Tutorium 1

16.November 2022

1.1 Optimal control of the Laplace equation

We expand the concept of finite-dimensional optimal control theory from the lecture to infinite dimensional optimal control problems using the Laplace equation.

Let us consider the minimization problem

$$\min J(y,u) \quad \text{s.t.} \begin{cases} -\Delta y = \langle u, \xi\rangle_{\mathbb{R}^{n_u}} & \text{on } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad u \in U_{ad} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^2$ a close and bounded Lipschitz domain, $J : Y \times \mathbb{R}^{n_u} \to \mathbb{R}$ with

$$J(y,u) := \frac{1}{2}\int\limits_{\Omega} |y - y_d|^2 \, dx + \frac{1}{2}\|u - u_d\|^2_{\mathbb{R}^{n_u}}$$

denotes the quadratic cost functional, $Y := H^1_0(\Omega)$ the state space with the following scalar product

$$\langle y, \varphi\rangle_{H^1_0(\Omega)} := \int\limits_{\Omega} \nabla y \nabla \varphi \, dx,$$

$U_{ad} \subset \mathbb{R}^{n_u}$ is the set of addmissible control parameters, $y_d \in Y$ the desired state, $u_d \in \mathbb{R}^{n_u}$ the desired control and $\xi \in L^2(\Omega, \mathbb{R}^{n_u})$.

We look for a control vector $u \in \mathbb{R}^{n_u}$ and a state $y \in Y$ which solve (1.1).

Remark 1.1.1. For $u \in U_{ad}$ the function $y \in Y = H^1_0(\Omega)$ is called a weak solution of

$$\begin{cases} -\Delta y = \langle u, \xi\rangle_{\mathbb{R}^{n_u}} & \text{on } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

if the weak formulation

$$a(y,\varphi) := \int\limits_{\Omega} \nabla y \nabla \varphi \, dx = \int\limits_{\Omega} \langle u, \xi\rangle_{\mathbb{R}^{n_u}} \varphi \, dx =: F(u,\varphi) \tag{1.3}$$

holds for all $\varphi \in Y$ with $a : Y \times Y \to \mathbb{R}$ and $F : U_{ad} \times Y \to \mathbb{R}$. $\lozenge$

Theorem 1.1.2 (Lax and Milgram)**.** *Let Y be a Hilbert space. Suppose that for the bilinear form $a : Y \times Y \to \mathbb{R}$ there exist $c, \alpha > 0$ with*

1. $|a(y,\varphi)| \leq \alpha \|y\|_Y \|\varphi\|_Y$ *(boundedness)*
2. $c\|y\|^2_Y \leq a(y,y)$ *(V-ellipticity).*

Then for every $F \in Y^*$ there exists a unique $y \in Y$ which solves

$$a(y,\varphi) = F(\varphi) \quad \textit{for all } \varphi \in Y$$

and there exists $c_a > 0$ with

$$\|y\|_Y \leq c_a \|F\|_{Y^*}.$$

The proof can be found in [1].

Theorem 1.1.3. *If Ω is a close and bounded Lipschitz domain, then for every $u \in U_{ad}$ and $\xi \in L^2(\Omega, \mathbb{R}^{n_u})$ there exists an unique weak solution $y \in Y$ of* (1.2) *with*

$$\|y\|_{H^1(\Omega)} \leq c\|u\|_2\|\xi\|_{L^2(\Omega,\mathbb{R}^{n_u})}. \tag{1.4}$$

Proof. **Exercise**. (Use Lax and Milgram and Poincaré inequality.) □

Remark 1.1.4. Induced by Theorem 1.1.3 we define the solution operator $\mathcal{S} : U_{ad} \to Y$ with $u \mapsto y(u)$. With this, we can define the reduced problem

$$\min \hat{J}(u) := J(\mathcal{S}(u), u) \text{ s.t. } u \in U_{ad}. \tag{1.5}$$

◊

Theorem 1.1.5. *Let $\Omega \subset \mathbb{R}^2$ be a closed and bounded Lipschitz domain, $Y := H_0^1(\Omega)$, $y_d \in Y$, $U_{ad} \subset \mathbb{R}^{n_u}$ a closed, nonempty, bounded and convex set, $\xi \in L^2(\Omega, \mathbb{R}^{n_u})$ and assume that the solution operator $\mathcal{S} : U_{ad} \to Y$ is linear and continuous. Then* (1.5) *has an optimal control $\bar{u}$ with an optimal state $\bar{y}$. If $\bar{u}$ is strictly convex, the optimal control $\bar{u}$ is unique.*

Proof. **Exercise**. □

Lemma 1.1.6. Assume that the conditions from Theorem 1.1.5 hold. Then the reduced cost functional $\hat{J}$ is continuously differentiable on U_{ad} and it holds

$$\nabla \hat{J}(u) = \left(\int\limits_\Omega \xi_1 p \, dx, ..., \int\limits_\Omega \xi_n p \, dx \right)^\top + (u - u_d) \tag{1.6}$$

and $p \in Y$ solves the adjoint equation

$$\begin{cases} -\Delta p = \mathcal{S}(u) - y_d & \text{on } \Omega \\ p = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

p is called the adjoint variable, associated with $(y, u$.

Proof. **Exercise**. Use the representation of (1.6) to show

$$\lim_{\|w\|_{\mathbb{R}^{n_u}} \to 0} \frac{\hat{J}(u + w) - \hat{J}(u) - \langle \nabla \hat{J}(u), w \rangle}{\|w\|_{\mathbb{R}^{n_u}}} = 0.$$

□

Theorem 1.1.7. *Assume that the conditions of Theorem* 1.1.5 *hold and assume that $\bar{u}$ solves the reduced problem* (1.5)*. Then the variational inequality*

$$\langle \nabla \hat{J}(\bar{u}), u - \bar{u} \rangle_{\mathbb{R}^{n_u}} \geq 0 \textit{ for all } u \in U_{ad} \tag{1.8}$$

holds.
If $\tilde{u} \in U_{ad}$ is a solution of the variational inequality (1.8) *and $\hat{J}$ is convex, then $\tilde{u}$ solves* (1.5).

Proof. **Exercise**. (Same arguments as in Optimization II). □

Remark 1.1.8. Utilizing the adjoint variable for the representation of the gradient of the reduced cost functional $\hat{J}$ the variational inequality (1.8) is equivalent to

$$0 \leq \langle \nabla \hat{J}(\bar{u}), u - \bar{u} \rangle_{\mathbb{R}^{n_u}} = \int\limits_\Omega \langle u - \bar{u}, \xi \rangle_{\mathbb{R}^{n_u}} p \, dx + \langle \bar{u} - u_d, u - \bar{u} \rangle_{\mathbb{R}^{n_u}} \quad \text{for all } u \in U_{ad}.$$

Now we can formulate an optimality system for the unknown variables $\bar{u}, \bar{y}$ and $\bar{p}$:

$$\begin{aligned} a(\bar{y}, \varphi) &= F(\bar{u}, \varphi) && \text{for all } \varphi \in Y, \\ a(\bar{p}, \varphi) &= F_p(\bar{y}, \varphi) && \text{for all } \varphi \in Y, \\ 0 &\leq \int\limits_\Omega \langle u - \bar{u}, \xi \rangle_{\mathbb{R}^{n_u}} p \, dx + \langle \bar{u} - u_d, u - \bar{u} \rangle_{\mathbb{R}^{n_u}} && \text{for all } u \in U_{ad} \end{aligned} \tag{1.9}$$

with

$$F_p(\bar{y},\varphi) := \int_\Omega (\bar{y} - y_d)\varphi \, dx.$$

Every solution $(\bar{y},\bar{u})$ to (1.1) must satisfy, together with the adjoint variable $\bar{p}$, the necessary conditions (1.9). Due to the convexity in this setting, the conditions are sufficient. $\Diamond$

Bibliography

[1] F. Tröltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010.