Tutorium 4

18. Januar 2023

4.1 Linear-Quadratic Optimal Control

Tracking problem- problem formulation

The goal is to find a state-feedback control law of the form

$$u(t) = \Psi(y(t), t) \text{ for } t \in [0, T]$$

with $u : [0, T] \to \mathbb{R}^{m_u}$, $y : [0, T] \to \mathbb{R}^{m_y}$ so that u minimizes the quadratic cost functional

$$J(y, u) = \int_0^T (y(t) - y^d(t))^\top Q(y(t) - y^d(t)) + u(t)^\top Ru(t)\, dt + (y(T) - y^d(T))^\top M(y(T) - y^d(T)) \tag{4.1}$$

subject to the linear initial value problem

$$\dot{y}(t) = Ay(t) + Bu(t) + f(t) \text{ for } t \in (0, T] \text{ and } y(0) = y_0. \tag{4.2}$$

The matrices Q, $M \in \mathbb{R}^{m_y \times m_y}$ are symmetric, positive semi-definite, $R \in \mathbb{R}^{m_u \times m_u}$ is symmetric, positive definite and in (4.2) we have $A \in \mathbb{R}^{m_y \times m_y}$, $B \in \mathbb{R}^{m_y \times m_u}$, $f \in C([0, T], \mathbb{R}^{m_y})$ and $y_0 \in \mathbb{R}^{m_y}$. The final time T is fixed. Thus, we aim to track the state to the state y^d as good as possible.

Exercise 1

Reformulate the minimization problem from 4.1 to obtain an optimization problem of the following form

$$J(x, u) = \int_0^T x(t)^\top Qx(t) + u(t)^\top Ru(t)\, dt + x(T)^\top Mx(T) \tag{4.3a}$$

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega(t) \text{ for } t \in (0, T] \text{ and } x(0) = x_0. \tag{4.3b}$$

Use $x(t) := y(t) - y^d(t)$.

Proof. □

Exercise 2

Use the same method as in Section 34.3 to find the optimal state-feedback law for (4.3). Instead of the quadratic ansatz in (34.11) use the ansatz

$$V^*(x_t, t) = x_t^\top P(t)x_t + 2b(t)^\top x_t + c(t)$$

for $x_t \in \mathbb{R}^{m_x}$, $t \in [0, T)$, $P(t) \in \mathbb{R}^{m_x \times m_x}$ symmetric, $b(t) \in \mathbb{R}^{m_x}$, $c(t) \in \mathbb{R}$ and $m_x := m_y$.
Assume that $A \in \mathbb{R}^{m_x \times m_x}$ is symmetric, to simplify the proof.

Proof. □

Exercise 3

Let us consider the problem

$$\min \int_0^T |x(t)|^2 + |u(t)|^2 \,\mathrm{d}t \quad \text{s.t.} \quad \dot{x}(t) = u(t) + t \text{ for } t \in (0, T].$$

Choosing $m_x = m_u = 1$, $A = M = 0$ and $B = Q = R = 1$.

1. Calculate the matrix Riccati equations.
2. Calculate the optimal state-feedback law.

Proof. □