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December 16th, 2013

Parabolic boundary value problems

Exercises and Questions

Question 1. Let X, Y be Banach spaces, and $T \in L(X,Y)$. Show that $\{T\}$ is $\mathcal{R}$-bounded.

Question 2. Are subsets of $\mathcal{R}$-bounded sets of operators $\mathcal{R}$-bounded? Is the union of two $\mathcal{R}$-bounded sets $\mathcal{R}$-bounded? What can be said on the $\mathcal{R}$-bounds?

Question 3. Let X, Y be Banach spaces. Show that $\mathbf{T} \in L(\mathrm{Rad}_2(X), \mathrm{Rad}_2(Y))$ holds if $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded (see Lemma 2.7).

Question 4. a) Let $X = L^p(\mathbb{R})$, $p \in (1,\infty)$, and define $T_s \in L(L^p(\mathbb{R}))$ by $(T_s f)(x) := e^{isx} f(x)$. Is $\{T_s : s \in \mathbb{R}\}$ $\mathcal{R}$-bounded?

b) Answer the same question for the operator family S_s defined by $S_s f := \mathscr{F}^{-1} e^{is\cdot} \mathscr{F} f$.

Question 5. Let Γ be a set, and let X, Y be Banach spaces. Define $\ell^\infty_{\mathcal{R}}(\Gamma, L(X,Y))$ as the space of all function $f\colon \Gamma \to L(X,Y)$ with $\mathcal{R}$-bounded range, and define the norm by $\|f\|_{\ell^\infty_{\mathcal{R}}} := \mathcal{R}(f(\Gamma))$.

Then, by definition, a family of operators $\{T_\gamma : \gamma \in \Gamma\}$ is $\mathcal{R}$-bounded if and only if $f \in \ell^\infty_{\mathcal{R}}(\Gamma, L(X,Y))$ for $f(\gamma) := T_\gamma$.

Show that $\ell^\infty_{\mathcal{R}}(\Gamma, L(X,Y))$ is a Banach space, and that the norm is submultiplicative (in the case $X{=}Y$).

Question 6. In the situation of Question 5, define the completed projective tensor product $\ell^\infty(\Gamma)\hat{\otimes}_\pi L(X,Y)$ of $\ell^\infty(\Gamma)$ and $L(X,Y)$ as the space of all functions $f\colon \Gamma \to L(X,Y)$ which can be represented as $f(\gamma) = \sum_{n\in\mathbb{N}} \lambda_n f_n(\gamma) A_n$ with $(\lambda_n)_{n\in\mathbb{N}} \subset \ell^1(\mathbb{N})$, $f_n \to 0$ in $\ell^\infty(\Gamma)$ and $A_n \to 0$ in $L(X,Y)$. Show that

$$\ell^\infty(\Gamma)\hat{\otimes}_\pi L(X,Y) \subset \ell^\infty_{\mathcal{R}}(\Gamma, L(X,Y)).$$

(This implies that the range of $C^\infty(M, L(X,Y))$-functions for a closed manifold M is $\mathcal{R}$-bounded as well as the range of $\mathscr{S}(\mathbb{R}^n, L(X,Y))$-functions due to $C^\infty(M, L(X,Y)) \cong C^\infty(M)\hat{\otimes}_\pi L(X,Y)$ and $\mathscr{S}(\mathbb{R}^n, L(X,Y)) \cong \mathscr{S}(\mathbb{R}^n)\hat{\otimes}_\pi L(X,Y)$.)

Question 7. A pseudodifferential operator (with constant coefficients) of order $\mu \in \mathbb{R}$ is defined as the operator $\mathrm{op}[m]$ with $m \in S^\mu(\mathbb{R}^n)$, where the (Hörmander) symbol class $S^\mu(\mathbb{R}^n)$ is defined as the set of all $m \in C^\infty(\mathbb{R}^n)$ with $\sup_{\xi\in\mathbb{R}^n} |\xi|^{|\alpha|-\mu} |D^\alpha m(\xi)| < \infty$ for all $\alpha \in \mathbb{N}_0^n$. Discuss the $\mathcal{R}$-boundedness of such operators (for an appropriate formulation, it is reasonable to define a topology on $S^\mu(\mathbb{R}^n)$).

Question 8. For fixed $M > 0$ define

$$\Phi := \Big\{\varphi \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} |x^\alpha|\, |\partial^\alpha \varphi(x)| dx \le M \ (\alpha \in \{0,1\}^n)\Big\}.$$

For $\varphi \in \Phi$ define the convolution operator

$$(K_\varphi f)(x) := \int_{\mathbb{R}^n} \varphi(x-y) f(y) dy \quad (f \in L^p(\mathbb{R}^n)).$$

Show that $\{K_\varphi : \varphi \in \Phi\} \subset L(L^p(\mathbb{R}^n))$ is $\mathcal{R}$-bounded.

Question 9. Show that X has property (α) if and only if for all $p \in [1,\infty)$ there exists $C_p > 0$ such that for all $\alpha_{ij} \in \mathbb{C}$, $|\alpha_{ij}| \le 1$, $N \in \mathbb{N}$, $x_{ij} \in X$ the inequality

$$\Big(\int_0^1 \int_0^1 \Big\| \sum_{i,j=1}^N r_i(u) r_j(v) \alpha_{ij} x_{ij} \Big\|_X^p dudv\Big)^{1/p} \le C_p \Big(\int_0^1 \int_0^1 \Big\| \sum_{i,j=1}^N r_i(u) r_j(v) x_{ij} \Big\|_X^p dudv\Big)^{1/p}.$$

Show that every Hilbert space has property (α).

Question 10. Let X be a Banach space with property (α) and $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space. Show that $L^p(\Omega; X)$ has property (α) for all $p \in [1, \infty)$.

Question 11. The operator $A_0(x, D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}^{N\times N}$, is called parameter-elliptic on the ray $S_\theta := \{re^{i\theta} : r \ge 0\}$ if there exists a $C > 0$ such that

$$\det\big(a_0(x,\xi) - \lambda\big) \ge C(|\xi|^{2m} + |\lambda|)^N \quad ((\xi,\lambda) \in (\mathbb{R}^n \times S_\theta) \setminus \{(0,0)\}).$$

Show that the set of angles θ for which $A_0(x, D)$ is parameter-elliptic in S_θ is open.

Question 12. For $f \in \mathscr{S}(\mathbb{R}^n)$, define

$$(Hf)(x) := [\mathrm{PV}(\tfrac{1}{x}) * f](x) := \lim_{\varepsilon \searrow 0} \int_{|x-y|\ge\varepsilon} \frac{f(y)}{x-y}\, dy.$$

Show that, up to constants, $H = \mathscr{F}^{-1} \operatorname{sign} \mathscr{F}$, i.e. H is the Hilbert transform.

Hint: Consider the function $\mathbb{R} \to \mathbb{R}$, $\xi \mapsto \frac{1}{\xi}\chi_{\varepsilon \le |\xi| \le R}$ (for small $\varepsilon > 0$ and large $R > 0$) and its Fourier transform.

Question 13. Let $\mathbb{R}_+ := (0, \infty)$ and define the one-sided Hilbert transform

$$(H_+ f)(x) := \int_0^\infty \frac{f(y)}{x+y}\, dy \quad (f \in L^p(\mathbb{R}_+)).$$

Show that $H_+ \in L(L^p(\mathbb{R}_+))$.