Universität Konstanz

PDE-Constrained Multiobjective Optimization

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Multiobjective Optimization [e.g., Ehrgott'05]

Competing goals in many applications: e.g.

- production (quality $\longleftrightarrow$ production costs)
- transport (travel costs $\longleftrightarrow$ comfort $\longleftrightarrow$ energy consumption)

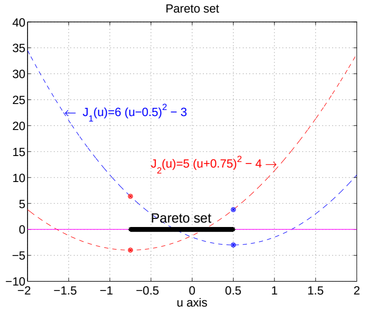
Multiobjective optimization: $\hat{J} \in C^1(\mathcal{U}_{\mathsf{ad}}, \mathbb{R}^k)$, $\mathcal{U}$ Hilbert space

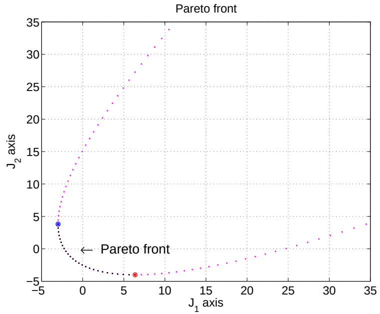
$$\min \hat{J}(u) = \big(\hat{J}_1(u), \ldots, \hat{J}_k(u)\big) \quad \text{subject to} \quad u \in \mathcal{U}_{\mathsf{ad}} \subset \mathcal{U} \qquad (\hat{\mathbf{P}})$$

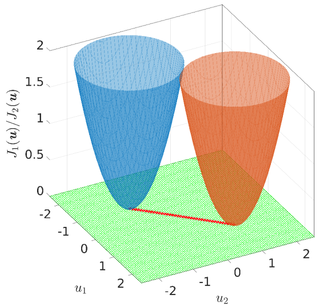
Pareto optimal points: $\bar{u} \in \mathcal{U}_{\mathsf{ad}}$ Pareto optimal, if there is no $u \in \mathcal{U}_{\mathsf{ad}}$ such that

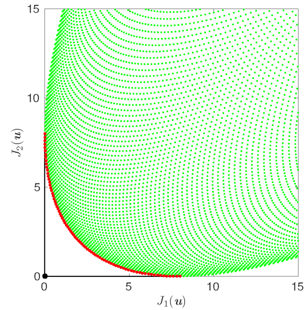
$$\hat{J}_j(u) \le \hat{J}_j(\bar{u}) \text{ for } 1 \le j \le k \quad \text{and} \quad \hat{J}_j(u) < \hat{J}_j(\bar{u}) \text{ for at least one } j \in \{1, \ldots, k\}$$

Sets: Pareto set $\mathcal{P}_\mathrm{s} = \big\{\bar{u} \in \mathcal{U}_{\mathsf{ad}} \,\big|\, \bar{u} \text{ Pareto optimal}\big\}$, Pareto front $\mathcal{P}_\mathrm{f} = \hat{J}(\mathcal{P}_\mathrm{s}) \subset \mathbb{R}^k$









Optimality Conditions and Optimization Methods

Karush-Kuhn-Tucker [Kuhn/Tucker'51]: $\bar{u} \in \mathcal{U}_{\text{ad}}$ Pareto optimal, then there is $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_k) \in \mathbb{R}^k$

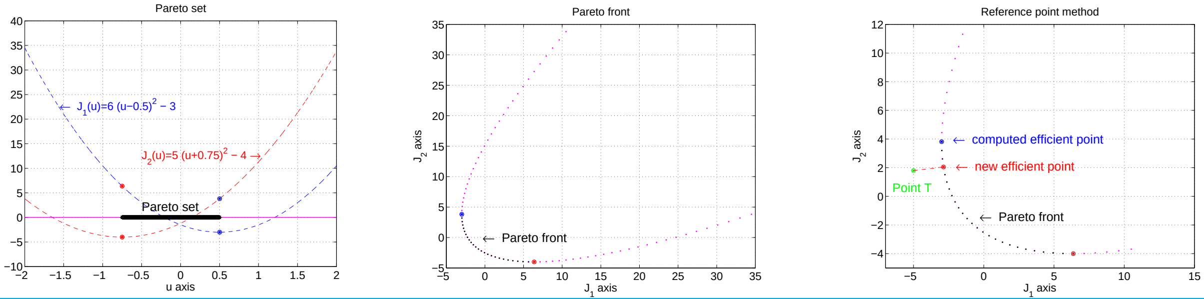
$$0 \leq \bar{\mu}_i \leq 1, \quad \sum_{i=1}^k \bar{\mu}_i = 1, \quad \sum_{i=1}^k \bar{\mu}_i \left\langle \hat{J}_i'(\bar{u}), u - \bar{u} \right\rangle_{\mathcal{U}} = \left\langle \sum_{i=1}^k \bar{\mu}_i \hat{J}_i'(\bar{u}), u - \bar{u} \right\rangle_{\mathcal{U}} \geq 0 \text{ for all } u \in \mathcal{U}_{\text{ad}}$$

$\to$ first-order (necessary) optimality conditions for $\min\limits_{u \in \mathcal{U}_{\text{ad}}} \hat{G}(u; \bar{\mu}) = \sum_{i=1}^k \bar{\mu}_i \hat{J}_i(u)$

Weighted sum method: solve $\bar{u}_\mu = \operatorname*{arg\,min}\limits_{u \in \mathcal{U}_{\text{ad}}} \hat{G}(u; \mu)$ with $0 \leq \mu_i \leq 1$ and $\sum_{i=1}^k \mu_i = 1$

Euclidean reference point method [Wierzbicki'79]: $\min\limits_{u \in \mathcal{U}_{\text{ad}}} \hat{F}_T(u) = \frac{1}{2} \| T - \hat{J}(u) \|_2^2$ for reference point $T = (T_1, \ldots, T_k)^\top$

Sets: Pareto set $\mathcal{P}_{\text{s}} = \left\{ \bar{u} \in \mathcal{U}_{\text{ad}} \,\middle|\, \bar{u} \text{ Pareto optimal} \right\}$, Pareto front $\mathcal{P}_{\text{f}} = \hat{J}(\mathcal{P}_{\text{s}}) \subset \mathbb{R}^k$



Motivating Example: Heat Equation with Convection

Bicriterial optimal control problem: minimize

$$J(y,u) = \frac{1}{2}\begin{pmatrix} \|y(T,\cdot)-y_\Omega\|^2_{L^2(\Omega)} \\ \sum_{i=1}^{m} \|u_i(t)\|^2_{L^2(0,T)} \end{pmatrix}$$

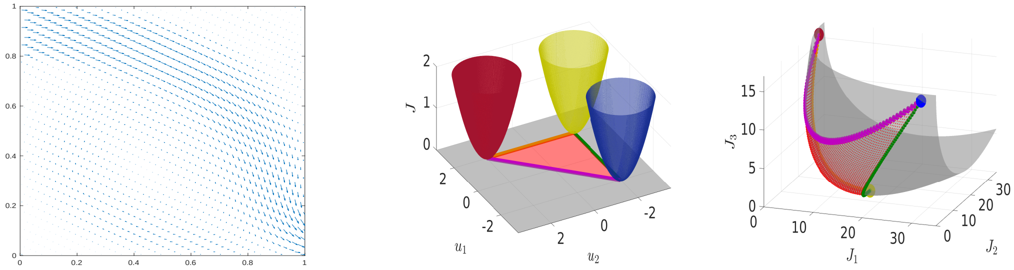
subject to the parabolic convection-diffusion PDE

$$\begin{aligned} y_t(t,\boldsymbol{x}) - \Delta y(t,\boldsymbol{x}) + b(\boldsymbol{x})\cdot\nabla y(t,\boldsymbol{x}) &= \sum_{i=1}^{m} u_i(t)\chi_{\Omega_i}(\boldsymbol{x}), && (t,\boldsymbol{x}) \in Q = (0,T)\times\Omega,\ \Omega = \Omega_1 \dot\cup \ldots \dot\cup \Omega_m \\ \frac{\partial y}{\partial \boldsymbol{n}}(t,\boldsymbol{x}) + \alpha_i y(t,\boldsymbol{x}) &= \alpha_i y_a(t), && (t,\boldsymbol{x}) \in \Sigma_i = (0,T)\times\Gamma_i,\ 1 \le i \le r \\ y(0,\boldsymbol{x}) &= y_\circ(\boldsymbol{x}), && \boldsymbol{x} \in \Omega \subset \mathbb{R}^d,\ d \in \{1,2,3\} \end{aligned} \qquad \textbf{(SE)}$$

Reduced cost: $\hat{J}(u) = J(\mathscr{S}(u),u)$, where $y = \mathscr{S}(u)$ solves (**SE**).

Diffusion dominated (Movie 1):
temperature $y(t_\mathsf{f},\cdot)$ (left) and Pareto front (right)

Convection dominated (Movie 2):
temperature $y(t_\mathsf{f},\cdot)$ (left) and Pareto front (right)



(a) Advection Field (b) Three Costs (c) Pareto Front

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