Exponentiation on the Surreals: An Overview with an Introduction to Surreal Integration

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Introduction

In his monograph *On Numbers and Games*, J. H. Conway introduced a real-closed field of *surreal numbers* embracing the reals and the ordinals as well as a great many less familiar numbers including

$$-\omega,\ \omega/2,\ 1/\omega,\ \sqrt{\omega},\ \ln\omega,\ e^{\omega} \text{ and } \sin(1/\omega)$$

to name only a few, where ω is the least infinite ordinal. This particular real-closed field, which Conway calls **No**, is so remarkably inclusive that, subject to the proviso that numbers—construed here as members of ordered fields—be individually definable in terms of sets of von Neumann-Bernays-Gödel set theory with global choice (NBG), it may be said to contain "*All Numbers Great and Small*." In this regard, **No** bears much the same relation to ordered fields that the ordered field $\mathbb{R}$ of real numbers bears to Archimedean ordered fields.

In addition to its inclusive structure as an ordered field, **No** has a rich *simplicity hierarchical* or *s-hierarchical* structure, that depends upon its structure as a *lexicographically ordered full binary tree* and arises from the fact that Conway's recursive definitions of the sums and products of members of **No** ensure that:

the sums and products of any two members of **No** are the simplest possible members of **No** consistent with **No**'s structure as an ordered group and an ordered field, respectively,

it being understood that x is *simpler than* y (written $x <_s y$) just in case x is a predecessor of y in the surreal number tree.

Among the most significant substructures A of **No** (or of its reducts and relational expansions) are those that are *initial*, i.e. those such that for each $x \in A$,

$$\{y \in A : y <_s x\} = \{y \in \mathbf{No} : y <_s x\}.$$

Unlike arbitrary substructures, the initial substructures inherit many the recursively generated canonical features of **No**, including canonical copies of subfields of the reals, value groups, integer parts and systems of ordinals to name only a few.

One the striking s–hierarchical features of **No** is that every surreal number can be assigned a canonical "proper name" (or *normal form*) that is a reflection of its characteristic s–hierarchical properties. These normal forms are formal sums of the form

$$\sum_{\alpha<\beta} \omega^{y_\alpha}.r_\alpha$$

where β is an ordinal, $(y_\alpha)_{\alpha<\beta}$ is a strictly decreasing sequence of surreals, and $(r_\alpha)_{\alpha<\beta}$ is a sequence of nonzero real numbers; every such sum is the normal form of a surreal, the normal form of an ordinal being just its *Cantor normal form*.

In the normal form of a surreal number, the r_α's are members of **No**'s canonical copy $\mathbb{R}$ of the reals, i.e. the unique Dedekind complete initial subfield of **No**; and the ω^{y_α}'s are *leaders* of **No**, a leader being the simplest member of the positive cone of an Archimedean subclass of **No**.

Every nonzero surreal is the sum of three components, each of which can be succinctly characterized in terms of its normal form:

the **purely infinite** component, whose terms solely have positive exponents;

the **real** component, whose sole term (if it is not the empty sum) has exponent 0;

the **infinitesimal** component, whose terms solely have negative exponents.

Making use of normal forms of surreal numbers, the following figure offers a glimpse of the some of the early stages of the recursive unfolding of **No**.

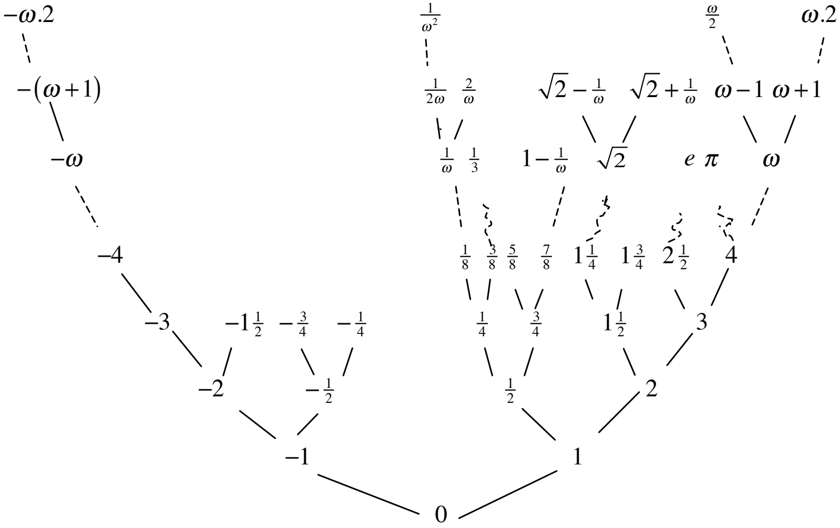

Figure: Early stages of the recursive unfolding of **No**

If L and R are sub*sets* of **No** for which every member of L precedes every member of R, there is simplest member of **No** lying between the members of L and the members of R, denoted

$$\{L \mid R\}.$$

In fact, every surreal x may be written in the canonical form

$$\{L_{s(x)} \mid R_{s(x)}\},$$

where

$$L_{s(x)} = \{a \in \mathbf{No} : a <_s x \text{ and } a < x\}$$

and

$$R_{s(x)} = \{a \in \mathbf{No} : a <_s x \text{ and } x < a\}.$$

The Exponential Ordered Field of Surreal Numbers

Employing the canonical representation of a surreal x, and inspired by Conway's definitions of sums and products, the Kruskal-Gonshor surreal exponential function exp may be defined by recursion as follows.

Definition 1

$$\exp(x) =$$

$$\left\{0,\ (\exp x^L)[x - x^L]_n,\ (\exp x^R)[x - x^R]_{2n+1} \;\middle|\; \frac{\exp x^L}{[x^L - x]_{2n+1}},\ \frac{\exp x^R}{[x^R - x]_n}\right\},$$

where x^L and x^R range over $L_{s(x)}$ and $R_{s(x)}$ respectively, n ranges over $\mathbb{N}$, $[y]_n := 1 + y + \frac{y^2}{2!} + ... + \frac{y^n}{n!}$ for all surreal y, and (for consistency sake) $[y]_{2n+1}$ is restricted to positive values.

While the definition of exp is quite complicated for the general case, it reduces to more revealing and manageable forms for the three theoretically significant cases.

Proposition 1 (Gonshor 1986)

(i) $\exp(x) = e^x$ *for all* $x \in \mathbb{R}$*;*

(ii) $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ *for all infinitesimal* x*;*

(iii) *if* x *is purely infinite, then*

$$\exp(x) = \left\{0, (\exp x^L)(x - x^L)^n \,\middle|\, \frac{\exp x^R}{(x^R - x)^n}\right\},$$

where x^L *and* x^R *range over all the purely infinite predecessors of* x *with* $x^L < x < x^R$.

Berarducci and Montova (2018) simplified Gonshor's characterization of exp for the purely infinite case by essentially establishing:

Proposition 2

For a purely infinite surreal $\sum_{\alpha<\beta} \omega^{y_\alpha}.r_\alpha$,

$$\exp(\sum_{\alpha<\beta} \omega^{y_\alpha}.r_\alpha) =$$

$$\{0, \exp(\sum_{\alpha<\nu} \omega^{y_\alpha}.r_\alpha + \omega^{y_\nu}.q_{\nu_L}) \mid \exp(\sum_{\alpha<\nu} \omega^{y_\alpha}.r_\alpha + \omega^{y_\nu}.q_{\nu_R})\}$$

where ν *ranges over the ordinals* $< \beta$, *and for each such* ν, q_{ν_L} *and* q_{ν_R} *range over the rational numbers for which* $q_{\nu_L} < r_\nu < q_{\nu_R}$.

The significance of cases (i)–(iii) accrues from the fact that for an arbitrary surreal number x,

$$\exp(x) = \exp(x_P) \cdot \exp(x_R) \cdot \exp(x_I)$$

where x_P, x_R and x_I are the purely infinite, real and infinitesimal components of x, respectively. It is already clear from (i) and (ii) what $\exp(x)$ is for real and infinitesimal values of x. The following additional result sheds further light on $\exp(x)$ when x is purely infinite.

Proposition 3 (Gonshor 1986)

The restriction of $\exp$ *to the class of purely infinite surreal numbers is an isomorphism of ordered groups onto* **No***'s class of leaders, i.e.* $\{\omega^x : x \in \mathbf{No}\}$.

By van den Dries, Macintyre and Marker's (1994), the elementary theory of the expansion of $\mathbb{R}_{\text{an}}$ by its exponential function e^x is axiomatized by T_{an} together with the "Ressayre" axioms which express the fact that the exponential function is an order preserving isomorphism from the additive group of the underlying ordered field onto its positive multiplicative group such that

(1) the exponential of any $x > n^2$ is greater than x^n for $(n = 1, 2, ...)$;

(2) the exponential of any x with $-1 \leq x \leq 1$ equals $E(x)$ where E is the function symbol of L_{an} corresponding to the power series $\sum(1/n!)X^n \in \mathbb{R}[[X]]$.

Appealing to the above, van den Dries and Ehrlich (2001) showed:

Proposition 4

The field of surreal numbers equipped with restricted analytic functions (defined via Taylor series expansion) and with exp *is an elementary extension of the field of real numbers with restricted analytic functions and real exponentiation.*

Corollary 1

The ordered exponential field of surreal numbers is an elementary extension of the ordered exponential field of real numbers.

Some Distinguished Initial Ordered Exponential Subfields of **No**

In our (2021), Elliot Kaplan and I provide necessary and sufficient conditions for an ordered exponential field to be isomorphic to an initial ordered exponential subfield of **No**. This will be the subject of Elliot's talk. In this and the next slide attention is simply drawn to some distinguished examples.

Definition 2

For each ordinal α, let $\mathbf{No}(\alpha) := \{x \in \mathbf{No}$ *: the tree-rank of* $x < \alpha\}$.

Proposition 5 (van den Dries and Ehrlich (2001))

Let α be an ε-number (which includes all uncountable cardinals). Then $\mathbf{No}(\alpha)$ *equipped with restricted analytic functions and exponentiation induced by* **No** *is an elementary substructure of* $(\mathbf{No}_{\mathrm{an}}, \exp)$ *and an elementary extension of* $(\mathbb{R}_{\mathrm{an}}, e^x)$.

Proposition 6 (Aschenbrenner, van den Dries and van der Hoeven (2019))

There is a canonical elementary embedding i of the ordered exponential field $\mathbb{T}$ of transseries into **No** *that sends x to ω.*

In their (2019), Berarducci and Mantova introduced the ordered exponential subfield

$$\mathbb{R}((\omega))^{LE}$$

of $(\mathbf{No}, \exp)$ and proved that it is the image of the embedding i. Using the revealing nature of their surprisingly simple construction, it was found that:

Proposition 7 (Ehrlich and Kaplan (2021))

$\mathbb{R}((\omega))^{LE}$ is initial.

Trigonometric-Exponential Ordered Subfields of **No**

Let

$$T_{\text{trig,exp}}$$

be the theory of the real field expanded by $\sin \restriction_{[0,2\pi]}$, the total exponential function $\exp$, and a constant symbol for each real number. Elliot Kaplan and I call a model of $T_{\text{trig,exp}}$ a

$$\textit{trigonometric} - \textit{exponential ordered field}.$$

Let K be such a field. Then K is real closed, so there is an *integer part* Z of K. Using this integer part, together with the fact that $\cos \restriction_{[0,2\pi]}$ is 0-definable in K, we may extend sine and cosine to all of K by setting

$$\sin(a + 2\pi d) := \sin a, \qquad \cos(a + 2\pi d) := \cos a$$

where $a \in [0, 2\pi)$ and where $d \in Z$. Since K may have many integer parts, the extension of $\sin$ and $\cos$ to K is not necessarily unique. However, if K is an initial subfield of **No**, K has a canonical integer part, namely $\mathbf{Oz} \cap K$, where **Oz** is **No**'s canonical integer part of *Omnific Integers*. $\mathbf{Oz} \cap K$ is the unique *initial integer part* of K.

Proposition 8 (Ehrlich and Kaplan (2021))

***No** is a trigonometric-exponential ordered field. Moreover, if K is an initial trigonometric-exponential ordered subfield of **No**, including **No** itself, then K admits canonical sine and cosine functions arising from its unique initial integer part.*

Making use of this result, the initial trigonometric-exponential natures of the *K*'s in question, and the corresponding map

$$a + ib \mapsto (\exp a)(\cos b + i \sin b) : K[i] \to K[i]^{\times},$$

one further obtains:

Proposition 9 (Ehrlich and Kaplan (2021))

*The exponential functions on initial trigonometric-exponential subfields of **No** extend to canonical exponential functions on their surcomplex counterparts. So, for example, **No**$[i]$, **No**$(\alpha)[i]$, for each epsilon number α, and $\mathbb{R}((\omega))^{LE}[i]$ admit canonical exponential functions extending* exp *or their corresponding restrictions thereof.*

Maximal Hardy Fields, etc.

In their (2018), Berarducci and Montova construct a “surreal derivation" ∂ on **No** in which exp plays a central role. Using the restriction ∂_{ω_1} of ∂ to $\mathbf{No}(\omega_1)$, Aschenbrenner, van den Dries and van der Hoeven (2023) proved:

Proposition 10

Assuming CH, every maximal Hardy field is isomorphic to $(\mathbf{No}(\omega_1), \partial_{\omega_1})$.

We draw this part of the talk to a close, by noting that in their ICM (2018) talk, Aschenbrenner, van den Dries and van der Hoeven outline the program they (along with Berarducci, Mantova, Bagayoko and Kaplan) are engaged in for developing an ambitious theory of asymptotic differential algebra for all of **No**, a theory in which $\exp$ would again play a critical role. Such a program, however, would require a derivation on **No** having compositional properties not enjoyed by ∂. If successful, such a program would provide the most dramatic advance towards interpreting growth rates as numbers since the pioneering work of Paul du Bois-Reymond, G. H. Hardy and Felix Hausdorff on “orders of infinity" in the decades bracketing the turn of the 20th century.

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In the second part of our talk we turn to a subject in which a derviation on surreal functions rather than on surreal numbers plays an important role.

Announcement/Advertisement

The remaining material is based on :

Integration on the Surreals

forthcoming

Ovidiu Costin and Philip Ehrlich

which, in turn, is a substantially revised and substantially expanded version of portions of the matharXiv (2015) paper

Integration on the Surreals :

A Conjecture of Conway, *Kruskal and Norton.*

Ovidiu Costin, *Philip Ehrlich*, *and Harvey Freidman*

There has been a longstanding program, initiated by Conway, Kruskal and Norton, to develop analysis on **No**, starting with a recursive definition of integration. The initial attempts at defining integration, in particular the recursive definition proposed by Norton, turned out, as Kruskal discovered, to have fundamental flaws. Despite this disappointment, the search for a theory of surreal integration has continued (Fornasiero 2004, Rubinstein-Salzedo and Swaminathan 2014), but remains largely open.

Observation

Making real progress towards developing a satisfactory theory of integration on the surreals, and more generally in interpreting divergent expansions by means of surreal analysis, requires finding a property-preserving operator that extends the members of a wide body of important classical functions from $\mathbb{R}$ to **No**. *The existence of such an extension operator would then in principle provide a theoretically satisfying and widely applicable definition of integration: in particular,* **the integral of an extension from $\mathbb{R}$ to No of a function on the reals would be defined to be the extension of its integral from $\mathbb{R}$ to No**.

Any such theory would have to keep in mind that functions whose behavior can be described in terms of exponentials and logarithms are remarkably ubiquitous. Indeed, as G. H. Hardy noted in 1910:

> *No function has presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmic-exponential terms.*

Accordingly, developing a satisfactory theory of integration on the surreals would require building on $(\mathbf{No}, \exp)$.

In the aforementioned paper, **it is shown that an extension operator as described above, and thereby extensions of integrals from $\mathbb{R}$ to No, exist for a large subclass $\mathcal{F}_{\mathcal{R}}$ of *Écalle's system of resurgent functions.***

Among other things, $\mathcal{F}_{\mathcal{R}}$ contains all real functions that at ∞ are semi-algebraic, semi-analytic, analytic, and functions with divergent but Borel summable series, as well as solutions of nonresonant linear or nonlinear meromorphic systems of ODEs or of difference equations. As such, most classical special functions, such as Airy, Bessel, Ei, erf, Gamma, and Painlevé transcendents, are covered by our analysis.

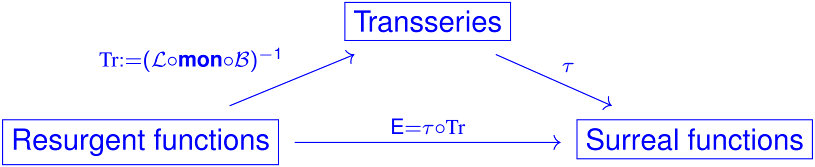
We work in **NBG less the Axiom of Choice (for both sets and proper classes)**, with the result that the extensions of functions and integrals that concern us here have a "constructive" nature in this sense.

Building on the structural similarity that exists between surreal numbers written in normal form and Écalle's transseries, which is typified by the aforementioned canonical embedding of

$$i : \mathbb{T} \rightarrow \textbf{No} \textit{ which sends } x \textit{ to } \omega,$$

there are two critical components that play roles in the theory.

The first is Écalle's theory of *resurgent transseries*, *resurgent functions*, and *Écalle-Borel summation* which relates the two.



The extension operator E for the *positive infinite case* is the composition of two intermediate isomorphisms: transseriation (i.e. $\mathrm{Tr} := (\mathcal{L} \circ \mathbf{mon} \circ \mathcal{B})^{-1}$) from a subspace of resurgent functions to a subspace of transseries, where $\mathcal{L} \circ \mathbf{mon} \circ \mathcal{B}$ is Écalle-Borel summation, and a map τ from transseries to surreal functions.

Écalle-Borel summation

In Écalle-Borel summation

$$\mathcal{L} \circ \textbf{mon} \circ \mathcal{B},$$

$\mathcal{L}$ is the *Laplace transform*, **mon** is a *well-behaved uniformizing average* in Écalle's sense, and $\mathcal{B}$ is the *Borel transform*.

Écalle introduced Écalle-Borel summation for the resummation of a large class of divergent series which do not fall in the scope of classical *Borel summation*

$$\mathcal{L} \circ \mathcal{B}.$$

The second component is the fact the surreals are closed under absolute convergence in the sense of Conway (or of B. H. Neumann 1949). That is:

Proposition (Conway 1976)

For each formal power series f in $n \geqslant 0$ variables with coefficients in $\mathbb{R}$, $f(a_1, ..., a_n)$ is absolutely convergent in **No** *for every choice of infinitesimals $a_1, ..., a_n$ in* **No**.

In fact:

Observation

A sufficient condition for developing our theory in an ordered exponential subfield of **No** *is that the exponential subfield be closed under absolute convergence in the sense of Conway.*

So, for example:

In addition to applying to **No**, our theory carries over to **No**(α), for each epsilon number α, and to $\mathbb{R}((\omega))^{LE}$.

Extension Operator E

Combining the two just-said components, here is the definition of our Extension Operator E.

Definition

Let $f \in \mathcal{F}_{\mathcal{R}}$, and let $c \in \mathbb{R}$ be such that f is real-analytic on (c, ∞). (Such a c always exists.) We extend f to $(c, \mathbf{On})$ as follows.

1. *For positive infinite $x \in \mathbf{No}$ we define $(\mathrm{E}f)(x) = (\tau \circ \mathrm{Tr}\ f)(x)$.*
2. *For finite $x \in \mathbf{No}$, where x_0 is the real part of x and ζ is the infinitesimal part of x, we define $(\mathrm{E}f)(x)$ by*

$$f(x_0 + \zeta) = f(x_0) + \sum_{k \geqslant 1} (k!)^{-1} f^{(k)}(x_0)\zeta^k,$$

the infinite sum being absolutely convergent in the sense of Conway.

Antidifferentiation Operator and Integral Operator

Making use of the extension operator E and an antidifferentiation operator A likewise defined on $\mathcal{F}_{\mathcal{R}}$, an antidifferentiation operator $\mathsf{A}_{\mathbf{No}}$ on $\mathsf{E}(\mathcal{F}_{\mathcal{R}})$ and a corresponding integral operator are defined as follows:

$$\mathsf{A}_{\mathbf{No}} := \mathsf{E}\mathsf{A}\mathsf{E}^{-1}.$$

$$\int_x^y f := \mathsf{A}_{\mathbf{No}}(f)(y) - \mathsf{A}_{\mathbf{No}}(f)(x).$$

The following result demonstrates that $\int_x^y f$ so-defined is worthy of the appellation "integral operator".

In the following proposition, $\alpha, \beta, a, b, a_1, a_2, a_3 \in \mathbf{No}$, and $f, g, fg, f \circ g, f', g'$ are understood to be elements of $\mathrm{E}(\mathcal{F}_{\mathcal{R}})$ on $[a, b]$, $[a_1, a_2]$, $[a_2, a_3]$ or $[a_1, a_3]$ where applicable.

Proposition 11 ($\int_x^y f$ *is an Integral Operator*)

$\int_x^y f$ as defined above is an *integral operator* on $\mathrm{E}(\mathcal{F}_{\mathcal{R}})$, meaning a function of three variables, $x, y \in \mathbf{No}$ and $f \in \mathrm{E}(\mathcal{F}_{\mathcal{R}})$, with the properties:

(a) $\left(\int_a^x f\right)' = f;$

(b) $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g;$

(c) $\int_a^b f' = f(b) - f(a);$

Proposition (Continued)

(d) $\int_{a_1}^{a_2} f + \int_{a_2}^{a_3} f = \int_{a_1}^{a_3} f$;

(e) $\int_a^b f'g = fg|_a^b - \int_a^b fg'$ if f and g are differentiable on (a, b);

(f) $\int_a^x (f \circ g)g' = \int_{g(a)}^{g(x)} f$ whenever $g \in \mathrm{E}(\mathcal{F}_{\mathcal{R}})$ is differentiable on (a, x).

(g) If f is a positive function and $b > a$, then $\int_a^b f > 0$.

Simple Example

The most trivial example is e^x, where $A_{\mathbf{No}}(e^x) = e^x$. By then applying the definition of $\int_x^y f$ to e^x, we obtain, for example,

$$\int_0^\omega e^x dx = e^\omega - 1,$$

as expected. This stands in contrast to Simon Norton's earlier proposed definition of integration which was shown by Kruskal to integrate e^x over the range $[0, \omega]$ to the wrong value e^ω.

Extension, Antidifferentiation and Integral Operators

Above we made use of the notions of extension and antidifferentiation operators without stating precisely what we mean by these notions. In some of the remaining time, we lend precision to these concepts. For this purpose, we require a generalization of the idea of a derivative of a function at a point.

Definition (Derivative)

Let K be an ordered field. A function f defined on an interval around a is differentiable at a if there is an $f'(a) \in K$ such that $(\forall \varepsilon > 0 \in K)(\exists \delta > 0 \in K)$ such that

$$(\forall x \in K)\left(|x - a| < \delta \Rightarrow \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \varepsilon\right).$$

As usual, $f'(a)$ is said to be *the derivative of* f *at* a and f is said to be *differentiable* if the derivative of f exists at each point of its domain.

Definition (Extension Operators)

Let I be an interval (i.e. a convex subclass) of $\mathbb{R}$ and J be an interval of **No** that contains I. Also, let $\mathcal{F}$ be a set of real-valued functions defined on intervals of $\mathbb{R}$.

By an *extension operator* E on $\mathcal{F}$ we mean a map that associates to each function $f : I \to \mathbb{R}$ in $\mathcal{F}$ a function $E\,f : J \to$ **No** in such a manner that

i. for all $f \in \mathcal{F}$, $E\,f$ is an extension of f;
ii. (Linearity) for all $g, h \in \mathcal{F}$ and $C \in \mathbb{R}$, $E(Cg) = CE\,g$ and $E(g+h) = E\,g + E\,h$;
iii. if $\beta, \lambda \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $g(x) = x^{\beta}e^{\lambda x}$ and $h(x) = x^n \log(x)$ for all $x \in I$, then $(E\,g)(x) = x^{\beta}e^{\lambda x}$ and $(E\,h)(x) = x^n \log(x)$ for all $x \in J$.
iv. $E\,f' = (E\,f)'$.

Definition (Real and Surreal Antidifferentiation Operators)

Let $\mathcal{F}$ be a set of real-valued (surreal-valued) functions whose domains are intervals of $\mathbb{R}$ (**No**). An *antidifferentiation operator* on $\mathcal{F}_1 \subseteq \mathcal{F}$ is a function $A : \mathcal{F}_1 \to \mathcal{F}$ such that for all $f, g \in \mathcal{F}_1$:

- i. $A\,f$ is differentiable and $(A\,f)' = f$;
- ii. For any $\lambda \in \mathbb{R}$ ($\lambda \in \textbf{No}$), $A(\lambda f) = \lambda A\,f$, $A(f+g) = A\,f + A\,g$;
- iii. If $y \geqslant x$ and $f \geqslant 0$, then $(A\,f)(y) - (A\,f)(x) \geqslant 0$.
- iv. $\forall n \in \mathbb{N}, A\,(x^n) = \frac{1}{n+1}x^{n+1}$ (the right side being the monomial in $\mathcal{F}$).
- v. $A\,(\exp)$ equals the real (surreal) exponential.
- vi. If $F \in \mathcal{F}_1$ and $F' = f \in \mathcal{F}_1$, then there is a $C \in \mathbb{R}$ ($C \in \textbf{No}$) such that $A\,f$ exists and equals $F + C$.

Definition (Integral Operators)

Let A be an antidifferentiation operator on $\mathcal{F}_1 \subseteq \mathcal{F}$, and let $f \in \mathcal{F}_1$ and $x, y \in \mathbf{No}$. Define

$$\int_x^y f := A(f)(y) - A(f)(x).$$

Integral operators thus defined have all of the aforementioned properties enjoyed by the surreal integral operator.

How Much Further Can the Theory be Developed?

Transseries are formal series built up from $\mathbb{R}$ and a variable $x > \mathbb{R}$ using powers, exponentiation, logarithms and infinite sums. Écalle's classical construction of the ordered differential field of transseries is inductive, beginning with log-free transseries. Transseries have (exponential) *heights* and *(logarithmic) depths* (for $n < \omega$) that emerge from their inductive construction, but in our theory of surreal integration thus far developed we are only concerned with *log-free, height one*, and *height one, depth one* transseries.

The theory of resurgent functions for the class of transseries we are concerned with has long been worked out in detail. However, in a far ranging recent work–The Natural Growth Scale (2020)–Écalle has provided what he describes as an "exploratory rather than systematic" presentation of an extension of his theory, including Écalle-Borel summability, to transseries having arbitrary heights and depths. This naturally suggests:

Question 1. Based on a rigorous theory of arbitrary height and depth transseries, is it possible to generalize our "constructive" treatment of extension and antidifferentiation operators to all resurgent functions?

A related and perhaps deeper question is:

Question 2. Do well-behaved extension and antidifferentiation operators exist for broad classes of functions that cannot be obtained by the inductive construction yielding transseries?

While these questions are wide open, we have shown:

There is a foundational obstruction to **constructively** extending many important larger families of functions (including, for example, the full class of smooth functions) to **No** and to defining integration thereon.

The definitions of the extension, antidifferentiation and integral operators E, $A_{\mathbf{No}}$ and $\int_x^y f$ mentioned above are *not* recursive. As such:

Question 3. Can the just-said operators be given simplicity-hierarchical formulations in the *recursive* sense that Conway's field operations and the Kruskal-Gonshor definition of exponentiation are?

Ovidiu and I do in fact know how to provide a simplicity-hierarchical account for much of the theory in terms of Conway's $\{L|R\}$ notation. However, the definitions in terms of Conway's $\{L|R\}$ notation employed in the account are not recursive.

Concluding Thoughts

In virtue of our combined positive and negative results one can say that whereas the Conway-Kruskal-Norton surreal integration program *succeeds in most practical cases*, and may very well be extended a good deal further, its success is limited in the full abstract generality that some, such as Kruskal, had originally hoped for. On the other hand, I suspect that what has been, and very well may be, accomplished in this direction goes beyond the expectations of some including Conway himself.

Thanks for Listening.