Parameter-Elliptic Boundary Value Problems and their Formal Asymptotic Solutions

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Abstract. We consider boundary value problems for mixed-order systems of partial differential operators which depend on a complex parameter but which are not parameter-elliptic in the sense of Agmon and Agranovich–Vishik. Such systems are closely related to the theory of singularly perturbed problems. Under the condition of so-called weak parameter-ellipticity it is possible to construct the formal asymptotic solution which shows, in particular, the existence of boundary layers.

1 Introduction

Let $A(D) = \big(A_{ij}(D)\big)_{i,j=1,\dots,N}$ be a matrix of partial differential operators and suppose that this matrix is elliptic in the sense of Douglis and Nirenberg. In this case there exist $2N$ integers $s_1,\dots,s_N,t_1,\dots,t_N$ such that $\operatorname{ord} A_{ij} \le s_i + t_j$. We will assume in the following that s_i and t_i are nonnegative. Without loss of generality we can suppose that the sequence $r_i := s_i + t_i$ is nonincreasing (in the opposite case we change the indexing of lines and rows). Let $A^0_{ij}(D)$ denote the principal part of the operator $A_{ij}(D)$ in the sense of Douglis–Nirenberg (we have $A^0_{ij} = 0$ if $\operatorname{ord} A_{ij} < s_i + t_j$). Ellipticity then means that $\det A^0(\xi) \ne 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ where $A^0(\xi) := \big(A^0_{ij}(\xi)\big)_{i,j}$ stands for the principal symbol of $A(D)$.

The aim of the present paper is to investigate boundary value problems for the parameter-dependent operator matrix given by

$$A(D,\lambda) := \begin{pmatrix} A_{11}(D) & \cdots & A_{1N}(D) \\ \vdots & & \vdots \\ A_{N1}(D) & \cdots & A_{NN}(D)-\lambda \end{pmatrix} = A(D) - \lambda E_N \tag{1.1}$$

and supplemented with general mixed-order boundary conditions. Here E_N stands for the $N \times N$ matrix which differs from the zero matrix only in the element at position (N,N) which equals 1.

There are several reasons for studying the parameter-dependent matrix (1.1) (see [DV00], Section 1). We only want to point out one reason. In the case of constant order matrices, the theory of ellipticity with parameter as it was developed in the

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sixties by Agmon [A62], Agranovich–Vishik [AV64] and others can be applied to the matrix $A(D) - \lambda I_N$ (where I_N denotes the N-dimensional identity matrix). It was also remarked by Agranovich in [A90] that in the case of mixed order systems for which all numbers r_i are equal we can adjust a definite weight to the parameter λ. This again makes it possible to apply the theory of parameter-ellipticity (see also the book of Roitberg [R96] for parameter-elliptic Douglis–Nirenberg systems).

The case where some of the numbers r_j are different is much more complicated and was treated by Kozhevnikov [K96] and by the authors [DMV98]. In these papers one can find several (equivalent) definitions of ellipticity with parameter for the matrix $A(D) - \lambda I_N$ which lead to solvability results and to a priori estimates. Roughly speaking, $A(D) - \lambda I_N$ is elliptic with parameter if all submatrices of the form $A_{(k)}(D) - \lambda E_k$ are weakly parameter-elliptic in the sense defined below, where we have set $A_{(k)}(D) := \big(A_{ij}(D)\big)_{i,j=1,\dots,k}$. This definition (also called the condition of elliptic principal minors) is essentially due to Kozhevnikov; for other descriptions we refer to [DMV98].

In [DV00] boundary value problems for systems with structure very close to (1.1) were investigated. Here the concept of weak parameter-ellipticity for such operator matrices and corresponding boundary value problems was introduced and a priori estimates in certain parameter-dependent Sobolev spaces were obtained. In the present paper we want to show that the conditions appearing in the definition of weak parameter-ellipticity are very natural from the point of view of singular perturbation theory (here λ is replaced by ε^{-1} with a small positive parameter ε), see , e.g., [VL57], [N81], [I89]. In particular, these conditions allow us to construct the so-called formal asymptotic solutions. For simplicity, we will only consider operators with constant coefficients and without lower order terms acting in the whole space or in the half-space $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. The same definitions and results hold for operators with variable coefficients acting on a bounded domain or on a compact manifold with boundary.

2 Weakly parameter-elliptic boundary value problems

We start with the definition of weak parameter-ellipticity for the matrix (1.1). As above, we assume that r_i are nonincreasing and that $r_{N-1} > r_N$. We fix a closed sector $\mathcal{L} \subset \mathbb{C}$ with vertex at the origin.

Definition 2.1. Let $A(D, \lambda)$ be of the form (1.1). Then $A(D, \lambda)$ is called weakly parameter-elliptic in $\mathcal{L}$ if the inequality

$$\big|\det(A^0(\xi) - \lambda E_N)\big| \geq C|\xi|^{r_1+\dots+r_{N-1}}(|\xi| + |\lambda|^{1/r_N})^{r_N} \quad (\xi \in \mathbb{R}^n,\ \lambda \in \mathcal{L}) \tag{2.1}$$

holds, where here and in the following the letter C denotes an unspecified constant independent of ξ and λ.

Scalar polynomials in ξ and λ satisfying an estimate of the form (2.1) were treated in [DMV00a], [DMV00b]. It is not difficult to see that (2.1) holds if and only if $A(\xi)$ is elliptic in the sense of Douglis–Nirenberg, the same holds for $A_{(N-1)}(\xi)$, and $\det(A^0(\xi) - \lambda E_N)$ does not vanish for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and all $\lambda \in \mathcal{L} \setminus \{0\}$. From this we obtain in the case $n \geq 3$ that the numbers r_j are even, for $n = 2$ we will assume this in the following.

Now let us assume that $A(D, \lambda)$ acts on the half-space $\mathbb{R}^n_+$. The partial Fourier transform with respect to the first $n-1$ variables reduces this operator to the ordinary differential operator $A(\xi', D_n, \lambda)$ (with $\xi' = (\xi_1, \dots, \xi_{n-1})$ and $D_n = -i\partial/\partial x_n$).

Lemma 2.1. *Let $A(D, \lambda)$ be weakly parameter-elliptic in $\mathcal{L}$.*
a) *For $\xi' \neq 0$ and $\lambda \in \mathcal{L}$ the ordinary differential equation on the half-line*

$$A^0(\xi', D_n, \lambda)w(x_n) = 0 \quad (x_n > 0)$$

has exactly R_N solutions which tend to zero for $x_n \to \infty$. Here we have set $R_j := (r_1 + \cdots + r_j)/2$ for $j = 1, \dots, N$.

b) *For $\lambda \in \mathcal{L} \setminus \{0\}$ the ordinary differential equation in $\mathbb{R}_+$*

$$A^0(0, D_n, \lambda)w(x_n) = 0 \quad (x_n > 0) \tag{2.2}$$

has exactly $r_N/2$ solutions which tend to zero for $x_n \to +\infty$.

Proof. The dimension of the space of asymptotically stable solutions (i.e. solutions which tend to zero for $x_n \to \infty$) in the case a) and b), respectively, coincides with the number of zeros of $\det A^0(\xi', \cdot, \lambda)$ and $\det A^0(0, \cdot, \lambda)$, respectively, with positive imaginary part, counted with multiplicities. As the first determinant has $2R_N$ zeros in $\mathbb{C}$ and no real zeros, it follows by standard homotopy argument that it has half of its zeros in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For the second determinant we use

$$\det A^0(0, \tau, \lambda) = \det A^0(0, \tau) - \lambda \det A^0_{(N-1)}(0, \tau)\,.$$

As both determinants are homogeneous and elliptic polynomials, there are constants $a_1, a_2 \in \mathbb{C}$ such that $\det A^0(0, \tau) = a_1 \tau^{r_1 + \cdots + r_N}$ and $\det A^0_{(N-1)}(0, \tau) = a_2 \tau^{r_1 + \cdots + r_{N-1}}$. Therefore the zeros in $\mathbb{C}_+$ of $\det A^0(0, \tau, \lambda)$ are the zeros of the equation

$$a_1 \tau^{r_N} - a_2 = 0 \quad \text{ in } \mathbb{C}_+.$$

As the last equation has no real zeros due to the condition of weak parameter-ellitpicity and as r_N is even, it has exactly $r_N/2$ zeros in $\mathbb{C}_+$. $\square$

Let us assume that we have a matrix of boundary operators of the form

$$B(D) = \big(B_{jk}(D)\big)_{\substack{j=1,\dots,R_N \\ k=1,\dots,N}}$$

where for the boundary conditions the inequality $\operatorname{ord} B_{jk} \le m_j + t_k$ holds, where $m_1, \dots, m_{R_N}$ are integer numbers satisfying $m_1 \le \dots \le m_{R_N}$ and

$$m_{R_k} < m_{R_k+1} \quad (k = 1, \dots, N-1)\,.$$

The principal part B^0 of B is defined in the same way as for A. We also set

$$B_{(N-1)}(D) := \big(B_{jk}(D)\big)_{\substack{j=1,\dots,R_{N-1}\\ k=1,\dots,N-1}}.$$

The following definition is essentially taken from [DV00].

Definition 2.2. The boundary value problem $(A(D,\lambda), B(D))$ is called weakly parameter-elliptic in $\mathcal{L}$ if the following conditions are satisfied:

(i) $A(D,\lambda)$ is weakly parameter-elliptic in $\mathcal{L}$ in the sense of Definition 2.1.

(ii) For every $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, $\lambda \in \mathcal{L}$ and every $g = (g_1, \dots, g_{R_N}) \in \mathbb{C}^{R_N}$ the boundary value problem in $\mathbb{R}_+$

$$\begin{aligned} A^0(\xi', D_n, \lambda) w(x_n) &= 0 \quad (x_n > 0)\,, && (2.3)\\ B^0(\xi', D_n) w(x_n)\big|_{x_n=0} &= g\,, && (2.4)\\ w(x_n) \to 0 &\quad \text{for } x_n \to \infty \end{aligned}$$

has a unique solution.

(iii) For every $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ and every $h \in \mathbb{C}^{R_{N-1}}$ the problem

$$\begin{aligned} A^0_{(N-1)}(\xi', D_n) v(x_n) &= 0 \quad (x_n > 0)\,, && (2.5)\\ B^0_{(N-1)}(\xi', D_n) v(x_n)\big|_{x_n=0} &= h\,, && (2.6)\\ v(x_n) \to 0 &\quad \text{for } x_n \to \infty \end{aligned}$$

has a unique solution.

(iv) For every every vector $h \in \mathbb{C}^{r_N/2}$ and every $\lambda \in \mathcal{L}$ with $|\lambda| = 1$ the problem

$$\begin{aligned} A^0(0, D_n, \lambda) v(x_n) &= 0 \quad (x_n > 0)\,, && (2.7)\\ B^0_{(N,1..N)}(0, D_n) v(x_n)\big|_{x_n=0} &= h, && (2.8)\\ v(x_n) \to 0 &\quad \text{for } x_n \to \infty \end{aligned}$$

has a unique solution. Here we have set

$$B_{(N,1..N)}(D) := \big(B_{jk}(D)\big)_{\substack{j=R_{N-1}+1,\dots,R_N\\ k=1,\dots,N}}.$$

Note that conditions (i) and (ii) in the above definition are very natural and correspond to similar conditions in traditional elliptic theory. Conditions (iii) and (iv) are connected with the case $\xi' = 0$ where the analog of (ii) does not hold. In the next section the meaning of these two conditions will become clear in the context of singular perturbation theory.

3 Formal asymptotic solutions

Consider the boundary value problem

$$A(D,\lambda)u = 0, \tag{3.1}$$
$$B(D)u = g \tag{3.2}$$

in the half-space $\mathbb{R}^n_+$ where we assume throughout this section that (A, B) are weakly parameter-elliptic in the sense of Definition 2.2 with $\mathcal{L} = [0,\infty)$. Setting $\lambda = \varepsilon^{-r_N}$ and multiplying the last equation of the system (3.1) by ε^{r_N}, we obtain the system

$$A_\varepsilon(D)u(x', x_n) = 0 \quad (x_n > 0) \tag{3.3}$$

where $A_\varepsilon(D) := \operatorname{diag}(1,\dots,1,\varepsilon^{r_N})A(D) - E_N$. For simplicity, let us now assume that all operators coincide with their principal parts. We are interested in the case of $\varepsilon \to 0$; more precisely, our goal is to find the formal asymptotic solution (FAS)

$$\sum_{l=0}^{\infty} \varepsilon^l u^{(l)}(x,\varepsilon),$$

i.e. the formal power series in ε for which the partial sums satisfy (3.3),(3.2) up to an arbitrary power of ε. The construction of the FAS will show the boundary layer structure of the solution of (3.3),(3.2) and give a deeper insight to the conditions of Definition 2.2. It seems to us that this construction cannot be found in literature for the boundary value problem considered here.

Following the Lyusternik-Vishik method, we will construct the FAS as the sum of the so-called exterior expansion

$$u(x,\varepsilon) = \sum_{l=0}^{\infty} \varepsilon^l u^{(l)}(x) \tag{3.4}$$

and the so-called interior expansion or boundary layer

$$v(x', x_n/\varepsilon, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{l_0+l} \operatorname{diag}(\varepsilon^{t_1},\dots,\varepsilon^{t_N})v^{(l)}(x', x_n/\varepsilon.) \tag{3.5}$$

The number l_0 will be chosen later.

We will now show that, due to the conditions of weak parameter-ellipticity, it is possible to describe $u^{(l)}$ and $v^{(l)}$ as the solutions of boundary value problems which appear in the definition of weak parameter-ellipticity and for which the right-hand sides can be computed recursively.

(i) *Differential equations for the exterior expansion.* Substituting (3.4) into (3.3) and posing $u'^{(l)} = (u_1^{(l)}, \dots, u_{N-1}^{(l)})$ we obtain

$$\sum_{l=0}^{\infty} \varepsilon^l \left[A_{(N-1)}(D) u'^{(l)} + \begin{pmatrix} A_{1N}(D) \\ \vdots \\ A_{N-1,N}(D) \end{pmatrix} u_N^{(l)} \right] = 0,$$

$$\sum_{l=0}^{\infty} \varepsilon^l \Big(\varepsilon^{r_N} \sum_{j=1}^{N} A_{Nj}(D) u_j^{(l)} - u_N^{(l)} \Big) = 0.$$

Equate to zero all terms corresponding to the same power of ε we obtain the relations

$$u_N^{(l)} = -\sum A_{Nj}(D) u_j^{(l-r_N)}, \tag{3.6}$$

$$A_{(N-1)}(D) u'^{(l)} = - \begin{pmatrix} A_{1N}(D) \\ \vdots \\ A_{N-1,N}(D) \end{pmatrix} u_N^{(l)} = \mathcal{F}(u^{(0)}, \dots, u^{(l-r_N)}). \tag{3.7}$$

(ii) *Differential equations for the interior expansion.* Pose $t = x_n/\varepsilon$. Then

$$\begin{aligned} &A_\varepsilon(D) v(x', x_n/\varepsilon, \varepsilon) = A_\varepsilon(D', \frac{1}{\varepsilon} D_t) v(x', t, \varepsilon) \\ &= \big[\operatorname{diag}(\varepsilon^{-s_1}, \dots, \varepsilon^{-s_{N-1}}, \varepsilon^{t_N}) A(\varepsilon D', D_t) \operatorname{diag}(\varepsilon^{-t_1}, \dots, \varepsilon^{-t_N}) - E_N \big] v(x', t, \varepsilon). \end{aligned}$$

Multiplying this by $\operatorname{diag}(\varepsilon^{s_1}, \dots, \varepsilon^{s_{N-1}}, \varepsilon^{-t_N})$ from the left and replacing $v(x, t, \varepsilon)$ by the expansion (3.5), we obtain from (3.3) the equation

$$\sum_{l=0}^{\infty} \varepsilon^{l_0 + l} v^{(l)} (A(\varepsilon D', D_t) - E_N) v^{(l)} = 0. \tag{3.8}$$

Now we use the Taylor expansion of $A(\varepsilon D', D_t)$ with respect to $\varepsilon D'$ which is of the form

$$A(\varepsilon D', D_t) = A(0, D_t) + \sum_{|\alpha| \geq 1} \varepsilon^{|\alpha|} A^{(\alpha)}(0, D_t) D'^{\alpha}/\alpha! = A(0, D_t) + \sum_{k \geq 1} \varepsilon^k C_k(D)$$

and substitute this into (3.8). We get the recurrence relations

$$A(0, D_t) v^{(l)}(x', t) = -\sum_{k \geq 1} C_k(D) v^{(l-k)}. \tag{3.9}$$

Note that the left-hand sides of (3.7) and (3.9) coincide with the operators appearing in conditions (iii) and (iv), respectively, of the definition of weak parameter-ellipticity. So we see that the vector functions $u'^{(l)}, u^{(l)}$ and $v^{(l)}$ can be found recursively, provided that we know the boundary values

$$g'_{lj} := B_j(D) u'^{(l)}(x', 0), \quad j = 1, \dots, R_{N-1}, \quad l = 0, 1, \dots$$

and

$$g''_{lj} := B_j(0, D_t)v^{(l)}(x', 0), \quad j = R_{N-1}+1, \dots, R_N, \quad l = 0, 1, \dots.$$

(Note that due to Definition 2.2 in the case of constant coefficients and no lower order terms the boundary value problems for $u^{(l)}$ and $v^{(l)}$ are uniquely solvable. In the case of variable coefficients the question of unique solvability is nontrivial; we hope to discuss this in a future paper.)

(iii) *Boundary conditions.* First of all note that

$$B_j(D)u(x', 0, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^l B_j(D)u^{(l)}(x', 0). \tag{3.10}$$

For the inner expansion we argue as before and get

$$\begin{aligned} B_j(D)v(x', 0, \varepsilon) &= \sum_{l=0}^{\infty} \varepsilon^{l_0+l} B_j(D', \frac{1}{\varepsilon}D_t)\operatorname{diag}(\varepsilon^{t_1}, \dots, \varepsilon^{t_N})v^{(l)}(x', 0) \\ &= \sum_{l=0}^{\infty} \varepsilon^{l+l_0-m_j} B_j(\varepsilon D', D_t)v^{(l)}(x', 0). \end{aligned}$$

Replacing $B_j(\varepsilon D', D_t)$ by

$$B_j(0, D_t) + \sum_{k=1} \varepsilon^k C_k(D)$$

and gathering terms with the same power of ε we finally obtain

$$\begin{aligned} B_j(D)v(x', 0, \varepsilon) = \sum_{l=l_0-m_j}^{\infty} \varepsilon^l \Big[&B_j(0, D_t)v^{(l-l_0+m_j)}(x', 0) \\ &+ C_1(D)v^{(l-l_0+m_j-1)}(x', 0) + \dots \Big]. \end{aligned} \tag{3.11}$$

Now we pose $l_0 = m_{R_{N-1}+1}$. According to our assumption $l_0 > m_j$ holds for $j = 1, \dots, R_{N-1}$, and the first R_{N-1} boundary conditions are of the form

$$\begin{aligned} B_j(D)u^{(l)}(x', 0) = \delta_{0l}g_j(x') &+ B_j(0, D_t)v^{(l-l_0+m_j)}(x', 0) \\ &+ C_1(D)v^{(l-l_0+m_j-1)}(x', 0) + \dots. \end{aligned} \tag{3.12}$$

If we already know $u^{(k)}$ and $v^{(k)}$ for $k = 1, \dots, l-1$ this gives us the value of

$$B_j(D){u'}^{(l)}(x', 0), \quad j = 1, \dots, R_{N-1}.$$

Using the system (3.7) and these boundary conditions we can define ${u'}^{(l)}$ and, consequently, $u^{(l)}$.

For $j = R_{N-1}+1$ equation (3.12) gives

$$B_j(0,D_t)v^{(l)}(x',0) = \delta_{0l}g_l - B_j(D)u^{(l)}(x',0) - \sum_{k\geq 1} C_k(D)v^{(l-k)}(x',0).$$

To find the boundary conditions for $j > R_{N-1}+1$, we apply the operator $B_j(D)$ to the term obtained from equating to zero the coefficient before $\varepsilon^{l+R_{N-1}+1-j}$. In this way we get for $j = R_{N-1}+2,\dots,R_N$

$$B_j(0,D_t)v^{(l)} = \delta_{0,l+R_{N-1}+1-j}g_l - B_j(D)u^{(l+R_{N-1}+1-j)} - \sum_{k\geq 1} C_k(D)v^{(l-k)}.$$

Now we can find $v^{(l)}$ and continue our process.

References

[A62] Agmon, S.: On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.* **15** (1962), 119-147.

[A90] Agranovich, M. S.: Nonselfadjoint problems with a parameter that are Agmon-Douglis-Nirenberg elliptic (Russian). *Funktsional. Anal. i Prilozhen.* **24** (1990), No. 1, 59–61. English transl. in *Functional Anal. Appl.* **24** (1990), No. 1, 50–53.

[AV64] Agranovich, M. S., Vishik, M. I.: Elliptic problems with parameter and parabolic problems of general form (Russian). *Uspekhi Mat. Nauk* **19** (1964), No. 3, 53-161. English transl. in *Russian Math. Surv.* **19** (1964), No. 3, 53-157.

[DMV98] Denk, R., Mennicken, R., Volevich, L.: The Newton polygon and elliptic problems with parameter. *Math. Nachr.* **192** (1998), 125-157.

[DMV00a] Denk, R., Mennicken, R., Volevich, L.: Boundary value problems for a class of elliptic operator pencils. To appear in *Integral Equations Operator Theory.*

[DMV00b] Denk, R., Mennicken, R., Volevich, L.: On elliptic operator pencils with general boundary conditions. To appear in *Integral Equations Operator Theory.*

[DV00] Denk, R., Volevich, L.: A priori estimate for a singularly perturbed mixed order boundary value problem. *Russian J. Math. Phys.* **7** (2000), 288–318.

[F79] Frank, L.: Coercive singular perturbations. I. A priori estimates. *Ann. Mat. Pura Appl.* (4) **119** (1979), 41–113.

[F97] Frank, L.: *Singular perturbations in elasticity theory.* Analysis and its Applications, 1. IOS Press, Amsterdam, 1997.

[G96] Grubb, G.: *Functional Calculus of Pseudodifferential Boundary Problems.* Second edition. Progress in Mathematics, 65, Birkhäuser, Boston, 1996.

[I89] Il'in, A. M.: *Matching of asymptotic expansions of solutions of boundary value problems* (Russian). "Nauka", Moscow, 1989. English transl. as Transl. Math. Monogr. **102**, Amer. Math. Soc., Providence, RI, 1992.

[K96] Kozhevnikov, A.: Asymptotics of the spectrum of Douglis–Nirenberg elliptic operators on a closed manifold. *Math. Nachr.* **182** (1996), 261–293.

[N81] Nazarov, S. A.: The Vishik–Lyusternik method for elliptic boundary value problems in regions with conic points. I. The problem in a cone (Russian). *Sibirsk. Mat. Zh.* **22** (1981), No. 4, 142-163. English transl. in *Siberian Math. J.* **22** (1982), 594-611.

[R96] Roitberg, Y.: *Elliptic Boundary Value Problems in the Spaces of Distributions.* Mathematics and its Applications, 384. Kluwer Academic Publishers, Dordrecht, 1996.

[VL57] Vishik, M. I., Lyusternik, L. A.: Regular degeneration and boundary layer for linear differential equations with small parameter (Russian). *Uspehi Mat. Nauk (N.S.)* **12** (1957), No. 5 (77), 3-122. English transl. in *Amer. Math. Soc. Transl.* (2) **20** (1962), 239-364.

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