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Stochastic partial differential equations

Exercises and Questions

Question 1. Show that subsets of separable metric spaces are separable but that this is not true in general topological spaces. (This was used in the proof of Lemma 2.2.)

Question 2. Give an example of a compact operator A in a separable real Hilbert space for which $\sum_{n\in\mathbb{N}}\langle Ae_n, e_n\rangle < \infty$ holds for one orthonormal basis of H but not for all orthonormal bases.

Question 3. Let $A \in L(H)$ be a compact operator in the separable real Hilbert space H. Show that if $\sum_{n\in\mathbb{N}} \|Ae_n\|^2 < \infty$ holds for one orthonormal basis, then this holds for all orthonormal bases, and the value is independent of the choice of the basis.

Question 4. Let H be a separable real Hilbert space, and let $A, B \in \mathscr{S}_2(H)$. Show that $AB \in \mathscr{S}_1(H)$ and $\|AB\|_{\mathrm{tr}} \le \|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}$. Show that equality holds if $B = A^*$. (This question maybe hard to solve without literature.)

Question 5. Let $\{\beta_n : n \in \mathbb{N}\}$ be a sequence of real random variables for which every finite linear combination $\sum_{n=1}^{N} \lambda_n\beta_n$ is a real Gaussian random variable. Assume that $\mathrm{E}\big(\beta_i - \mathrm{E}\beta_i)(\beta_j - \mathrm{E}\beta_j)\big) = 0$ holds for all $i \neq j$. Show that $\{\beta_n : n \in \mathbb{N}\}$ is independent.

Question 6. Let H be an infinite-dimensional separable real Hilbert space. Show that $L(H)$ (with operator norm) is not separable.

Question 7. Let H be a separable real Hilbert space, and let $X\colon \Omega \to H$ be integrable. Is it true that $X \otimes X\colon \Omega \to L(H)$ is integrable?

Question 8. Show that all E-valued stochastic processes with parameter set J can be characterized by their induced measures in the following sense: For every process $X\colon (\Omega, \mathscr{F}, P) \to S^J$ there exists an equivalent process $Y_0\colon (\Omega_0, \mathscr{F}_0, P_X) \to S^J$ with Ω_0, $\mathscr{F}_0$ and Y_0 being independent of X.

Question 9. Prove Lemma 2.25.

Question 10. Let $A \in L(H)$ be compact. Show that the following statements are equivalent:

(i) $A \in \mathscr{S}_1(H)$.

(ii) There exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} \|Ae_n\| < \infty$.

(iii) For all orthonormal bases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ we have $\sum_{n\in\mathbb{N}} |\langle Ax_n, y_n\rangle| < \infty$.

(iv) There are $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset H$ with $A = \sum_{n\in\mathbb{N}} \langle a_n, \cdot\rangle b_n$ and $\sum_{n\in\mathbb{N}} \|a_n\|\,\|b_n\| < \infty$.

(This is an addendum to Question 2 and may also be used for the proof of Question 4.)

Question 11. Let H_1, H_2, H be separable real Hilbert spaces and $T_i \in L(H_i, H)$ for $i = 1, 2$, and assume that $\|T_1^* x\|_{H_1} = \|T_2^* x\|_{H_2}$ $(x \in H)$. Show that $R(T_1) = R(T_2)$ and $\|T_1^{-1}x\|_{H_1} = \|T_2^{-1}x\|_{H_2}$ $(x \in R(T_1))$, where T_i^{-1} denotes the pseudo-inverse of T_i.

Question 12. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ be a probability space with normal filtration. Define

$$\begin{aligned}
\mathscr{L} &:= \{X\colon [0,T]\times\Omega \to H \mid X \text{ adapted, left continuous with bounded paths}\},\\
\mathscr{C} &:= \{X\colon [0,T]\times\Omega \to H \mid X \text{ adapted, continuous}\},\\
\mathscr{R} &:= \big\{\{0\}\times A : A \in \mathscr{F}_0\big\} \cup \big\{(s,t]\times A : s<t, A \in \mathscr{F}_s\big\}.
\end{aligned}$$

Show that for the generated σ-algebras on $[0,T]\times\Omega$ we have $\sigma(\mathscr{L}) = \sigma(\mathscr{C}) = \sigma(\mathscr{R})$.

Question 13. a) Define $H_0^1((0,\pi)) := \{u \in H^1((0,\pi)) : u(0) = u(\pi) = 0\}$ and the dual space

$$H^{-1}((0,\pi)) := \{\varphi\colon (H_0^1((0,\pi)), \|\cdot\|_{H^1((0,\pi))}) \to \mathbb{R} \mid \varphi \text{ continuous, linear}\}.$$

Let Δ_D be the Dirichlet Laplacian in $L^2((0,\pi))$, i.e. $D(\Delta_D) := H^2((0,\pi)) \cap H_0^1((0,\pi))$ and $\Delta_D u := \Delta u$. Compute the eigenvalues and eigenvectors of $-\Delta_D$.

b) Use $A := (-\Delta_D)^{1/2}$ to show that the canonical embedding

$$J\colon L^2((0,\pi)) \to H^{-1}((0,\pi)),\ u \mapsto \varphi_u \text{ with } \varphi_u(v) := \int_0^\pi u(x)v(x)dx$$

is a Hilbert-Schmidt operator.