Exercise 4 - MFCQ and Complementarity Constraints (5 Points)

For \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), \( g, h : \mathbb{R}^n \rightarrow \mathbb{R}^m \) consider the complementarity constrained problem

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \geq 0 \\
& \quad h_i(x) \geq 0 \quad \forall i = 1, \ldots, m. \\
& \quad g_i(x)h_i(x) = 0
\end{align*}
\]

Prove that the Mangasarian Fromovitz constraint qualification is violated at every feasible point. (I.e., that there are no feasible points that have linearly independent gradients of the equality constraints and a direction in the kernel of the linearized equality constraints that is a first order ascent direction for all active inequality constraints.)

Exercise 5 - Linear Optimization (5 Points)

Given initial data: \( b \in \mathbb{R}^m \), \( A = [B|N] \in \mathbb{R}^{m \times n} \) with \( B \in \mathbb{R}^{m \times m} \) invertible and \( N \in \mathbb{R}^{m \times (n-m)} \) and \( c = [c_B \ c_N]^\top \in \mathbb{R}^n \), consider the following linear programming problems:

\[
\begin{align*}
\min & \quad c^\top x \quad \text{s.t.} \quad Ax = b, x \geq 0 \tag{P}
\end{align*}
\]

and

\[
\begin{align*}
\max & \quad b^\top \lambda \quad \text{s.t.} \quad A^\top \lambda + s = c, s \geq 0 \tag{D}
\end{align*}
\]

where: \( x = [x_B \ x_N]^\top \in \mathbb{R}^n \), \( s = [s_B \ s_N]^\top \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^m \). Further, let matrices \( Y \) and \( Z \) be given by

\[
Y = \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix}.
\]

a) Show that the columns of \([Y|Z]\) are linearly independent.

b) Show that the constraint of problem (P) implies

\( x_B = B^{-1}b - B^{-1}Nx_N \)

with objective value

\( \zeta = c_B^\top B^{-1}b - ((B^{-1}N)^\top c_B - c_N)^\top x_N. \)

By setting \( x_N = 0 \), we call the solution \( x^* = [x_B^* \ 0]^\top \) the basic solution of (P) to the pair \((x_B, \zeta)\).
c) Verify that for problem (D) we have
\[
\lambda = (B^{-1})^\top (c_B - s_B)
\]
\[
s_N = -(B^{-1} N)^\top c_B + c_N + (B^{-1} N)^\top s_B
\]
with objective value
\[
\xi = (B^{-1} b)^\top c_B - (B^{-1} b)^\top s_B.
\]
By setting \( s_B = 0 \), we call the solution \( \lambda^* = (B^{-1})^\top c_B \) and \( s^* = [0 \ s_N^*]^\top \) the basic solution of (D) to the pair \( (s_N, \xi) \).

Exercise 6 - Linear Optimization
Consider the optimization problem

\[\begin{align*}
\text{min} & \quad 6x_1 - 32x_2 + 9x_3 \\
\text{s.t.} & \quad \begin{cases} 
    x_1 - 7x_2 + 2x_3 + x_4 = 4 \\
    -2x_1 + 10x_2 - 3x_3 + x_5 = -6 \\
    x_1, x_2, x_3, x_4, x_5 \geq 0
\end{cases}
\end{align*}\]

a) Define the matrices \( B \) and \( N \) (see exercise 5) for the problem above. Then, transform (1) into the form of (D).

b) Write down the vector \( x_B \) and \( s_N \) for (1) and compute the corresponding basic solutions \( x_B^* \) and \( \lambda^*, s_N^* \). Are they feasible?