$\exp(\partial) = \sigma$

by Vincent Bagayoko (imj-prg)

Joint work with L. S. Krapp, S. Kuhlmann, D. C. Panazzolo & M. Serra

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Answer: you are *wrong*.

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Furthermore, we have

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The same applies for the algebra $\mathbb{C}[[x]] \supset \mathcal{A}$ of formal power series.

Fix a field k with $\mathrm{char}(k)=0$. Given an algebra A and an endomorphism $\phi : A \longrightarrow A$, we want to make sense of the exponential

$$\exp(\phi) = \sum_{n \geqslant 0} \frac{1}{n!}\, \phi^{[n]}$$

and logarithm

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Ideas:

- In finite dimensional Lie group theory: notions of convergence, e.g. taking exponentials of matrices.
- On fields of generalised power series (e.g. Hahn series): notions of summability $\longrightarrow$ formal axiomatic approach?

Algebras with infinite sums

Ideal context: an algebra $\mathcal{A}$ with a notion of infinite sum such that the formal power series

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Furthermore $\mathcal{A}$ should be an algebra of linear maps on another algebra A, such that

$$\exp(\mathcal{A} \cap \mathrm{Der}(A)) = \mathcal{A} \cap \mathrm{Aut}(A).$$

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Finite sums in vector spaces

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For $\boldsymbol{v} \in V^I$, write $\operatorname{supp} \boldsymbol{v} := \{i \in I : \boldsymbol{v}(i) \neq 0\}$. We have a subspace

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and a linear summation operator

$$\begin{aligned} \Sigma_I^{\text{fin}} : V^{(I)} &\longrightarrow V \\ \boldsymbol{v} &\longmapsto \sum_{i \in \operatorname{supp} \boldsymbol{v}} \boldsymbol{v}(i). \end{aligned}$$

What are the properties of the family $(\Sigma_I^{\text{fin}})_{I \in \mathbf{Set}}$?

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Properties of finitely supported summation

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We call $(V, (\Sigma_I)_{I \in \mathbf{Set}})$ a **summability space**. For instance (V, Σ^{fin}) is a summability space.

I) (V, Σ): summability space; Ω: non-empty set; $\mathfrak{q}$: ideal in the Boolean algebra $\mathcal{P}(\Omega)$ containing all finite subsets. We have a subspace $V[\mathfrak{q}] := \{\boldsymbol{v} \in V^{\Omega} : \operatorname{supp} \boldsymbol{v} \in \mathfrak{q}\}$ of V^{Ω}.

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We define a summability structure $\Sigma^{\mathfrak{q}}$ on $V[\mathfrak{q}]$ by setting $\boldsymbol{v} \in \operatorname{dom} \Sigma^{\mathfrak{q}}_I$ if and only if

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Examples of summability spaces

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III) The category of summability spaces with suitable morphisms is complete and cocomplete.

Strong linearity

Strongly linear maps

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- For $\boldsymbol{\phi} \in \operatorname{dom} \Sigma_I^{\mathrm{Lin}}$, define

$$\Sigma_I^{\mathrm{Lin}} \boldsymbol{\phi} := v \longmapsto \Sigma_I(\boldsymbol{\phi}(i)(v))_{i \in I}.$$

Definition: summability algebra

Let $(A, +, 0, ., \cdot)$ *be an algebra over* k, *and* Σ *a summability structure on* $(A, +, 0, .)$. *Then* (A, Σ) *is a* **summability algebra** *if for all sets* I, J *and all* $(\boldsymbol{a}, \boldsymbol{b}) \in \operatorname{dom} \Sigma_I \times \operatorname{dom} \Sigma_J$, *we have*

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- complete algebras for Hausdorff $\mathfrak{p}$-adic topologies;
- given a summability space (V, Σ), the summability space $\operatorname{Lin}^+(V)$ under composition;
- quotients of summability algebras by ideals which are closed under arbitrary sums.

Let (A,Σ) be a summability algebra. Write

$$\mathrm{Der}^{+}(A)=\{\delta\in\mathrm{Lin}^{+}(A):\forall a,b\in A,\delta(a\cdot b)=\delta(a)\cdot b+a\cdot\delta(b)\}.$$

$$\mathrm{Aut}^{+}(A):=\{\sigma\in\mathrm{Lin}^{+}(A)\cap\mathrm{GL}(A):\forall a,b\in A,\sigma(a\cdot b)=\sigma(a)\cdot\sigma(b)\}.$$

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We can now ask: does the exponential

$$\delta \mapsto \sum_{n \in \mathbb{N}} \frac{\delta^{[n]}}{n!}$$

define an isomorphism

$$(\mathrm{Der}^+(A), +) \simeq (\mathrm{Aut}^+(A), \circ) \quad ?$$

Finite words: Let $I \in \mathbf{Set}$. Write $I^\star := \bigcup_{n\in\mathbb{N}} I^n$ for the monoid of finite words (including the empty one $\varnothing$) over I under concatenation

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Then $k\langle\langle I \rangle\rangle$ is a unital summability algebra.

Summability algebras with evaluations

Let (A, Σ) be a unital summability algebra of the form $A = k + \mathfrak{m}$ where $\mathfrak{m}$ is a (two-sided) proper ideal which is closed under arbitrary sums. Then A has **evaluations** *if:*

For all sets I and all families $\boldsymbol{a} \in \operatorname{dom} \Sigma_I$ with $\boldsymbol{a}(i) \in \mathfrak{m}$ for all $i \in I$, the family $(\boldsymbol{a}(i_1) \cdots \boldsymbol{a}(i_n))_{n \in \mathbb{N} \wedge (i_1, \ldots, i_n) \in I^n}$ is summable.

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We can then define, for each such $(I, \boldsymbol{a})$, a strongly linear evaluation morphism

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A j-variant

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$$\begin{aligned}
\hat{j}(\gamma \cdot \phi) &= \hat{j}(\Gamma_\tau[\phi - \tau]) \\
&= J_{\gamma \cdot \tau}[\Gamma_\tau[\phi - \tau] - \gamma \cdot \tau] \\
&= J_{\gamma \cdot \tau}[(\Gamma_\tau - \gamma \cdot \tau)[\phi - \tau]] && (16) \\
&= (J_{\gamma \cdot \tau}[\Gamma_\tau - \gamma \cdot \tau])[\phi - \tau] && (17) \\
&= J_\tau[\phi - \tau] && (18) \\
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We get a "modular function" $\hat{j}$ on $\mathbb{H}+\mathfrak{m}$! (maybe)

Back to exp-log

Let $I = \{0,1\}$. In $k\langle\langle I\rangle\rangle$, we have formal series

$$\exp(X_i) := \sum_{n\in\mathbb{N}} \frac{1}{n!} X_i^n, i \in \{0,1\} \qquad \text{and} \qquad \log(1+X_0) := \sum_{n>0} \frac{(-1)^{n+1}}{n} X_0^n.$$

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Given a summability algebra (A, Σ) with evaluations, with maximal ideal $\mathfrak{m}$, define

$$\begin{aligned} \exp\colon \mathfrak{m} &\longrightarrow 1 + \mathfrak{m} \\ \varepsilon &\longmapsto \mathrm{ev}_\varepsilon(\exp(X_0)) = \sum_{n\in\mathbb{N}} \frac{1}{n!} \varepsilon^n. \end{aligned}$$

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Routine computations give $\exp(\log(1+X_0)) = 1+X_0$ and $\log(\exp(X_0)) = X_0$. Thus $\exp$ is bijective with inverse

$$\begin{aligned} \log : 1+\mathfrak{m} &\longrightarrow \mathfrak{m} \\ 1+\varepsilon &\longmapsto \mathrm{ev}_\varepsilon(\log(1+X_0)) = \sum_{n>0} \frac{(-1)^{n+1}}{n}\varepsilon^n. \end{aligned}$$

Less routine computations give that the series

$$\boldsymbol{X_0 * X_1} := \log(\exp(X_0)\cdot\exp(X_1)) \in k\langle\langle I\rangle\rangle$$

is a sum of elements in the Lie subalgebra of $k\langle\langle I\rangle\rangle_0$ generated by X_0 and X_1.

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By evaluation, we obtain that

- $\exp(\varepsilon_0) \cdot \exp(\varepsilon_1) = \exp(\varepsilon_0 * \varepsilon_1)$
- $\varepsilon_0 * \varepsilon_1$ is a sum of elements in the Lie subalgebra of $\mathfrak{m}$ generated by ε_0 and ε_1 (in particular $*$ preserves derivations).
- $\exp : (\mathfrak{m}, *) \longrightarrow (1 + \mathfrak{m}, \cdot)$ is an isomorphism.

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A $\delta \in \mathfrak{m}$ is a derivation on A if and only if $\exp(\delta)$ is an automorphism of A. Therefore

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is an isomorphism.

As a corollary, the group $\mathrm{Aut}^+(A) \cap (1 + \mathfrak{m})$ is divisible and torsion-free.

Now assume that $k + \mathfrak{m}$ is a summability algebra with evaluations, which is a subalgebra of $(\mathrm{Lin}^+(A), +, ., \circ)$ for a given summability algebra (A, Σ).

Theorem A

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How can we find examples of such situations?

Let $(M, +, 0, <)$ be an ordered monoid. A subset of M is said **Noetherian** (or w.q.o) if it has no infinite antichain and no strictly decreasing infinite sequence.

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- *If $(I, <)$ is a Noetherian ordered set and $M = (I^\star, :, \varnothing, <^\star)$ for Higman's ordering $<^\star$ on $I^\star$, then $k((M)) = k\langle\langle I \rangle\rangle$.*

Write $\mathbb{A} = k((M))$. Given $a, b \in \mathbb{A}, b \neq 0$, we write

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if for all $m_a \in \operatorname{supp} a$, there is an $m_b \in \operatorname{supp} b$ with $m_a > m_b$.

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Corollary

We have an isomorphism

$$\begin{array}{rcl} \exp : (\operatorname{Der}^+(\mathbb{A}) \cap \operatorname{Lin}^+_{\prec}(\mathbb{A}), *) & \longrightarrow & (1\text{-}\operatorname{Aut}^+_k(\mathbb{A}), \circ) \\ \partial & \longmapsto & \displaystyle\sum_{n \in \mathbb{N}} \frac{1}{n!} \partial^{[n]}. \end{array}$$

One can study properties of the group $1\text{-}\mathrm{Aut}^{+}_{k}(\mathbb{A})$ by looking at $\mathrm{Der}^{+}_{\prec}(\mathbb{A})$ instead.

Lie homomorphism theorem

One can study properties of the group $1\text{-}\mathrm{Aut}_k^+(\mathbb{A})$ by looking at $\mathrm{Der}_{\prec}^+(\mathbb{A})$ instead.

Theorem C

Let $\Phi : \mathrm{Der}_{\prec}^+(\mathbb{A}) \longrightarrow \mathrm{Der}_{\prec}^+(\mathbb{B})$ *be a strongly linear Lie algebra endomorphism. There is a unique group morphism* $\Psi : 1\text{-}\mathrm{Aut}_k^+(\mathbb{A}) \longrightarrow 1\text{-}\mathrm{Aut}_k^+(\mathbb{B})$ *with* $\exp \circ \Phi = \Psi \circ \exp$*.*

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Question. The group $1\text{-}\mathrm{Aut}_k^+(\mathbb{A})$ can be equipped with infinite ordered products in a precise sense. If $\Psi : 1\text{-}\mathrm{Aut}_k^+(\mathbb{A}) \longrightarrow 1\text{-}\mathrm{Aut}_k^+(\mathbb{B})$ preserves infinite products, does it induce a strongly linear Lie algebra homomorphism?

(don't look at the picture)