn group** $(\mathbb{R}^\Gamma)_\kappa \subseteq \mathbb{R}^\Gamma$ consists of all maps of which support has cardinality $< \kappa$.

• The κ**-bounded power series field** $\mathbb{R}((G))_\kappa \subseteq \mathbb{R}((G))$ consists of all series of which support has cardinality $< \kappa$.

Theorem 0.1 *Let* Γ *be a chain,* $G = (\mathbb{R}^\Gamma)_\kappa$ *and* $K = \mathbb{R}((G))_\kappa$. *Assume that*

$$l : \Gamma \to G^{<0}$$

is an **isomorphism** *of chains. Then* l *induces a* *log* $: (K^{>0}, \cdot, 1, <) \longrightarrow (K, +, 0, <)$ *as follows: given* $a \in K^{>0}$, *write* $a = t^g r(1+\varepsilon)$ *with* $g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$, $r \in \mathbb{R}^{>0}$, ε *infinitesimal. Set*

$$\log(a) := -\sum_{\gamma\in\Gamma} g_\gamma t^{l(\gamma)} + \log r + \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{\varepsilon^i}{i}$$

Moreover, log *satisfies* **Growth Axiom Scheme** *if and only if*

$$l(\min \operatorname{support} g) > g \qquad \text{for all } g \in G^{<0} \ . \quad (1)$$

Getting such isomorphisms l:

Let Γ be any chain, $G = (\mathbb{R}^\Gamma)_\kappa$ and $K = \mathbb{R}((G))_\kappa$. Then

$$\iota : \Gamma \to G^{<0} \text{ defined by } \gamma \mapsto -\mathbf{1}_\gamma$$

is an **embedding** of chains, and gives rise to prelogaritm on K. However, this prelogarithm neither satisfies **GA** nor is it **surjective**.

• To get a prelogarithm satisfying **GA**, we choose $\sigma \in \text{Aut}\,(\Gamma)$ a **right shift**, i.e.

$$\sigma(\gamma) > \gamma \text{ for all } \gamma \in \Gamma \qquad (2)$$

We set $l = \iota \circ \sigma$. Now

$$l : \Gamma \to G^{<0} \text{ defined by } \gamma \mapsto -\mathbf{1}_\sigma(\gamma)$$

is an embedding of chains satisfying (1), so gives rise to a prelogaritm on K satisfying **GA**

To get a surjective prelogarithm, we have to modify Γ as follows:

Proposition 0.2 *Fix a regular uncountable cardinal* κ *and let* Γ *be a given chain. There is a canonically constructed chain* $\Gamma_\kappa \supseteq \Gamma$ *together with an* **isomorphism** *of ordered chains*

$$\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

where $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$. *Moreover, every right shift* $\sigma \in \operatorname{Aut}(\Gamma)$ *extends canonically to a right shift* $\sigma_\kappa \in \operatorname{Aut}(\Gamma_\kappa)$.

We call the pair $(\Gamma_\kappa, \iota_\kappa)$ the κ-th **iterated lexicographic power** of Γ.

Summarizing the procedure of constructing the Exponential-Logarithmic field of κ-bounded series over (Γ, σ):

• Fix a regular uncountable cardinal κ, a chain Γ and σ a right shift automorphism of Γ.

• Define $\Gamma_\kappa, \quad G_\kappa, \quad \iota_\kappa, \quad$ and σ_κ as above.

• Set $K := \mathbb{R}((G_\kappa))_\kappa$ and $l := \iota_\kappa \circ \sigma_\kappa$. For $a \in K^{>0}$ write $a = t^g r(1+\varepsilon)$ where $g = \sum_{\gamma \in \Gamma} g_\gamma \mathbf{1}_\gamma$, $r \in \mathbb{R}^{>0}$, ε infinitesimal, then

$$\log(a) := -\sum_{\gamma \in \Gamma} g_\gamma t^{-\mathbf{1}_{\sigma_\kappa(\gamma)}} + \log r + \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{\varepsilon^i}{i}$$

• Set $\exp = \log^{-1}$.

• $(K, \exp)$ is a model of $T_{\mathrm{an,exp}}$.

Derivations

We want a "Kaplansky embedding Theorem" for ordered differential fields. The κ-bounded fields of power series are good candidates as "universal domains". But for this to make sense, we need first had to endow them with a good differential structure.

We want to endow the field of κ-bounded series over (Γ, σ) with a derivation d satisfying the following properties:

• d is strongly linear, that is

$$d \sum_g r_g t^g \;=\; \sum_g r_g dt^g \;. \tag{3}$$

• d satisfies strong Leibniz rule:

$$d(t^g) = d(\prod t_\gamma^{g_\gamma}) = t^g \sum g_\gamma (d(t_\gamma)/t_\gamma) \tag{4}$$

where $g = \sum_{\gamma \in \Gamma} g_\gamma \mathbf{1}_\gamma$ and $t_\gamma := t^{\mathbf{1}_\gamma}$ here and below to simplify notations.

• d satisfies the rule for the logarithmic derivative:

$$d \log a = da/a, \text{ for } a > 0. \tag{5}$$

Given $d : t_\Gamma \to K$ the problem is to find a criterion so that the series defined via the strong Leibniz rule and strong linearity make sense, i.e. the family of terms is **summable**. Using Ramsey theory type arguments we show:

Theorem: d extends to a series derivation on K iff both of the following conditions hold:

(C1) for any strictly increasing sequence $\gamma_n \subset \Gamma$ and any sequence $\tau_n \subset G$ s.t. τ_n in the support of $d(t_{\gamma_n})/t_{\gamma_n}$ for all n, τ_n cannot be decreasing.

(C2) for any strictly decreasing sequences $\gamma_n \subset \Gamma$ and $\tau_n \subset G$ s.t. τ_n in the support of $d(t_{\gamma_n})/t_{\gamma_n}$ for all n, there is N such that $v_G(\tau_{N+1} - \tau_N) > \gamma_{N+1}$.

On the decidability of $T_{\exp}$.

A. Macintyre and A. Wilkie showed that $T_{\exp}$ *is decidable if the real* **Schanuel conjecture** *has a positive solution.*

S. Schanuel conectured that if $y_1, \cdots, y_n \in \mathbb{R}$ *are linearly independent over* $\mathbb{Q}$*, then the transcendence degree over* $\mathbb{Q}$ *of the field*

$$\mathbb{Q}(y_1, \cdots, y_n\,;\, \exp(y_1), \cdots, \exp(y_n))$$

is at least n*.*

J. Ax proved Schanuel's conjecture for formal Laurent series without constant term.

Theorem[J. Ax]
Let $y_i \in \mathbb{R}[[t]]$ such that $y_i - y_i(0)$ are $\mathbb{Q}$-linearly independent, $i = 1, \cdots, n$. Then

$$\mathrm{td}_{\mathbb{R}}\mathbb{R}(y_1, \cdots, y_n, \exp(y_1), \cdots, \exp(y_n)) \geq n + 1$$

.

With M. Matusinski and A. Shkop we show that this result holds for κ- bounded exponential-logarithmic series.

Integer Parts and Models of Arithmetic.

An **integer part** for an ordered field R is a discretely ordered subring Z such that for each $r \in R$, there exists some $z \in Z$ with $z \leq r < z + 1$.

Shepherdson shows that the class of integer parts of real closed fields coincides with the class of models of **open induction***. He constructs an integer part of the field of Puiseux series, in which primes are not cofinal. Many open questions about integer parts of real closed fields, and their primes and irreducibles arise naturally.*

Theorem [J.-P. Ressayre and M.-H. Mourgues]
$Z := \mathbb{R}((\Gamma^-)) \oplus \mathbb{Z}$ is an integer part of the real closed field $\mathbb{R}((\Gamma))$.

Proof: Clearly, Z is a discrete subring. Let $s \in \mathbb{R}((\Gamma))$. Let $\lfloor s_0 \rfloor \in \mathbb{Z}$ be the integer part of $s_0 \in \mathbb{R}$. Define

$$z_s = \begin{cases} s_{<0} + s_0 - 1 & \text{if } s_0 \in \mathbb{Z} \text{ and } s_{>0} < 0, \\ s_{<0} + \lfloor s_0 \rfloor & \text{otherwise.} \end{cases}$$

Clearly, $z_s \leq s < z_s + 1$.

• M. Kotchetov we studied primes and irreducibles in integer parts of real closed fields, we showed that this integer part has a cofinal set of primes.

• With M. Carl, P. D'Aquino, we considered other fragments of Peano Arithmetic:

Does $\mathbb{R}((\Gamma))$ *admit an integer part which is a model of normal open induction, of full Peano Arithmetic*?

Conwa's field of Surreal Numbers

Of particular interest to us is the analysis of certain equivalence relations on the surreal numbers. Conway introduced and studied the ω-map to give a complete system $\omega^{\mathbf{No}}$ (:= the image of **No** under this map) of representatives of the Archimedean additive equivalence relation. Exploiting the convexity of the subclass of positive elements of each equivalence class, Gonshor describes such a representative ω^a as the *unique* surreal of minimal length in a given class. By a simple modification of their arguments, we first describe a complete system $\omega^{\omega^{\mathbf{No}}}$ of representatives of the Archimedean multiplicative equivalence relation.

We then introduce and study what we call the κ-map to give a complete system $\kappa_{\mathbf{No}}$ (:= the image of $\mathbf{No}$ under this map) of representatives of the exponential equivalence relation. We observe that:

$$\epsilon_{\mathbf{No}} \subset \kappa_{\mathbf{No}} \subset \omega^{\omega^{\mathbf{No}}} \subset \omega^{\mathbf{No}} \subset \mathbf{No}.$$

Finally, we introduce the notion of exp-log transseries (ELT) fields, which unifies the notion of transseries and exp-log series. We conjectured that $\mathbf{No}$ is an ELT field.

Notation and Terminology:

• **No** is endowed with a partial ordering called the simplicity ordering : a is simpler than b, write $a <_s b$, iff a is a proper initial sign subsequence of b.

• We use Conway "cut" notation for surreals. For a pair (L, R) with $L < R$ of "left" and "right" subsets of **No** we denote by $\langle L|R\rangle :=$ $a \in \mathbf{No}$ the unique surreal of minimal length representing the cofinality class of the cut. We call it the cut between L and R.

• For any surreal number a, the representation $a = \langle L_a \mid R_a\rangle$ of a is called the canonical cut of a. We also denote the canonical cut by $a = \langle a^L \mid a^R\rangle$ where a^L and a^R are general elements of the canonical sets $L_a := \{b \in \mathbf{No}\ ;\ b < a, b <_s a\}$ and $R_a := \{b \in \mathbf{No}\ ;\ b > a, b <_s a\}$.

• (Conway Normal Form) The map Ω sending a to ω^a extends exponentiation in base ω of ordinals. For $a \in \mathbf{No}$, ω^a is the representative of minimal length of its Archimedean equivalence class.

Corollary For any $a \in \mathbf{No}$, ω^{ω^a} is the representative of minimal length in its equivalence class of comparability.

• (generalised epsilon numbers) $\epsilon(\mathbf{No})$ is the proper class of all the fixed points of the map $\Omega : \forall a \in \mathbf{No},\ \omega^{\epsilon_a} = \epsilon_a$.

We shall now introduce a new class strictly between the class of epsilon numbers and that of representatives of the comparability classes.

The kappa map.

We study the exponential (logarithmic) equivalence relation for surreal numbers as we did for EL series, when we considered the exponential rank.

Theorem The recursive formula

$$\forall a \in \mathbf{No},\ \kappa(a) = \kappa_a := \langle \exp^n(0),\ \exp^n(\kappa_{a^L}) \mid \log^n(\kappa_{a^R}) \rangle$$

(where it is understood that $n \in \mathbb{N}$) defines a map

$$\begin{array}{rccc} \kappa : & \mathbf{No} & \to & \mathbf{No} \\ & a & \mapsto & \kappa(a) := \kappa_a \end{array}$$

with values in $\mathbf{No}^{\gg 1}_{>0}$ and such that:

(i) for any $a, b \in \mathbf{No}$, $a < b \Rightarrow \kappa_a \ll_{\exp} \kappa_b$

(ii) there is a uniformity property for this formula (i.e. the recursive formula does not depend on the choice of the cut for a).

The End