Tutorium 3

14. Dezember 2022

3.1 Proper Orthogonal Decomposition

Exercise 1

1. Show, that the eigenvalues $\lambda_1^n, ..., \lambda_m^n$ of the operator $\mathcal{R}^n\psi = \sum\limits_{j=1}^{n} \alpha_j \langle y(t_j), \psi\rangle_2 y(t_j)$ fulfil

$$\sum_{j=1}^{n} \alpha_j \|y(t_j)\|_2^2 = \sum_{i=1}^{m} \lambda_i^n.$$

2. Show that the normed eigenvectors $\psi_1, ..., \psi_\ell \in \mathbb{R}^m$ of the matrix $YY^\top \in \mathbb{R}^{m\times m}$ with $Y \in \mathbb{R}^{m\times n}$ for the ℓ biggest eigenvalues $\lambda_1, ..., \lambda_\ell$ fulfil

$$\sum_{i=1}^{\ell}\sum_{j=1}^{n} \langle y(t_j), \psi_i\rangle_2^2 = \sum_{i=1}^{\ell} \lambda_i.$$

Compare Section 31.1 and 33.2.

Exercise 2

Consider a trajectory $y \in C([0,T];\mathbb{R}^m)$ and the minimization problem

$$\min_{\tilde{u}_1,...,\tilde{u}_\ell \in \mathbb{R}^m} \int_0^T \left\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \tilde{u}_i\rangle_2 \, \tilde{u}_i \right\|_2^2 \mathrm{d}t \quad \text{s.t.} \quad \langle \tilde{u}_i, \tilde{u}_j\rangle_2 = \delta_{ij},\ 1 \le i,j \le \ell. \tag{3.1}$$

1. Rewrite (3.1) in the standard form of

$$\max_{u\in\mathbb{R}^N} J(u) \text{ s.t. } e(u) = 0 \tag{3.2}$$

with a cost function $J : \mathbb{R}^N \to \mathbb{R}$ and a constraint function $e : \mathbb{R}^N \to \mathbb{R}^p$ with $N = m\ell$ and $p = \frac{1}{2}\ell(\ell+1)$. Compare section 33.1..

2. Compute the gradient $\nabla J(u) \in \mathbb{R}^N$ and the derivative $\nabla e(u) \in \mathbb{R}^{N\times p}$.

3. Prove by using the first-order necessary optimality condition of constrained optimization that every solution $\bar{u} \in \mathbb{R}^N$ of (3.2) is an eigenvector of the operator

$$\mathcal{R} : \mathbb{R}^N \to \mathbb{R}^N,\ \mathcal{R}u = \int\limits_0^T \langle y(t), u\rangle_2 y(t)\, dt$$

Exercise 3

For $i \in \mathbb{N}_0$, consider the data points in $\mathbb{R}^2$ given by $y^{(i)} = (i,1)^\top$. The task of approximating the first n points with a linear subspace $U := \langle u \rangle$ can be described by the typical POD problem

$$\min_{u \in \mathbb{R}^2} \sum_{i=0}^{n} \|y^{(i)} - \langle y^{(i)}, u \rangle_2 u\|_2^2 \text{ s.t. } \|u\|_2 = 1. \tag{3.3}$$

Show that for $n \in \mathbb{N}_0$ there is a solution $u^{(n)} \in \mathbb{R}^2$ of (3.3) satisfying $u^{(n)} \to (1,0)^\top$ as $n \to \infty$.

Hint: The eigenvector to the larger eigenvalue of a symmetric matrix

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

is given by

$$\psi = \begin{pmatrix} 1 & \frac{b-a+\sqrt{(a-b)^2+4c^2}}{2c} \end{pmatrix}^\top .$$

It holds $x + \sqrt{x^2+1} \to 0$ for $x \to -\infty$.

Exercise 4

1. Assume the matrix $A \in \mathbb{R}^{n \times n}$ has full range and assume that $A = U\Sigma V^\top$ is the singular value decomposition of A. Compute the singular value decomposition of A^{-1}.
2. Compute the singular value decomposition of

$$A = \begin{pmatrix} -2 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}.$$