Exercise 11 (Programming) (7 Points)

Let \( b \in \mathbb{R}^m \), \( A = [B|N] \in \mathbb{R}^{m \times n} \) with \( B \in \mathbb{R}^{m \times m} \) invertible, \( N \in \mathbb{R}^{m \times (n-m)} \) and \( c = [c_B \ c_N]^T \in \mathbb{R}^n \) be given and we consider again the following primal-dual pair of linear programming problems:

\[
\min c^T x \quad \text{subject to} \quad Ax = b, x \geq 0 \quad (P)
\]

and

\[
\max b^T \lambda \quad \text{subject to} \quad A^T \lambda + s = c, s \geq 0, \quad (D)
\]

where: \( x = [x_B \ x_N]^T \in \mathbb{R}^n \), \( s = [s_B \ s_N]^T \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^m \).

a) Implement the function

\([x, \text{iter}, \text{stat}] = \text{mydualsimplex}(A, b, c, B, N, \text{maxiter})\)

for solving the problem (P) using the dual simplex method. That is, the primal simplex method is applied on the dual problem (D) (cf. Algorithm 1 and Exercise 8).

b) Explain when the dual simplex algorithm can be the better choice compared to the primal version. Consider, e.g., feasibility of initial values and what happens when constraints are added to the initial problem.

**Hint:** Your well-commented code is required but no report is needed.

Exercise 12 (8 Points)

Consider the following linear programming problem

\[
\min -5x_1 - 8x_2 \quad \text{subject to} \quad \begin{bmatrix} x_1 + x_2 \leq 6 \\ 5x_1 + 9x_2 \leq 45 \\ x_1, x_2 \geq 0 \end{bmatrix}. \quad (1)
\]
a) Solve the problem (1) with either primal simplex method or dual simplex method. Write down the values of $x_B$, $\lambda$ and $s_N$ for each iteration of your simplex method if you use your code.

Now we want to modify the problem, that is, we restrict the variables to take only integer values:

$$\begin{align*}
\text{min} & \quad -5x_1 - 8x_2 \\
\text{subject to} & \quad \begin{bmatrix} x_1 + x_2 \leq 6 \\
5x_1 + 9x_2 \leq 45 \\
x_1, x_2 \geq 0 \\
x_1, x_2 \in \mathbb{Z} \end{bmatrix}.
\end{align*}$$

b) Show that rounding the solution of (1) to the next integer values is not a good strategy to solve (2).

As the first step to solve (2), we consider the following child problems:

$$\begin{align*}
\text{min} & \quad -5x_1 - 8x_2 \\
\text{subject to} & \quad \begin{bmatrix} x_1 + x_2 \leq 6 \\
5x_1 + 9x_2 \leq 45 \\
x_2 \leq -\lceil \bar{x}_2 \rceil \\
x_1, x_2 \geq 0 \end{bmatrix}, \quad (2a)
\end{align*}$$

and

$$\begin{align*}
\text{min} & \quad -5x_1 - 8x_2 \\
\text{subject to} & \quad \begin{bmatrix} x_1 + x_2 \leq 6 \\
5x_1 + 9x_2 \leq 45 \\
x_2 \leq \lfloor \bar{x}_2 \rfloor \\
x_1, x_2 \geq 0 \end{bmatrix}, \quad (2b)
\end{align*}$$

where $\bar{x}_2$ is the optimal value by solving (1). Notice that, if $\bar{x}_2$ takes fractional value, it will become infeasible for the child problems.

c) Provided the solution of (1) from part a), which simplex method will be better for solving the child problems? Explain your answer.

d) Solve the child problems (2a) and (2b) by your choice of simplex method. Write down the values of $x_B$, $\lambda$ and $s_N$ for each iteration of your simplex method if you use your code.

Hint: You need to rewrite the problems with inequality constraints into those with equality constraints to perform simplex methods. Your codes for solving the problems are not required to be submitted.
**Algorithm 1 Dual Simplex method**

Given: $A, b$ and $c; s_B = 0, \lambda = (B^{-1})^\top c_B, s_N = c_N - (B^{-1} N)\top c_B \geq 0; x_B = B^{-1} b, x_N = 0$;

while $x_B \not\geq 0$ do

  Set $q = \min(i \in B : x_i < 0)$;
  Compute $\Delta s_N = -(B^{-1} N)\top e_q$ and $\Delta \lambda = -(B^{-1})\top e_q$;
  if $\Delta s_N \leq 0$ then
    STOP and dual problem is unbounded;
  end

  Compute $s_q = \min \{i : (\Delta s_N)_i > 0\} (s_N)_i / (\Delta s_N)_i$;

  Set $p = \min \left( i \in \arg \min \{i : (\Delta s_N)_i > 0\} (s_N)_i / (\Delta s_N)_i \right)$;

  Compute $\Delta x_B = (B^{-1} A) e_p$ and $x_p = (x_B)_q / (\Delta x_B)_q$;

  Update $s_N = s_N - s_q \Delta s_N, \lambda = \lambda + s_q \Delta \lambda, s_B = s_q e_q$;

  Update $x_B = x_B - x_p \Delta x_B, x_N = x_p e_p$;

  Update indices $B = \{p\} \cup B \setminus \{q\}, N = \{q\} \cup N \setminus \{p\}$;

end