Received 1 December 1995 (559)

Exponentiation in power series fields

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27. 11. 1995

We introduce the notion of a logarithmic cross-section and show that the power series field $\mathbb{R}((G))$ admits such, for a large class of ordered abelian groups G. We use this to construct models of real exponentiation which are unions over countable chains of such power series fields. We show that this construction is best possible in the following sense: There is no archimedean ordered field k and no ordered abelian group $G \neq 0$ such that $k((G))$ would admit an exponential. We prove even stronger nonexistence results. These results are based on a structure theorem for lexicographic products of ordered sets.

1 Introduction

In [DR–MI–MK2], L. van den Dries, A. Macintyre and D. Marker modify an approach of Dahn and Göring ([D–G]) in order to construct nonarchimedean exponential fields which are models of real exponentiation, i.e., have the same elementary properties as $(\mathbb{R}, \exp)$.

In the present paper, we will give a simplified construction. Using some structure theory of nonarchimedean exponential fields, we eliminate one of the two limit processes used in [DR–MI–MK2]. The so constructed models are unions over countable chains of power series fields, each of them admitting a non-surjective logarithm.

[†] Supported by a Deutsche Forschungsgemeinschaft fellowship / The second author is suppor

[‡] Research partially supported by the Edmund Landau Center for research in Mathematical Analysis, supported by the Minerva Foundation (Germany). Publication number 601 / The third author's research was

If k is an ordered field and $G \neq 0$ is an ordered abelian group, then the ("generalized") power series field $k((G))$ admits at least one nonarchimedean order. Further, it is real closed if and only if k is real closed and G is divisible. This provides a very simple and elegant method of constructing nonarchimedean ordered real closed fields. We show that this method is not available for exponential fields. Recall that an exponential on the ordered field $(K,<)$ is in particular an isomorphism between its ordered additive group $(K,+,0,<)$ and its multiplicative group $(K^{>0},\cdot,1,<)$ of positive elements. In this paper, we will prove:

Theorem 1 *Let k be an archimedean ordered field and $G \neq 0$ be an ordered abelian group. Then for every order $<$ on the power series field $K = k((G))$,*

$$(K,+,0,<) \not\simeq (K^{>0},\cdot,1,<)\,. \tag{1}$$

In particular, K cannot be expanded to a model of the theory of the reals with exponentiation.

If we drop the condition that k be archimedean, we can still prove that there is no isomorphism $(K,+,0,<) \simeq (K^{>0},\cdot,1,<)$ which induces a similar isomorphism on k (Theorem 12).

Theorem 1 shows that the construction of models of real exponentiation as unions over chains of power series fields is the best possible among all that involve power series fields. However, there is an "instant" construction producing models of real exponentiation which are almost power series fields. If κ is a regular uncountable cardinal and the group G is suitably chosen, then the κ-restricted power series field $K = k((G))_\kappa$ (which consists of all power series whose support has cardinality $< \kappa$) will admit an exponential, by which we mean just an isomorphism $(K,+,0,<) \simeq (K^{>0},\cdot,1,<)$. ($G$ has to be an exponential group, cf. [KS1], *and* a κ-restricted Hahn product). If G satisfies an additional condition (strong exponential group, cf. [K–K1]), then $k((G))_\kappa$ admits an exponential with which it is a model of real exponentiation. We will consider this and related constructions in a subsequent paper. In Remark 16 below, we will describe how to obtain κ-restricted power series fields with exponentials as unions over chains of length κ.

In Section 2 of this paper, we will recall some notions and summarize some results from the papers [KS1] and [K–K1]. Then we define a **logarithmic cross-section** of an ordered field $(K,<)$ with natural valuation v and value group $G = vK$ to be an embedding h of G into an additive group complement of the valuation ring. Every exponential induces a logarithmic cross-section which is surjective (i.e., $h(G)$ is an additive group complement to the valuation ring). A logarithmic cross-section will be called **strong** if it satisfies $vh(g) > g$ for all $g \in G, g < 0$. Suppose that an exponential field (K,f) is a model of restricted real exponentiation, that is, $(K,f|_{[0,1]})$ has the same elementary properties as $(\mathbb{R},\exp|_{[0,1]})$. Then Theorem 5 states that (K,f) is a model of real exponentiation if and only if f induces a strong logarithmic cross-section. From this, one obtains Corollary 7 which says that a model of restricted real exponentiation can be turned into a model of real exponentiation if and only if it admits a strong logarithmic cross-section which is surjective. We show that, for a large class of groups G, the power series field $\mathbb{R}((G))$ admits a strong logarithmic cross-section.

On the other hand, we prove Theorem 1 by showing that the power series field $k((G)$ does not admit any surjective logarithmic cross-section, and hence no exponential. The key to this result is the fact that every group complement of the valuation ring in $k((G))$ is a lexicographic product of ordered abelian groups. Let us recall the definition of lexicographic products. Let Γ and Δ_γ , $\gamma \in \Gamma$ be totally ordered sets. For every $\gamma \in \Gamma$, we fix a distinguished element $0 \in \Delta_\gamma$. The **support** of $(\delta_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} \Delta_\gamma$ is the set of all $\gamma \in \Gamma$ for which $\delta_\gamma \neq 0$. We denote it by support(a). As a set, we define $\mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma$ to be the set of all $(\delta_\gamma)_{\gamma\in\Gamma}$ with well ordered support. The **lexicographic order** on $\mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma$ is defined as follows. Given $a = (\delta_\gamma)_{\gamma\in\Gamma}$ and $b = (\delta'_\gamma)_{\gamma\in\Gamma} \in \mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma$, observe that supp$(a) \cup$ supp(b) is well ordered. Let γ_0 be the least of all elements $\gamma \in$ supp$(a) \cup$ supp(b) for which $\delta_\gamma \neq \delta'_\gamma$. We set $a < b :\Leftrightarrow \delta_{\gamma_0} < \delta'_{\gamma_0}$. Then $(\mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma, <)$ is a totally ordered set, the **lexicographic product** (or **Hahn product**) of the ordered sets Δ_γ . If all Δ_γ are totally ordered abelian groups, then we can take the distinguished elements 0 to be the neutral elements of the groups Δ_γ . Defining addition on $\mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma$ componentwise, we obtain a totally ordered abelian group $(\mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma, +, 0 <)$.

In Section 3, we prove the following

Theorem 2 *Let* Γ *and* Δ_γ , $\gamma \in \Gamma$ *be totally ordered sets without greatest element, and fix an element* 0 *in every* Δ_γ . *Suppose that* Γ' *is a cofinal subset of* Γ *and that* $\iota : \Gamma' \to \mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma$ *is an order preserving embedding. Then the image* $\iota\Gamma'$ *is not convex in* $\mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma$.

If we drop the condition that Γ has no greatest element, the situation changes drastically. Suitably chosen ordered sets Γ, Δ_γ will even admit an isomorphism $\Gamma \simeq \mathbf{H}_{\gamma\in\Gamma}\,\Delta_\gamma$. We will study this situation and related questions in a subsequent paper [K–K–S].

In Section 3, we shall also apply Theorem 2 to show the nonexistence of surjective logarithmic cross-sections on the power series field $k((G))$. A non-surjective logarithmic cross-section on $\mathbb{R}((G))$, in combination with the logarithm defined on the valuation ring $\mathbb{R}[[G]]$ by the logarithmic power series, only gives rise to a non-surjective logarithm, defined on all positive elements of $\mathbb{R}((G))$. By taking the union over a suitable countable ascending chain of such power series fields, we obtain a surjective logarithm. This is done in Section 4. For the case of models of the theory $T_{\rm an}(\exp)$ of the reals with exponential function and restricted analytic functions, we will prove:

Theorem 3 *Every model* (K, f) *of* $T_{\rm an}(\exp)$ *can be elementarily embedded in a model* (K_ω , f_ω) *of* $T_{\rm an}(\exp)$ *which is a countable union of power series fields.*

In a subsequent paper [K–K2], we will consider the exponential rank of an exponential field. It is defined to be the order type of the ordered set of all valuations coarser than the natural valuation whose residue fields carry an induced exponential. Our construction given in Section 4 below will be used in [K–K2] to show the existence of exponential fields with arbitrary given exponential rank. Cf. also Remark 16 in Section 4.

2 Preliminaries on left logarithms

Let G be a totally ordered abelian group. Recall that the set of archimedean classes of all nonzero elements of G is endowed with a total ordering given by the rule that $[a] < [b]$ if $|a| \gg |b|$. The chain thus obtained is the **rank** of G. The **natural valuation** v_G on G is the surjective map which associates to every element $a \neq 0$ its archimedean class $[a]$. Thus, the rank of G will be denoted by $v_G G$. Similarly, given a totally ordered field K, we consider the natural valuation v on its additive group $(K,+,0,<)$. In this case, the rank carries an extra structure: it forms a totally ordered abelian group G (denoted by vK) if endowed with the addition $[a]+[b] := [ab]$. The natural valuation is now a field valuation, with **value group** G. The **valuation ring** $R = \{a \in K \mid va \geq 0\}$ is the convex hull of $\mathbb{Q}$ in K. The **valuation ideal** $I = \{a \in K \mid va > 0\}$ consists of all elements whose absolute value is smaller than all positive rationals. Such elements are called **infinitesimals**. Their inverses, i.e. the elements $a \in K$ of value $va < 0$, are those whose absolute value is bigger than all rationals. Such elements are called **infinite**.

Like field valuations, also group valuations satisfy the triangle inequality and the law $v_G(-g) = v_G g$. For more information on natural valuations, see [KS1]. Here, let us mention only the following fact. If $g_1, g_2 \in G^{<0} = \{g \in G \mid g < 0\}$, then $v_G g_1 < v_G g_2$ says that $|g_1| \gg |g_2|$, hence it implies that $g_1 < g_2$. Analogously, the natural valuation v of an ordered field K acts on its negative elements. This yields

$$a, b \in K^{>0} \wedge a \geq b \Rightarrow va \leq vb\,. \tag{2}$$

For an arbitrary valued field (K,v), its value group will be denoted by vK and its residue field will be denoted by Kv or by $\overline{K}$. In this paper, K will always be a real closed field with natural valuation v, valuation ring R and value group G. The natural valuation of G will be denoted by v_G. Further, $\mathcal{U}^{>0} := \{a \in K \mid va = 0 \wedge a > 0\}$ denotes the set of positive units in K. It is a convex subgroup of $(K^{>0},\cdot,1,<)$. Recall that we have the following representations for $(K,+,0,<)$ and $(K^{>0},\cdot,1,<)$ (see [KS1], Lemma 3.4 and Theorem 3.8). The former admits a representation as a lexicographic product

$$(K,+,0,<) \simeq \mathbf{A} \amalg (R,+,0,<) \tag{3}$$

where $\mathbf{A}$ is an arbitrary group complement of R in $(K,+)$. Endowed with the restriction of the ordering, it is unique up to isomorphism. The archimedean components of $\mathbf{A}$ are all isomorphic to the ordered additive group of $\overline{K}$, and its rank is the ordered set $G^{<0}$.

An analogous representation of $(K^{>0},\cdot,1,<)$ can be given:

$$(K^{>0},\cdot,1,<) \simeq \mathbf{B} \amalg (\mathcal{U}^{>0},\cdot,1,<) \tag{4}$$

where $\mathbf{B}$ is an arbitrary group complement of $\mathcal{U}^{>0}$ in $(K,\cdot)$. Endowed with the restriction of the ordering, it is again unique up to isomorphism. Moreover, there is something special about $\mathbf{B}$: In view of (2) and the fact that $v(-a) = va$, the map

$$(K^{>0},\cdot,1,<) \to (G,+,0,<)\,, \qquad a \mapsto -va = va^{-1} \tag{5}$$

is a surjective group homomorphism preserving $\leq$, with kernel $\mathcal{U}^{>0}$. We find that every complement $\mathbf{B}$ is isomorphic to $(G,+,0,<)$ through the map $-v$.

Every isomorphism $(K,+,0,<) \simeq (K^{>0},\cdot,1,<)$ of ordered groups gives rise to an isomorphism

$$f:\ (K,+,0,<)\ \rightarrow\ (K^{>0},\cdot,1,<)\,, \tag{6}$$

which also satisfies $f(R) = \mathcal{U}^{>0}$ and $f(I) = 1{+}I$ (cf. [KS1]). Such an isomorphism f will be called an **exponential on** K. The inverse of an exponential is a **logarithm.** We see that f decomposes into an isomorphism

$$f_R:\ (R,+,0,<)\ \rightarrow\ (\mathcal{U}^{>0},\cdot,1,<)$$

of ordered groups, on the one hand, and an isomorphism

$$f_L:\ \mathbf{A}\ \rightarrow\ \mathbf{B}$$

of ordered groups, on the other hand. Such an isomorphism f_R is called a **right exponential**, whereas an isomorphism f_L is a **left exponential**. Conversely, in view of (3) and (4), a right and a left exponential can be put together to obtain an exponential of K. (The indices "L" and "R" refer to the left hand summand resp. the right hand summand of the lexicographic products (3) and (4)). The inverse f_L^{-1} of a left exponential will be called a **(surjective) left logarithm**.

Through the isomorphism $-v$, every isomorphism

$$h:\ (G,+,0,<)\ \rightarrow\ \mathbf{A}$$

gives rise to a surjective left logarithm $h \circ -v$. Conversely, given a surjective left logarithm f_L^{-1}, the map $f_L^{-1} \circ (-v)^{-1}$ is such an isomorphism h. That is, there is a one to one correspondence between surjective left logarithms and isomorphisms of G onto $\mathbf{A}$.

Since we are interested in models of real exponentiation, the constructed exponentials have to satisfy certain conditions. We employ a theorem of J.-P. Ressayre [RE], which can be stated as follows:

Theorem 4 (J.-P. Ressayre)
Let $(K,<)$ be a real closed ordered field and let $f:\ (K,+,0,<) \simeq (K^{>0},\cdot,1,<)$. If (K,f) is a model of restricted real exponentiation and if f satisfies the axiom scheme

$$x > n^2 \implies f(x) > x^n \qquad (n \in \mathbb{N})\,, \tag{7}$$

then (K,f) is a model of real exponentiation.

This result still holds if one adds restricted analytic functions to (K,f) and $(\mathbb{R},\exp)$, cf. [DR–MI–MK1], (4.10).

Because of the condition "$x > n^2$", axiom scheme (7) is void for infinitesimals. That is, it gives information only in the case of $vx \leq 0$. It holds in the case $vx = 0$ if the exponential $\overline{f}$ induced by f on $\overline{K}$ satisfies (7) in the place of f (e.g. if $\overline{f}$ is the usual exp on $\overline{K} = \mathbb{R}$), the proof is simple, see e.g. [K–K1], Lemma 2.10.

Now we have to consider the case of $vx < 0$. In this case, "$x > n^2$" holds for all $n \in \mathbb{N}$ if only x is positive. Restricted to $K \setminus R$, axiom scheme (7) is thus equivalent to the assertion

$$vx < 0 \wedge x > 0 \implies \forall n \in \mathbb{N}: \ f(x) > x^n \ . \tag{8}$$

But "$\forall n \in \mathbb{N} : f(x) > x^n$" means that $f(x)$ is infinitely bigger than x in the ordered multiplicative group $(K^{>0}, \cdot, 1, <)$. Through the isomorphism f^{-1}, this is equivalent to x being infinitely bigger than $f^{-1}(x)$ in $(K, +, 0 <)$, or in other words, $vf^{-1}(x) > vx$. Hence, (8) is equivalent to

$$vx < 0 \wedge x > 0 \implies vf^{-1}(x) > vx \ . \tag{9}$$

(Observe that the condition "$vx < 0 \wedge x > 0$" implies that $f^{-1}(x)$ exists and that $vf^{-1}(x) < 0$.)

Now every $x \in K^{>0}$ can be written as $x = b \cdot c$ where $b \in \mathbf{B}$ and $c \in \mathcal{U}^{>0}$, and $vx = vb$. Then $vf^{-1}(x) = v(f^{-1}(b) + f^{-1}(c)) = vf^{-1}(b)$ since $c \in \mathcal{U}^{>0}$ implies that $vf^{-1}(c) \geq 0 > vf^{-1}(b)$. So (9) holds if and only if it holds for f_L in the place of f. Hence, (9) is equivalent to

$$x \in \mathbf{B} \wedge x > 0 \implies vf_L^{-1}(x) > vx \ . \tag{10}$$

With $g = vx$ and the isomorphism $h = f_L^{-1} \circ (-v)^{-1} : G \to \mathbf{A}$, and in view of $(-v)^{-1}(vx) = x^{-1}$ and $vf_L^{-1}(x^{-1}) = v(-f_L^{-1}(x)) = vf_L^{-1}(x)$, condition (10) translates to

$$vh(g) > g \quad \text{for all } g \in G^{<0} \ . \tag{11}$$

In view of Ressayre's Theorem, we have proved the following:

Theorem 5 *Let f be an exponential on $(K, <)$ such that (K, f) is a model of restricted real exponentiation. Then (K, f) is a model of real exponentiation if and only if (10) holds, or equivalently, if and only if (11) holds.*

Here again, the result still holds if one adds restricted analytic functions.

Remark 6 The following holds:
If on a nonarchimedean ordered field $(K, <)$, an isomorphism (6) satisfies $f(a) > a$ for all infinite elements $a \in K^{>0}$, then it satisfies $f(a) > a^n$ for all infinite elements $a \in K^{>0}$ and all $n \in \mathbb{N}$.

Indeed, $f(a) > a$ implies $vf(a) \leq va$. Suppose that $vf(a) = va$ for some infinite $a \in K$ (recall that $va < 0$). By assumption, $f(a/2) > a/2$, which yields that $vf(a/2) \leq v(a/2)$. But then, $va = vf(a) = vf(a/2)^2 = 2vf(a/2) \leq 2v(a/2) = 2va < va$, a contradiction. Hence, $vf(a) < va$ and thus $f(a) > a^n$ for all infinite elements $a \in K^{>0}$ and all $n \in \mathbb{N}$.

In view of the above theorem, it is natural to ask for the existence of isomorphisms h satisfying equation (11). Not every real closed field will admit such an isomorphism. For instance, if G is a countable divisible ordered agelian group, then the real closed power series field $\mathbb{R}((G))$ does not admit such an isomorphism, since every additive group complement to the valuation ring is uncountable (having $\mathbb{R}$ as its components, cf. the next section).

So, we will rather start by asking for an *embedding* h of the value group G into an additive complement to the valuation ring. Such an embedding will be called a **logarithmic cross-section**. If in addition it satisfies condition (11), then we call it a **strong logarithmic cross-section**. Every logarithmic cross-section h gives rise to a *not necessarily surjective* left logarithm $f_L^{-1} = h \circ -v$, and vice versa. Then h is surjective if and only if f_L^{-1} is. Following the terminoloy introduced in [K–K1], a left logarithm f_L^{-1} (respectively, left exponential f_L) satisfying (10) will be called a **strong left logarithm** (respectively, **strong left exponential**). Hence, h is strong if and only if f_L^{-1} is.

If a real closed field K admits a surjective strong logarithmic cross-section, then it admits a strong left exponential f_L. If it also admits some exponential with which it is a model of restricted real exponentiation, then we let f_R be the right part of this exponential. Note that $f|_{[0,1]} = (f_R)|_{[0,1]}$ for every exponential f since $[0,1] \subset R$. So we can put f_L and f_R together to obtain an exponential f such that (K, f) is a model of restricted real exponentiation. By Theorem 5, (K, f) is then a model of real exponentiation. We have proved:

Corollary 7 *If an exponential field is a model of restricted real exponentiation and admits a surjective strong logarithmic cross-section, then it admits an exponential with which it is a model of real exponentiation.*

Now recall that every embedding (resp. isomorphism) of ordered abelian groups induces canonically an embedding (resp. isomorphism) of their ranks as ordered sets (c.f. [KS1]). In particular, such an embedding h induces an embedding $\tilde{h}$ such that the following diagram commutes, i.e. h is a **lifting** of $\tilde{h}$:

$$\begin{array}{ccc}
\mathbf{A} & \xleftarrow{\;h\;} & G \\
\Big\downarrow v & & \Big\downarrow v_G \\
G^{<0} & \xleftarrow{\;\tilde{h}\;} & v_G G
\end{array}$$

and we have

Lemma 8 *For every $g \in G^{<0}$,*

$$\tilde{h}(v_G g) > g \iff v h(g) > g\,.$$

That is, h is a strong logarithmic cross-section if and only if

$$\tilde{h}(v_G g) > g \quad \text{for all } g \in G^{<0}\,. \tag{12}$$

If h is an isomorphism, then so is $\tilde{h}$ (in this case, it is just the inverse of a "group exponential" as defined in [KS1]).

Note that every ordered abelian group admits an embedding $s: v_G G \to G^{<0}$ of ordered sets such that $v_G \circ s$ is the identity on $v_G G$ (for $\alpha \in v_G G$, we just have to set $s\alpha = g$ where $g \in G^{<0}$ is an arbitrary element of value $v_G g = \alpha$). We will call such a map a **group cross-section** of the valued group (G, v_G). It can be used to get

Lemma 9 *Let G be any ordered abelian group such that $v_G G$ admits an automorphism ζ satisfying $\zeta\alpha > \alpha$ for all $\alpha \in v_G G$. Then for every group cross-section s of G, the embedding $\tilde{h} := s \circ \zeta : v_G G \to G^{<0}$ will satisfy condition (12).*

Indeed, $v_G \tilde{h}(v_G g) = \zeta v_G g > v_G g$ and thus a fortiori $\tilde{h}(v_G g) > g$ if $g \in G^{<0}$. Note that there are plenty of groups satisfying the hypothesis of the lemma. For instance, this is the case if $v_G G$ is isomorphic to an arbitrary nontrivial ordered abelian group, as an ordered set.

Now the question arises whether an embedding (resp. isomorphism) $\tilde{h}$ can be lifted to an embedding (resp. isomorphism) h. (Cf. the related notion of "lifting property" as used in [K–K1].) Such a lifting always exists if $\mathbf{A}$ is rich enough, i.e. if it is a Hahn product. This in turn is the case if the field K is a suitable power series field.

Let k be an archimedean ordered real closed field. If G is an arbitrary ordered abelian group, then the power series field $K := k((G))$ is a formally real field, and it is real closed if and only if G is divisible (which we shall always assume here). Further, K carries a canonical valuation v which associates to every formal power series the minimum of its support. It also carries a natural ordering $<$ such that v is the natural valuation of the ordered field $(K, <)$. The residue field of (K, v) is k, and its value group is G. The valuation ring R of (K, v) is the power series ring $k[[G]]$ which consists of all formal power series whose support is a subset of $G^{\geq 0} = \{g \in G \mid g \geq 0\}$.

Here, we can take the additive group complement $\mathbf{A}$ of the valuation ring R to be the ordered ring $k((G^{<0})) := \{a \in k((G)) \mid \text{support}(a) \subset G^{<0}\}$. As an ordered abelian group, it is canonically isomorphic to $\mathbf{H}_{G^{<0}}(k, +, 0)$. For the case of $k = \mathbb{R}$, we can show:

Theorem 10 *Assume that the rank $v_G G$ of the ordered abelian group G admits an automorphism ζ satisfying $\zeta\alpha > \alpha$ for all $\alpha \in v_G G$. Then the power series field $\mathbb{R}((G))$ admits a strong logarithmic cross-section.*

Proof: According to Lemma 9, we can choose an embedding $\tilde{h}: v_G G \to G^{<0}$ which satisfies condition (12). Note that $\mathbf{A}$ is archimedean-complete (that is, it is maximal and all its components are $\mathbb{R}$). Hence by Hahn's embedding theorem, the embedding $\tilde{h}$ of $v_G G$ into $G^{<0} = v\mathbf{A}$ lifts to an embedding h of G into $\mathbf{A}$. Moreover, since $\tilde{h}(v_G g) > g$, Lemma 8 shows that $vh(g) > g$ for all $g \in G^{<0}$, as required. □

Remark 11 More generally, the power series field $k((G))$ admits a logarithmic cross-section if and only if G is divisible and every archimedean component of G embeds in the ordered additive group of k.

In the next section we will show that the logarithmic cross-sections of $k((G))$ cannot be surjective, that is, they give rise only to non-surjective logarithms.

3 Lexicographic products and logarithmic cross-sections

In this section, we wish to prove and apply Theorem 2.

Proof of Theorem 2: Let us assume that Γ and Δ_γ, $\gamma \in \Gamma$ are totally ordered sets, and let us fix an element 0 in every Δ_γ. Let us further assume that no Δ_γ has a greatest element, so that we can choose maps $\tau_\gamma : \Delta_\gamma \to \Delta_\gamma$ such that $\tau_\gamma \delta > \delta$ for all $\delta \in \Delta_\gamma$. For every well ordered set $S \subset \Gamma$ and every $d = (d_\gamma)_{\gamma\in\Gamma} \in \mathbf{H}_{\gamma\in\Gamma}\Delta_\gamma$ we set

$$d \oplus S = (d'_\gamma)_{\gamma\in\Gamma} \ \text{ where } d'_\gamma := \begin{cases} d_\gamma & \text{if } \gamma \notin S \\ \tau_\gamma d_\gamma & \text{if } \gamma \in S\,. \end{cases}$$

Observe that the support of $d \oplus S$ is contained in support$(d) \cup S$ and thus, it is again well ordered. Further, if $S, S' \subset \Gamma$ are well ordered sets (or empty), then

$$S \subsetneq S' \ \Rightarrow\ d \oplus S < d \oplus S'\,. \tag{13}$$

Now suppose that Γ has no greatest element, Γ' is a cofinal subset of Γ and $\iota : \Gamma' \to \mathbf{H}_{\gamma\in\Gamma}\Delta_\gamma$ is an order preserving embedding such that the image $\iota\Gamma'$ is convex in $\mathbf{H}_{\gamma\in\Gamma}\Delta_\gamma$. We wish to deduce a contradiction.

By induction on $n \in \mathbb{N}$, we define elements $\gamma_0^{(n)} \in \Gamma'$. We choose an arbitrary $\gamma_0^{(1)} \in \Gamma'$. Having already constructed $\gamma_0^{(n)}$, we carry through the following induction step. Since Γ has no greatest element, the same holds for Γ', and there is some $\alpha^{(n)} \in \Gamma'$ such that $\gamma_0^{(n)} < \alpha^{(n)}$. Hence, $\iota\gamma_0^{(n)} < \iota\alpha^{(n)}$. Let $\beta^{(n)} \in \Gamma$ be the least element of support$(\iota\gamma_0^{(n)}) \cup$ support$(\iota\alpha^{(n)})$ for which

$$(\iota\gamma_0^{(n)})_{\beta^{(n)}} < (\iota\alpha^{(n)})_{\beta^{(n)}}\,.$$

Since Γ has no greatest element and Γ' is a cofinal subset, we can choose $\gamma_0^{(n+1)} \in \Gamma'$ such that $\beta^{(n)} < \gamma_0^{(n+1)}$.

Let us observe the following fact. If $S \subset \Gamma$ is a well ordered set with least element $\gamma_0^{(n+1)}$, then

$$\iota\gamma_0^{(n)} < \iota\gamma_0^{(n)} \oplus S < \iota\alpha^{(n)}\,. \tag{14}$$

Indeed, $(\iota\gamma_0^{(n)} \oplus S)_\beta = (\iota\gamma_0^{(n)})_\beta$ for every $\beta < \gamma_0^{(n+1)}$. In particular, since $\beta^{(n)} < \gamma_0^{(n+1)}$,

$$(\iota\gamma_0^{(n)} \oplus S)_{\beta^{(n)}} = (\iota\gamma_0^{(n)})_{\beta^{(n)}} < (\iota\alpha^{(n)})_{\beta^{(n)}}\,,$$

which implies the second inequality of (14). The first inequality of (14) follows from (13).

The image of Γ' in $\mathbf{H}_{\gamma\in\Gamma}\Delta_\gamma$ being convex, (14) yields that also $\iota\gamma_0^{(n)} \oplus S$ lies in this image. Thus, $\iota^{-1}(\iota\gamma_0^{(n)} \oplus S)$ is a well defined element of Γ'.

Suppose now that for some ordinal number $\mu \geq 1$ we have chosen elements $\gamma_\nu^{(n)} \in \Gamma'$, $\nu < \mu$, $n \in \mathbb{N}$, such that for every fixed n, the sequence $(\gamma_\nu^{(n)})_{\nu<\mu}$ is strictly increasing. Then we set

$$\gamma_\mu^{(n)} := \iota^{-1}(\iota\gamma_0^{(n)} \oplus \{\gamma_\nu^{(n+1)} \mid \nu < \mu\}) \in \Gamma'$$

for every $n \in \mathbb{N}$. If $\lambda < \mu$, then $\{\gamma_\nu^{(n+1)} \mid \nu < \lambda\} \subsetneq \{\gamma_\nu^{(n+1)} \mid \nu < \mu\}$ and thus, $\gamma_\lambda^{(n)} < \gamma_\mu^{(n)}$ by (13). We see that for every ordinal number μ, the sequences $(\gamma_\nu^{(n)})_{\nu<\mu}$ can be extended. Thus, we obtain strictly increasing sequences of arbitrary length. This is a contradiction since their length is bounded by the cardinality of Γ'. This contradiction completes the proof of Theorem 2. □

Now we wish to apply Theorem 2 to logarithmic cross-sections of $(K,<)$. Suppose that w is a **coarsening** of v, that is, its valuation ring R_w contains the valuation ring R of v (we do not exclude equality) and its value group $\mathcal{G} = wK$ is the quotient of G by a convex subgroup. We set $\mathcal{U}_w^{>0} := \{a \in K \mid wa = 0 \wedge a > 0\}$. Then $(K^{>0},\cdot,1,<)/\mathcal{U}_w^{>0} \simeq \mathcal{G}$ as ordered groups. This isomorphism is induced by the map $-w$ which has convex kernel $\mathcal{U}_w^{>0}$. Note that every group complement of $\mathcal{U}_w^{>0}$ with its induced order is isomorphic to the ordered group $(K^{>0},\cdot,1,<)/\mathcal{U}_w^{>0}$ and thus to $\mathcal{G}$. If f is an exponential on K then we will say that f is **compatible with** w if $f(R_w) = \mathcal{U}_w^{>0}$ and $f(I_w) = 1 + I_w$. Such an exponential induces an exponential $fw : (Kw,+,0,<) = (R_w/I_w,+,0,<) \to (\mathcal{U}_w^{>0}/1+I_w,\cdot,1,<) = (Kw^{>0},\cdot,1,<)$ on the residue field Kw. It also induces an (order preserving) isomorphism of any given group complement of R_w onto a group complement of $\mathcal{U}_w^{>0}$ and thus also an isomorphism onto $\mathcal{G}$.

Now assume that $K = k((\mathcal{G}))$ is a power series field (k not necessarily archimedean) and that w is its canonical valuation. Then w is henselian (cf. [RI] or [KF4]). Consequently, w is compatible with every order on K (cf. [KN–WR]). It follows that w is a coarsening of v. We will say that f is **a compatible exponential of** $k((\mathcal{G}))$ if f is an exponential which is compatible with w.

In the power series field $K = k((\mathcal{G}))$, one of the complements for the valuation ring $R_w = k[[\mathcal{G}]]$ is the power series ring $k((\mathcal{G}^{<0}))$. As an ordered group, it is isomorphic to the Hahn product $\mathbf{H}_{\mathcal{G}^{<0}}(k,+,0,<)$. Consequently, a compatible exponential of K would induce an isomorphism between $\mathbf{H}_{\mathcal{G}^{<0}}(k,+,0,<)$ and $\mathcal{G}$. This in turn would give rise to an embedding of $\mathcal{G}^{<0}$ in $\mathbf{H}_{\mathcal{G}^{<0}}(k,+,0,<)$ with convex image. By virtue of Theorem 2, this is impossible. So we have proved:

Theorem 12 *Let k be an ordered field and $\mathcal{G} \neq 0$ an ordered abelian group. Let $<$ be any order on the power series field $K = k((\mathcal{G}))$. Then $(K,<)$ does not admit any compatible exponential.*

If k is archimedean ordered, then w coincides with the natural valuation v. In this situation, every exponential is compatible already by its definition, cf. Section 2. There we have also mentioned that every isomorphism $(K,+,0,<) \simeq (K^{>0},\cdot,1,<)$ gives rise to an exponential. Consequently, Theorem 1 follows from Theorem 12. In other words,

Corollary 13 *If k is an archimedean ordered field and G a nontrivial ordered abelian group, then $k((G))$ admits no surjective logarithmic cross-section.*

A little additional argument may show that the attempt to construct an exponential power series field fails "by far". For our attempt to construct an exponential power series field $k((G))$ over an archimedean field k, we would certainly choose a group G which is a Hahn product of the form $\mathbf{H}_\Gamma(k,+,0,<)$ with Γ having no greatest element (otherwise, we could not expect it to be isomorphic to

$\mathbf{H}_{G^{<0}}(k,+,0,<)$). In this case, every logarithmic cross-section h yields an embedding of the rank Γ of $G = \mathbf{H}_\Gamma(k,+,0,<)$ in the rank $G^{<0}$ of $\mathbf{H}_{G^{<0}}(k,+,0,<)$. By Theorem 2, no final segment Γ' of Γ can have a convex image under this embedding. This means: *every interval* $(\alpha, 0)$ *in* G *contains an element* β *such that* $va \neq \beta$ *for every* $a \in h(G)$. In other words: *there are infinite elements arbitrarily near to* 0 *whose archimedean classes are disjoint from* $h(G)$. In this sense, every logarithm on $k((G))$ is highly non-surjective.

The foregoing argument in fact shows that the group G does not admit a surjective group cross-section. That is, it is not an exponential group in the sense of [KS1] and can thus not be the natural value group of an exponential field. Further consequences of Theorem 2 for exponential groups will be stated in [KS2].

Remark 14 I. Kaplansky [KA] has shown that a valued field is maximal (i.e., admits no proper immediate extensions) if and only if every pseudo Cauchy sequence admits a limit. The same principle was proved by I. Fleischer [F] for valued abelian groups. In [KF4], it is proved for certain classes of valued modules. At the first glimpse one might believe that this principle holds for all (reasonable) valued structures. But the nonarchimedean exponential fields with their natural valuation constitute a counterexample to this principle. This is seen as follows.

There are maximal naturally valued exponential fields (i.e., they do not admit proper immediate extensions to which also the exponential extends). In fact, it can be shown that these are precisely the exponential fields whose natural valuation v is complete: On the one hand, it was remarked in [KS1] that if $(L,v) \supset (K,v)$ is immediate and the exponential extends from K to L, then (K,v) is dense in (L,v). On the other hand, if (K,v) is dense in (L,v), then an exponential of K extends to L by continuity. Hence, the completion of a nonarchimedean exponential field with respect to its natural valuation is the maximal immediate extension as a naturally valued exponential field. But by our nonexistence result, it cannot be a power series field. On the other hand, Kaplansky has also shown in [KA] that a valued field (K,w) of residue characteristic 0 is a power series field with canonical valuation w if and only if every pseudo Cauchy sequence admits a limit. (Note that the natural valuation has residue characteristic 0 since the residue field is ordered.) Hence, every maximal naturally valued exponential field admits pseudo Cauchy sequences without a limit, or in other words, it is not maximal as a valued field.

Another counterexample is given by contraction groups with their natural valuation. (Contraction groups are the natural value groups of exponential fields with a peculiar map induced by the exponential – for their definition, see [K–K1]. They have been studied in detail in [KF1] and [KF2].) Here, the situation is even more rigid. Every such group is maximal as a naturally valued contraction group. But by an application of Theorem 2, it can be shown that naturally valued contraction groups cannot be Hahn products. By virtue of Hahn's Embedding Theorem and Fleischer's result it follows that they must admit pseudo Cauchy sequences without a limit. For details, see [KS2] or [KF3].

For the conclusion of this section, we wish to generalize Theorem 12 a bit further. First, we observe that under the hypothesis of Theorem 12 one can prove that there is not even an exponential which is compatible with some nontrivial coarsening of w. To see this, we use Kaplansky's results mentioned in the foregoing remark. Since K is assumed to be a power series field with canonical valuation w, it follows that (K,w) is maximal (as a valued field). If w' is a coarsening of w, then also (K,w') is maximal (cf. [RI], or [KF4]). Since w has residue characteristic 0, the same holds for w'. Hence K can also be written as a power series with canonical valuation w', and from Theorem 12 it follows that no exponential can be compatible with w', provided that w' is nontrivial.

We have seen that we can actually talk about maximal valuations instead of power series fields. This leads to the following reformulation of Theorem 12: *If the ordered field K admits an exponential f, then there is no nontrivial coarsening of its natural valuation v which is maximal and with which f is compatible.*

We prove the following generalization:

Theorem 15 *Let f be an exponential on the ordered field K and w a coarsening of the natural valuation v of K such that f is compatible with w. Then there is no coarsening $\tilde{w}$ of w such that the valuation $\overline{w} = w/\tilde{w}$ induced by w on the residue field $K\tilde{w}$ is nontrivial and $(K\tilde{w},\overline{w})$ is maximal.*

Proof: Suppose to the contrary that there is a coarsening $\tilde{w}$ of w such that $\overline{w} = w/\tilde{w}$ is nontrivial and $(K\tilde{w},\overline{w})$ is maximal. Since $\tilde{w}$ is a coarsening of w, we have that $R_w \subset R_{\tilde{w}}$. Let $\overline{\mathbf{A}}$ be a group complement of R_w in $R_{\tilde{w}}$ and $\tilde{\mathbf{A}}$ a group complement of $R_{\tilde{w}}$ in $(K,+,0,<)$. Then the lexicographic product $\tilde{\mathbf{A}} \amalg \overline{\mathbf{A}}$ is an group complement to R_w in $(K,+,0,<)$. Further, f induces an isomorphism h from $\mathcal{G} = wK$ onto $\tilde{\mathbf{A}} \amalg \overline{\mathbf{A}}$ as ordered groups.

By general valuation theory, the value group of $\overline{w}$ is isomorphic to a nontrivial convex subgroup $\overline{\mathcal{G}}$ of $\mathcal{G}$. Since $(K\tilde{w},\overline{w})$ is maximal, it is isomorphic to the power series field $Kw((\overline{\mathcal{G}}))$ since $Kw = (K\tilde{w})w/\tilde{w}$ is the residue field of $(K\tilde{w},\overline{w})$. Consequently, $\overline{\mathbf{A}}$ is isomorphic to a Hahn product $\mathbf{H}_{\overline{\mathcal{G}}^{<0}}(Kw,+,0,<)$. Hence, we obtain an embedding of the nontrivial convex subgroup $\mathcal{H} := \overline{\mathcal{G}} \cap h^{-1}(\overline{\mathbf{A}})$ of $\overline{\mathcal{G}}$ in $\mathbf{H}_{\overline{\mathcal{G}}^{<0}}(Kw,+,0,<)$. Under this embedding, the image of $\mathcal{H}^{<0}$ is convex in $\mathbf{H}_{\overline{\mathcal{G}}^{<0}}(Kw,+,0,<)$. But $\mathcal{H}^{<0}$ is a final segment of $\overline{\mathcal{G}}^{<0}$. We have obtained a contradiction to Theorem 2, which proves our theorem. □

4 Going to the limit

Using the above Theorem 10, we shall now construct nonarchimedean models of real exponentiation which are countable unions of power series fields. Indeed, a common method to obtain surjectivity of a map is to construct the union over a suitable countably infinite chain of fields. In the following, we will apply such a construction to strong logarithmic cross-sections.

- **Construction of the left exponential.**

To get started, let G be as in Theorem 10. Set $G_0 :=$ and $K_0 = \mathbb{R}((G_0))$. Let $\mathbf{A}_0$ be a group complement of $\mathbb{R}[[G_0]]$ in K_0 and $h_0 : G_0 \to \mathbf{A}_0$ a strong logarithmic cross-section of K_0. Now assume that we have already constructed G_{n-1}, K_{n-1}, $\mathbf{A}_{n-1}$ and the strong logarithmic cross-section

$$h_{n-1} : G_{n-1} \to \mathbf{A}_{n-1}$$

i.e. satisfying that

$$vh_{n-1}(g) > g \quad \text{for all } g \in G_{n-1}^{<0}\,. \tag{15}$$

Since G_{n-1} is isomorphic to a subgroup of $\mathbf{A}_{n-1}$ through h_{n-1}, we can take G_n to be a group containing G_{n-1} as a subgroup and admitting an isomorphism h_n onto $\mathbf{A}_{n-1}$ which extends h_{n-1}. We set $K_n := \mathbb{R}((G_n))$. Hence, $K_{n-1} \subset K_n$ canonically (the elements of K_{n-1} being those elements of K_n whose support is a subset of G_{n-1}). Further, we choose a group complement $\mathbf{A}_n$ for the valuation ring $\mathbb{R}[[G_n]]$ such that $\mathbf{A}_n$ contains $\mathbf{A}_{n-1}$. In this way, h_n appears as an embedding of G_n into $\mathbf{A}_n$ which extends h_{n-1}. We show that h_n is again a strong logarithmic cross-section. For $g \in G_n$, the image $h_n(g)$ lies in $\mathbf{A}_{n-1}$, and $vh_n(g)$ lies in its value set $G_{n-1}^{<0}$. Consequently, in (15) we may replace $g \in G_{n-1}^{<0}$ by $vh_n(g)$ for $g \in G_n^{<0}$. But $vh_{n-1}(vh_n(g)) > vh_n(g)$ implies $h_{n-1}(vh_n(g)) > h_n(g)$, because $h_n(g) < 0$ and $h_{n-1}(vh_n(g)) < 0$. Since h_n extends h_{n-1}, this may be read as $h_n(vh_n(g)) > h_n(g)$. Since h_n is order preserving, this in turn implies $vh_n(g) > g$. Thus, we have proved that (15) holds with n in the place of $n-1$.

By our induction on n, we obtain a chain of fields K_n, $n \in \mathbb{N}$. Now we take $K_\omega := \bigcup_{n\in\mathbb{N}} K_n$ and $h_\omega := \bigcup_{n\in\mathbb{N}} h_n$. Also the groups G_n form a chain, and their union $G_\omega := \bigcup_{n\in\mathbb{N}} G_n$ is the value group of K_ω. Similarly, the group complements $\mathbf{A}_n$ form a chain, and their union $\mathbf{A}_\omega := \bigcup_{n\in\mathbb{N}} \mathbf{A}_n$ is a group complement for the valuation ring $\mathbb{R}[[G_\omega]]$ in K_ω. By construction, we have $\mathbf{A}_{n-1} = h_n(G_n)$ for all n. Consequently, $h_\omega : G_\omega \to \mathbf{A}_\omega$ is surjective. Moreover, h_ω satisfies property (11). It follows that the surjective map $f_{L,\omega} := (h_\omega \circ -v)^{-1}$ is a left exponential satisfying condition (10).

- **Construction of the right exponential.**

Let $n \in \mathbb{N}$ and a be an element of the valuation ring $\mathbb{R}[[G_n]]$ of K_n. Then we can write $a = r + \varepsilon$ with $r \in \mathbb{R}$ and $v\varepsilon > 0$. We set

$$f_{R,n}(a) := \exp(r) \cdot \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!}\,;$$

note that the second factor is again an element of $\mathbb{R}[[G_n]]$. This definition yields that $(K_n\,,\, f_{R,n})$ is a model of restricted real exponentiation; this follows from Corollary (2.11) of [DR–MI–MK1]. Further, $f_{R,n+1}$ is immediately seen to be an extension of $f_{R,n}$, and since also $(K_{n+1}\,,\, f_{R,n+1})$ is a model of restricted real exponentiation, it follows from Wilkie's theorem on the model completeness of the restricted exponential function (cf. [W]) that

$$(K_n\,,\, f_{R,n}|_{[0,1]}) \subset (K_{n+1}\,,\, f_{R,n+1}|_{[0,1]})$$

is an elementary extension. Setting $f_{R,\omega} := \bigcup_{n\in\mathbb{N}} f_{R,n}$, we obtain that $(K_\omega\,,\, f_{R,\omega})$ is the union over an elementary chain and thus is itself a model of restricted real exponentiation. Moreover, our definition of the $f_{R,n}$ yields that $f_{R,\omega}$ coincides with exp on the subfield $\mathbb{R}$ of K_ω.

Now we let f_ω be the exponential on K_ω which is induced by $f_{L,\omega}$ and $f_{R,\omega}$. Then f_ω satisfies (7), and in view of $f_{R,\omega}|_{[0,1]} = f_\omega|_{[0,1]}$, we see that $(K_\omega\,,\, f_\omega)$ is a model of restricted real exponentiation. From Theorem 4 we conclude that $(K_\omega\,,\, f_\omega)$ is a model of real exponentiation. This completes our construction.

Note that for every $n \in \mathbb{N}$, the two homomorphisms $h_n \circ -v$ and $f_{R,n}^{-1}$ induce a non-surjective logarithm ℓ_n of the power series field K_n. The surjective logarithm $\ell_\omega := f_\omega^{-1}$ is equal to the union $\bigcup_{n\in\mathbb{N}} \ell_n$.

Remark 16

1) The smallest groups G satisfying the conditions of Theorem 10 have rank $v_G G = \mathbb{Z}$. If we start with such a group, then the constructed exponential field will have the following property. Let a_1 be any positive infinite element. Define by induction $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ is cofinal in K. This property is equivalent to the condition that there is no nontrivial coarsening of v with which f is compatible. We say that v **has exponential rank 1 (with respect to f)**. For details, see [K–K2].

2) The above construction can be iterated in order to obtain unions over chains indexed by an arbitrary limit ordinal κ. If $\lambda \leq \kappa$ is a limit ordinal and we have constructed G_ν, K_ν, $\mathbf{A}_\nu$ and h_ν for every $\nu < \lambda$, then we take for G_λ, K_λ and h_λ the respective unions in the same manner as before. If $\lambda < \kappa$, then we replace K_λ by $\mathbb{R}((G_\lambda))$, which by virtue of Theorem 1 is a proper extension of $\bigcup_{\nu<\lambda} K_\nu$. We choose a group complement $\mathbf{A}_\lambda$ to its valuation ring $\mathbb{R}[[G_\lambda]]$ which contains $\bigcup_{\nu<\lambda} \mathbf{A}_\nu$. Thus, h_λ is a non-surjective logarithmic cross-section of K_λ with image in $\mathbf{A}_\lambda$. The induction step for successor ordinals works as before.

If κ is a regular cardinal, then the exponential field (K_κ, f_κ) obtained by this construction is almost a power series field. In fact, it is the restricted power series field $\mathbb{R}((G_\kappa))_\kappa$. Indeed, since κ is assumed to be regular and $G_\kappa = \bigcup_{\nu<\kappa} G_\nu$, every power series with support of cardinality $< \kappa$ is already an element of $\mathbb{R}((G_\nu)) = K_\nu$ for some $\nu < \kappa$. Hence, it lies in $K_\kappa = \bigcup_{\nu<\kappa} K_\nu$.

3) In our construction, we have worked with non-surjective logarithms on power series fields $\mathbb{R}((G))$. In [K–K–S] we will show that for every archimedean ordered field k there are (arbitrarily large) strong exponential groups G such that $k((G))$ admits a non-surjective exponential. By a union over an infinite ascending chain of power series fields similar to our above construction, one can obtain surjectivity of the exponential.

Now let (K, f) be a model of $T_{\text{an}}(\exp)$, and set $G = vK$. By [DR–MI–MK1], $\mathbb{R}((G))$ is a model of the theory T_{an} of the reals with restricted analytic functions. Moreover, there is an embedding of K in $\mathbb{R}((G))$ which respects the restricted analytic functions. Now the left logarithm of K induces canonically a strong logarithmic cross-section h_0 on $K_0 = \mathbb{R}((G))$. We continue the construction as above. The so obtained exponential f_ω on K_ω extends f. By [DR–MI–MK1], the embedding of (K, f) in (K_ω, f_ω) is elementary. This proves Theorem 3. (In the same way, one can use the fields (K_κ, f_κ) described in the foregoing remark.)

References

[A] Alling, N. L.: On exponentially closed fields, *Proc. A.M.S.* **13** No. **5** (1962), 706–711

[D–G] Dahn, B. I. – Göring, P.: Notes on exponential logarithmic terms, *Fund. Math.* **127** (1986), 45–50

[D-WO] Dahn, B. I. – Wolter, H.: On the theory of exponential fields, *Zeitschr. f. math. Logik und Grundlagen d. Math.* **29** (1983), 465–480

[DR-MI-MK1] van den Dries, L. – Macintyre, A. – Marker, D.: The elementary theory of restricted analytic functions with exponentiation, *Annals of Math.* **140** (1994), 183–205

[DR-MI-MK2] van den Dries, L. – Macintyre, A. – Marker, D.: Logarithmic–exponential power series, *to appear*

[F] Fleischer, I.: Maximality and ultracompleteness in normed modules, *Proc. Amer. Math. Soc.* **9** (1958), 151–157

[KA] Kaplansky, I.: Maximal fields with valuations I, *Duke Math. Journ.* **9** (1942), 303–321

[KN-WR] Knebusch, M. – Wright, M.: Bewertungen mit reeller Henselisierung, *J. reine angew. Math.* **286/287** (1976), 314–321

[K-K1] Kuhlmann, F.-V. – Kuhlmann, S.: On the structure of nonarchimedean exponential fields II, *Comm. in Algebra* **22(12)** (1994), 5079–5103

[K-K2] Kuhlmann, F.-V. – Kuhlmann, S.: The exponential rank of nonarchimedean exponential fields, *in preparation*

[K-K-S] Kuhlmann, F.-V. – Kuhlmann, S. – Shelah, S.: Functional equations for lexicographic products, *in preparation*

[KF1] Kuhlmann, F.-V.: Abelian groups with contractions I, in: eds. R. Göbel, P. Hill and W. Liebert, *Abelian Group Theory and Related Topics, Proceedings of the Oberwolfach Conference on Abelian Groups 1993, AMS Contemporary Mathematics* **171** (1994)

[KF2] Kuhlmann, F.-V.: Abelian groups with contractions II: weak o-minimality, to appear in: *Proceedings of the Padua Conference on Abelian Groups 1994*

[KF3] Kuhlmann, F.-V.: Some remarks about contractions on abelian groups, *in preparation*

[KF4] Kuhlmann, F.-V.: Valuation theory of fields, abelian groups and modules, *to appear in the* "Algebra, Logic and Applications" *series*, eds. A. MacIntyre and R. Göbel, Gordon & Breach, New York

[KS1] Kuhlmann, S.: On the structure of nonarchimedean exponential fields I, *Archive for Math. Logic* **34** (1995), 145–182

[KS2] Kuhlmann, S.: *Habilitation Thesis, in preparation*

[N] Neumann, B. H.: On ordered division rings, *Trans. A.M.S.* **66** (1949), 202–252

[RE] Ressayre, J.-P.: Integer parts of real closed exponential fields, in: eds. P. Clote and J. Krajicek, *Arithmetic, Proof Theory and Computational Complexity*, Oxford University Press, New York (1993)

[RI] Ribenboim, P.: Théorie des valuations, *Les Presses de l'Université de Montréal*, Montréal (1964)

[W] Wilkie, A. J.: Model completeness results for expansions of the real field, *to appear in Amer. J. Math.*