Pseudodifferential operators and maximal $L^p$-regularity

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A first example

We want to solve the heat equation

\[ \partial_t u(t, x) - \Delta u(t, x) = f(t, x) \quad \text{in } [0, \infty) \times \mathbb{R}^n, \]
\[ u(0, x) = 0 \quad \text{in } \mathbb{R}^n \]

and its parameter-dependent version

\[ \lambda u(x) - \Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n. \]
A first example

We want to solve the heat equation

\[ \frac{\partial_t}{t} u(t, x) - \Delta u(t, x) = f(t, x) \quad \text{in } [0, \infty) \times \mathbb{R}^n, \]
\[ u(0, x) = 0 \quad \text{in } \mathbb{R}^n \]

and its parameter-dependent version

\[ \lambda u(x) - \Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n. \]

Main idea: apply Fourier transform

\[ \mathcal{F}: u(x) \mapsto \hat{u}(\xi), \quad \partial_{x_j} u(x) \mapsto i \xi_j \hat{u}(\xi) \]

and obtain the solution

\[ u = \mathcal{F}^{-1} \frac{1}{\lambda + |\xi|^2} \mathcal{F} f. \]

pseudodifferential operator (PsDO)!
Solution spaces: \( L^p \)-theory

**Elliptic case:**

\[
\lambda u(x) - \Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n.
\]

**Question:** Is the operator

\[
\lambda - \Delta : W^2_p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)
\]

an isomorphism of Banach spaces?
Solution spaces: $L^p$-theory

**Elliptic case:**

$$\lambda u(x) - \Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n.$$ 

**Question:** Is the operator

$$\lambda - \Delta : W^{2}_p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

an isomorphism of Banach spaces?

**Parabolic case:**

$$\partial_t u(t, x) - \Delta u(t, x) = f(t, x) \quad \text{in } [0, \infty) \times \mathbb{R}^n,$$

$$u(0, x) = 0 \quad \text{in } \mathbb{R}^n$$

**Question:** Is the operator

$$\partial_t - \Delta : W^{1}_p((0, \infty); L^p(\mathbb{R}^n)) \cap L^p((0, \infty); W^{2}_p(\mathbb{R}^n)) \to L^p((0, \infty) \times \mathbb{R}^n)$$

an isomorphism of Banach spaces?
Consider the Dirichlet boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$(\lambda - \Delta)u = f \quad \text{in } \Omega,$$

$$\gamma_0 u = g \quad \text{on } \partial \Omega$$

with $\gamma_0 u := u|_{\partial \Omega}$ (trace of $u$).
Boundary value problems

Solution to boundary value problems

Consider the Dirichlet boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^n$:

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$$\gamma_0 u = g \quad \text{on } \partial \Omega$$

with $\gamma_0 u := u|_{\partial \Omega}$ (trace of $u$).

**Question:** Is the operator

$$(\lambda - \Delta, \gamma_0): W^2_p(\Omega) \to L^p(\Omega) \times W^{2-1/p}_p(\partial \Omega)$$

an isomorphism of Banach spaces?

- How can we compute $u$?
- How can we compute the normal trace $h := (\partial_\nu u)|_{\partial \Omega}$?
We want to solve the boundary value problem

$$(\lambda - \Delta)u = f \quad \text{in } \Omega,$$

$$\gamma_0 u = g \quad \text{on } \partial \Omega.$$
How to solve boundary value problems

We want to solve the boundary value problem

\[(\lambda - \Delta)u = f \quad \text{in } \Omega,\]
\[\gamma_0 u = g \quad \text{on } \partial\Omega.\]

Standard approach:

- Reduce to the case \(f = 0\) by subtracting a whole space-solution,
- freeze the coefficients, take local coordinates ("model problem"),
- take partial Fourier transform in tangential directions,
- solve the ordinary differential equation in normal direction,
- take inverse partial Fourier transform.
Reduction to the boundary

Model problem in the half-space $\mathbb{R}^n_+$:

$$(\lambda - \Delta)u = 0 \quad \text{in } \mathbb{R}^n_+, $$

$$u = g \quad \text{on } \mathbb{R}^{n-1}. $$

Partial Fourier transform $\mathcal{F}' (x' \sim \xi')$ in the tangential variables $x' = (x_1, \ldots, x_{n-1})$ gives the ODE

$$(\lambda + |\xi'|^2 - \partial^2_{x_n})w(x_n) = 0 \quad (x_n > 0),$$

$$w(0) = (\mathcal{F}' g)(\xi').$$
Reduction to the boundary

Model problem in the half-space $\mathbb{R}^+_n$:

\[
(\lambda - \Delta) u = 0 \quad \text{in } \mathbb{R}^+_n,
\]

\[
u = g \quad \text{on } \mathbb{R}^{n-1}.
\]

Partial Fourier transform $\mathcal{F}' (x' \sim \xi')$ in the tangential variables $x' = (x_1, \ldots, x_{n-1})$ gives the ODE

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(\lambda + |\xi'|^2 - \partial^2_{x_n}) w(x_n) = 0 \quad (x_n > 0),
\]

\[
w(0) = (\mathcal{F}' g)(\xi').
\]

Stable (!) solution of is given by

\[
w(x_n) = w(\xi', \lambda, x_n) = \exp(-\sqrt{\lambda + |\xi'|^2} x_n)(\mathcal{F}' g)(\xi')
\]

\[
= - \int_0^\infty \partial y_n \left[ \exp(-\sqrt{\lambda + |\xi'|^2}(x_n + y_n))(\mathcal{F}' \tilde{g})(\xi', y_n) \right] dy_n.
\]
The Dirichlet-Neumann operator

The normal trace of \( w \) is given by the symbol

\[
(\partial_\nu w)(\xi', \lambda, 0) = \sqrt{\lambda + |\xi'|^2} (\mathcal{F}'g)(\xi').
\]

We obtain the Dirichlet-Neumann operator

\[
g \mapsto h = (\mathcal{F}')^{-1} \sqrt{\lambda + |\xi'|^2} (\mathcal{F'}g)(\xi').
\]
The Dirichlet-Neumann operator

The normal trace of $w$ is given by the symbol

$$(\partial_{\nu} w)(\xi', \lambda, 0) = \sqrt{\lambda + |\xi'|^2}(\mathcal{F}' g)(\xi').$$

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→ pseudodifferential operator (PsDO) on the boundary $\partial \Omega$!

(DIRICHLET-NEUMANN OPERATOR, LOPATINSKII MATRIX)
The Dirichlet-Neumann operator

The normal trace of $w$ is given by the symbol

$$(\partial_\nu w)(\xi', \lambda, 0) = \sqrt{\lambda + |\xi'|^2}(\mathcal{F}'g)(\xi').$$

We obtain the Dirichlet-Neumann operator

$$g \mapsto h = (\mathcal{F}')^{-1}\sqrt{\lambda + |\xi'|^2}(\mathcal{F}'g)(\xi').$$

pseudodifferential operator (PsDO) on the boundary $\partial \Omega$!

(Dirichlet-Neumann operator, Lopatinskii matrix)

More general:

- higher order operators ($\leadsto$ systems of PsDOs),
- additional equations on the boundary (e.g., free boundary value problems),
Summary: Why PsDOs?

1. Solution of \((\lambda - \Delta)u = f\) in the whole space:

\[
u = \mathcal{F}^{-1} \frac{1}{\lambda + |\xi|^2} \mathcal{F} f.
\]

2. Solution of \((\lambda - \Delta)u = 0, \gamma_0 u = g\) in the half-space:

\[
u = (\mathcal{F}')^{-1} \int_0^\infty \partial_n \left[ \exp\left(-\sqrt{\lambda + |\xi'|^2}(x_n + y_n)\right) (\mathcal{F}' \tilde{g})(\xi', y_n) \right] dy_n.
\]

3. Dirichlet-Neumann operator \(g := u|_{\partial \Omega} \mapsto h := (\partial_\nu u)|_{\partial \Omega}:

\[h = (\mathcal{F}')^{-1} \sqrt{\lambda + |\xi'|^2} \mathcal{F}' g.
\]
Reduction to the boundary: an example

**Spin-coating process** (Geissert-Hieber-Saal-Sawada-D. 2011):

The spin-coating model leads to the following generalized Dirichlet-Neumann operator (Lopatinskii matrix):

\[
L(\xi', \tau) = \begin{pmatrix}
i\xi_1 & i\xi_2 & -\omega & 0 & 0 \\
0 & 0 & 1 & \frac{|\xi'|}{\omega(\omega + |\xi'|)} & \lambda \\
\omega & 0 & -i\xi_1 & -\frac{i\xi_1(\omega - |\xi'|)}{\omega(\omega + |\xi'|)} & 0 \\
0 & \omega & -i\xi_2 & -\frac{i\xi_2(\omega - |\xi'|)}{\omega(\omega + |\xi'|)} & 0 \\
0 & 0 & -2\omega & -1 & \sigma|\xi'|^2
\end{pmatrix}
\]

with \( \omega := \sqrt{\lambda + |\xi'|^2} \).
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3 Non-local boundary value problems
Maximal $L^p$-regularity

One approach to solve nonlinear PDEs is to show maximal $L^p$-regularity for the linearized problems. Consider

$$\partial_t \nu + A \nu = f \quad (t \in [0, T]),$$
$$\nu(0) = 0.$$  \quad (1)

where $A$ is a closed operator in the space $X$ with domain $D(A)$. 
Maximal $L^p$-regularity

One approach to solve nonlinear PDEs is to show maximal $L^p$-regularity for the linearized problems. Consider

\[ \partial_t v + Av = f \quad (t \in [0, T]), \]
\[ v(0) = 0. \]  

(1)

where $A$ is a closed operator in the space $X$ with domain $D(A)$.

**Definition**

The linear operator $A$ has maximal $L^p$-regularity if for every $f \in L^p([0, T], X)$ equation (1) has a unique solution

\[ v \in W^1_p([0, T], X) \cap L^p([0, T], D(A)) \]

depending continuously on $f$. 
R-boundedness and maximal $L^p$-regularity

How to prove maximal $L^p$-regularity?

**Theorem (Weis 2001)**

Let $X$ be a UMD space, $1 < p < \infty$, and $A$ be a sectorial operator. Then $A$ has maximal $L^p$-regularity (\(=\) well-posedness in $L^p$-Sobolev spaces) if and only if $A$ is $R$-sectorial, i.e. the resolvent is $R$-bounded:

$$R(\{\lambda(\lambda - A)^{-1} : \text{Re } \lambda \geq 0\}) < \infty.$$  

- UMD space: valid for many spaces, e.g. reflexive $L^p$-spaces,
- $R$-bounded: similar to bounded in operator norm (but stronger),
- in Hilbert spaces: $R$-bounded \(=\) bounded.
Mikhlin’s theorem

Let $M = \text{op}(m) := \mathcal{F}^{-1}m\mathcal{F}$. How to prove boundedness of $M$ in $L^p(\mathbb{R}^n)$?

a) $p = 2$: $M \in L(L^2(\mathbb{R}^n))$ if and only if $m \in L^\infty(\mathbb{R}^n)$ by Plancherel’s theorem.

b) $p \neq 2$:
Mikhlin’s theorem

Let \( M = \text{op}(m) := \mathcal{F}^{-1} m \mathcal{F} \). How to prove boundedness of \( M \) in \( L^p(\mathbb{R}^n) \)?

a) \( p = 2 \): \( M \in L(L^2(\mathbb{R}^n)) \) if and only if \( m \in L^\infty(\mathbb{R}^n) \) by Plancherel’s theorem.

b) \( p \neq 2 \):

**Theorem (Mikhlin 1962 – Lizorkin 1963)**

Let \( 1 < p < \infty \) and \( m \in C^n(\mathbb{R}^n \setminus \{0\}) \) with

\[
\sup \left\{ |\xi^\beta \partial_\xi^\beta m(\xi)| : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0, 1\}^n \right\} < \infty.
\]

Then the associated Fourier multiplier \( \mathcal{F}^{-1} m \mathcal{F} \) is a bounded operator in \( L^p(\mathbb{R}^n) \).
\( R \)-boundedness of Fourier multipliers

The following result is the “\( R \)-bounded version” of Mikhlin’s theorem:

**Theorem (Girardi-Weis 2003)**

Let \( 1 < p < \infty \), \( X \) be a UMD Banach spaces with property \((\alpha)\), \( \Lambda \) a set, and let \( \{m_\lambda : \lambda \in \Lambda\} \) with \( m_\lambda \in C^n(\mathbb{R}^n \setminus \{0\}, L(X)) \) with

\[
R \left( \{\xi^\beta \partial_\xi^\beta m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0, 1\}^n, \lambda \in \Lambda\} \right) < \infty.
\]

Then the set of associated Fourier multipliers \( \{\mathcal{F}^{-1}m_\lambda \mathcal{F} : \lambda \in \Lambda\} \) is \( R \)-bounded in \( L(L^p(\mathbb{R}^n; X)) \).

- property \((\alpha)\): valid for many spaces, e.g. \( L^p \)-spaces.
Pseudodifferential operators: scalar symbols

Let $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ and $\mu \in \mathbb{R}$.

**Definition**

$S^\mu(\mathbb{R}^n)$ is the space of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for which

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{\mu - |\alpha|} \quad (x, \xi \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n)$$

(space of all symbols of order $\mu$).
Pseudodifferential operators: scalar symbols

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(space of all symbols of order $\mu$).

Set $\Psi^\mu(\mathbb{R}^n) := \{\text{op}(a) : a \in S^\mu(\mathbb{R}^n)\}$ with

$$[\text{op}(a)u](x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i x \xi} a(x, \xi)(\mathcal{F} u)(\xi) d\xi.$$ 

$\text{op}(a)$ is called a pseudodifferential operator (PsDO) with symbol $a$. 
Vector-valued PsDOs: $\mathcal{R}$-bounded symbols

Let $X$ be a Banach space and $\mu \in \mathbb{R}$.
First we consider symbols which do not depend on $x$:

**Definition**

Let $a \in C^\infty(\mathbb{R}^n, L(X))$, $\xi \mapsto a(\xi)$. Then $a \in S^\mu_R(\mathbb{R}^n, L(X))$ if for all $\alpha \in \mathbb{N}^n_0$

$$p^{(\alpha)}_R(a) := \mathcal{R}\left(\{\langle \xi \rangle^{-\mu+|\alpha|} \partial_\xi^\alpha a(\xi) : \xi \in \mathbb{R}^n\}\right) < \infty.$$

Fréchet space $S^\mu_R(\mathbb{R}^n, L(X))$ of $\mathcal{R}$-bounded symbols.
Vector-valued PsDOs: $\mathcal{R}$-bounded symbols

Let $X$ be a Banach space and $\mu \in \mathbb{R}$.
First we consider symbols which do not depend on $x$:

**Definition**

Let $a \in C^\infty(\mathbb{R}^n, L(X))$, $\xi \mapsto a(\xi)$. Then $a \in S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$ if for all $\alpha \in \mathbb{N}_0^n$

$$p^{(\alpha)}(a) := \mathcal{R}\left(\{\langle \xi \rangle^{-\mu+|\alpha|} \partial_\xi^\alpha a(\xi) : \xi \in \mathbb{R}^n\}\right) < \infty.$$  

Fréchet space $S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$ of $\mathcal{R}$-bounded symbols.

- Let $S^{-\infty} := \bigcap_{\mu \in \mathbb{R}} S^\mu$ (smoothing symbols).
- There is a calculus for $\mathcal{R}$-bounded symbols.
Vector-valued PsDOs: constant coefficients

Some symbols are automatically $\mathcal{R}$-bounded:

**Theorem (Krainer-D. 2007)**

a) **Constant smoothing symbols:**

$$S^{-\infty} (\mathbb{R}^n, L(X)) \subset S^{-\infty}_\mathcal{R} (\mathbb{R}^n, L(X)).$$

b) **Constant scalar symbols:**

$$S^\mu (\mathbb{R}^n) \subset S^\mu_\mathcal{R} (\mathbb{R}^n, L(X))$$

via $a(\xi) \mapsto a(\xi) \text{id}_X$. 

Idea of proof:

a) $S^{-\infty} (\mathbb{R}^n, L(X)) \sim = S(\mathbb{R}^n) \hat{\otimes} \pi L(X)$ (completed projective tensor product).

b) Kahane's inequality.
Vector-valued PsDOs: constant coefficients

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a) *Constant smoothing symbols:*

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b) *Constant scalar symbols:*

$$S^\mu(\mathbb{R}^n) \subset S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X))$$

via $a(\xi) \mapsto a(\xi) \text{id}_X$.

**Ideas of proof:**

a) $S^{-\infty}(\mathbb{R}^n, L(X)) \cong \mathcal{S}(\mathbb{R}^n) \hat{\otimes}_\pi L(X)$

(completed projective tensor product).

b) Kahane’s inequality.
Vector-valued PsDOs: classical symbols

\[ a \in S^\mu_{\text{cl}} \iff a \sim \sum_{j=0}^{\infty} a_j \text{ with } a_j \text{ homogeneous of order } \mu - j \] (classical symbols)

Theorem (Krainer-D. 2007)

a) Every \( a \in S^\mu_{\text{cl}}(\mathbb{R}^n, L(X)) \) belongs to \( S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X)) \).

b) Every

\[ a \in S^0_{\text{cl}}(\mathbb{R}^n_x, S^\mu_{\text{cl}}(\mathbb{R}^n_\xi, L(X))) \]

belongs to \( S^\mu_{\mathcal{R}}(\mathbb{R}^n, L(X)) \).

c) Let \( X \) be a UMD Banach spaces with property (\( \alpha \)), and let

\[ \{a_\lambda : \lambda \in \Lambda \} \subset S^0_{\text{cl}}(\mathbb{R}^n_x, S^\mu_{\text{cl}}(\mathbb{R}^n_\xi, L(X))) \]

be a bounded family of symbols. Then

\[ \{\text{op}(a_\lambda) : \lambda \in \Lambda \} \subset L(W^s_p(\mathbb{R}^n, X), W^{s-\mu}_p(\mathbb{R}^n, X)) \]

is \( \mathcal{R} \)-bounded.
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1. Solution of \((\lambda - \Delta)u = f\) in the whole space:

\[ u = \mathcal{F}^{-1} \frac{1}{\lambda + |\xi|^2} \mathcal{F} f. \]

2. Solution of \((\lambda - \Delta)u = 0, \ \gamma_0 u = g\) in the half-space:

\[ u = (\mathcal{F}')^{-1} \int_0^\infty \partial_n \left[ \exp(-\sqrt{\lambda + |\xi'|^2}(x_n + y_n)) (\mathcal{F}' \tilde{g})(\xi', y_n) \right] dy_n. \]

3. Dirichlet-Neumann operator \(g := u|_{\partial \Omega} \mapsto h := (\partial_\nu u)|_{\partial \Omega}:

\[ h = -(\mathcal{F}')^{-1} \sqrt{\lambda + |\xi'|^2} \mathcal{F}' g. \]
The Stokes equation

The pseudodifferential approach to the Stokes system:

\[ \partial_t u - \Delta u + \nabla p = 0 \quad \text{in } (0, \infty) \times \Omega, \]
\[ \text{div } u = 0 \quad \text{in } (0, \infty) \times \Omega, \]
\[ u = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \]
\[ u|_{t=0} = u_0 \quad \text{in } \Omega. \]
The Stokes equation

The pseudodifferential approach to the Stokes system:

\[\begin{align*}
\partial_t u - \Delta u + \nabla p &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
\text{div } u &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
u &= 0 \quad \text{on } (0, \infty) \times \partial \Omega, \\
u|_{t=0} &= u_0 \quad \text{in } \Omega.
\end{align*}\]

Applying \text{div} to the first line, we obtain an elliptic boundary value problem for the pressure:

\[-\Delta p = 0 \quad \text{in } \Omega, \]
\[\partial_{\nu\Omega} p = \gamma_0 \nu_\Omega \cdot \Delta u \quad \text{on } \partial \Omega\]

where \(\nu_\Omega\) is the outer normal to \(\partial \Omega\) and \(\gamma_0 u := u|_{\partial \Omega}\).

Let \(p = G[\gamma_0 \nu_\Omega \cdot \Delta u]\) with the solution operator \(G\).
The Stokes system

Inserting the solution operator into the first equation, we obtain

\[(\partial_t - \Delta + \nabla G_{\gamma_0 \nu \Omega} \cdot \Delta)u = 0 \quad \text{in} \ (0, \infty) \times \Omega,\]
\[u = 0 \quad \text{on} \ (0, \infty) \times \partial \Omega,\]
\[u\big|_{t=0} = u_0 \quad \text{in} \ \Omega.\]
The Stokes system

Inserting the solution operator into the first equation, we obtain

\[(\partial_t - \Delta + \nabla G_{\gamma_0 \nu} \cdot \Delta)u = 0 \text{ in } (0, \infty) \times \Omega,\]
\[u = 0 \text{ on } (0, \infty) \times \partial\Omega,\]
\[u\big|_{t=0} = u_0 \text{ in } \Omega.\]

Remarks:

- This is a nonlocal boundary value problem for \(u\),
- the equality \(\text{div } u = 0\) is satisfied automatically,
- the nonlocal term \(\nabla G_{\gamma_0 \nu} \cdot \Delta\) is not of lower order.

(Grubb-Kokholm 1993), (Grubb-Solonnikov 1991)
The structure of the solution operator

We have to solve the Neumann-Laplace equation for the pressure:

\[-\Delta p = 0 \quad \text{in } \Omega,\]
\[\partial_{\nu_\Omega} p = \gamma_0 \nu_\Omega \cdot \Delta u =: \gamma_0 g \quad \text{on } \partial \Omega.\]
The structure of the solution operator

We have to solve the Neumann-Laplace equation for the pressure:

\[-\Delta p = 0 \quad \text{in} \ \Omega,\]
\[\partial_{\nu_{\Omega}} p = \gamma_0 \nu_{\Omega} \cdot \Delta u =: \gamma_0 g \quad \text{on} \ \partial\Omega.\]

Locally, the solution operator can be written in the form

\[(Gg)(x) = (\mathcal{F}')^{-1}\left[ \int_{0}^{\infty} k(x', \xi', x_n, y_n)(\mathcal{F}'g)(\xi', y_n)dy_n \right]\]

with some kernel function \(k\) containing the fundamental solution of the ODE.
Singular Green operators

We have to study the resolvent of the non-local operator

$$A := -\Delta + \nabla G_{\gamma_0 \nu_\Omega} \cdot \Delta \quad \text{in } L^p(\mathbb{R}^n)$$

(model problem). The last term is an example of a singular Green operator.
Singular Green operators

We have to study the resolvent of the non-local operator

\[ A := -\Delta + \nabla G \gamma_0 \nu_\Omega \cdot \Delta \quad \text{in } L^p(\mathbb{R}^n_+) \]

(model problem). The last term is an example of a singular Green operator.

**Definition**

A singular Green operator (of type 0) has the form

\[
(Gf)(x) = \left[ \operatorname{op}_G(k)f \right](x) = (\mathcal{F}')^{-1} \left[ \int_0^\infty k(x', \xi', x_n, y_n)(\mathcal{F}' f)(\xi', y_n) dy_n \right]
\]

with a kernel \( k \) belonging to the symbol class \( S^d_G \) defined as all \( C^\infty \)-functions for which

\[
\left\| x_\ell D_{x_n}^\ell y_n^m D_{y_n}^{m'} D_{x'}^{\alpha'} D_{\xi'}^{\beta'} k(x', \xi', x_n, y_n) \right\|_{L^2((0, \infty)_x \times (0, \infty)_y)} \leq C \langle \xi' \rangle^{d-\ell+\ell'-m+m'} - |\beta'|.
\]

(see (Grubb 1996), (Schrohe 2001), also parameter-dependent versions)
The Boutet-de Monvel calculus

Singular Green operators appear in the calculus of PsDO boundary value problems (Boutet-de Monvel calculus). Here the operators (and their inverses!) have the form

\[
\left( \begin{array}{cc}
P + G & K \\ T & S \end{array} \right)
\]

with

- \( P \): PsDO in \( \Omega \),
- \( G \): singular Green operator in \( \Omega \),
- \( T \): boundary operators (trace) \( \Omega \hookrightarrow \partial \Omega \),
- \( K \): Poisson operators \( \partial \Omega \hookrightarrow \Omega \),
- \( S \): PsDO on \( \partial \Omega \).
The resolvent $R(\lambda) := (\lambda + A)^{-1}$ of the non-local operator $A$ is again of the form

$$R(\lambda) = P(\lambda) + \tilde{G}(\lambda)$$

with $P$ being a **PsDO** and $\tilde{G}$ being a **singular Green operator**.

Proof: This follows directly from the (parameter-dependent) Boutet-de Monvel calculus.

Using the boundedness of PsDOs and of singular Green operators, one can show unique solvability in $L^p$-Sobolev spaces for the Stokes equation.

(Grubb-Kokholm 1993), (Grubb-Solonnikov 1991)
The Stokes equation in a cylindrical domain

Let \( \Omega = \Omega' \times \mathbb{R} \subset \mathbb{R}^{n+1} \) be a cylindrical domain with \( \Omega' \subset \mathbb{R}^n \) being a bounded smooth domain.

Consider the Stokes equation in \( \Omega \):

\[
\begin{align*}
\partial_t u - \Delta u + \nabla p &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
\text{div } u &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
u &= 0 \quad \text{on } (0, \infty) \times \partial \Omega, \\
u|_{t=0} &= u_0 \quad \text{in } \Omega.
\end{align*}
\]
The Stokes equation in a cylindrical domain

Idea to treat the Stokes system in the cylindrical domain $\Omega = \Omega' \times \mathbb{R}$: use Fourier transform $\mathcal{F}_{x_{n+1} \rightarrow \xi_{n+1}}$ in unbounded direction!
The Stokes equation in a cylindrical domain

\[ \Omega \]

\[ x_{n+1} \in \mathbb{R} \]

Idea to treat the Stokes system in the cylindrical domain \( \Omega = \Omega' \times \mathbb{R} \):
use Fourier transform \( \mathcal{F}_{x_{n+1} \to \xi_{n+1}} \) in unbounded direction!

obtain a parameter-dependent problem with an additional parameter \( \tau = \xi_{n+1} \) in a bounded domain:

\[
\begin{align*}
\lambda u - \Delta u + \tau^2 u + \nabla p &= f \quad \text{in } \Omega', \\
\text{div}' \tilde{u} + i\tau u_{n+1} &= 0 \quad \text{in } \Omega', \\
u &= 0 \quad \text{on } \partial \Omega'.
\end{align*}
\]

The parameters are \( \lambda \leftrightarrow \partial_t \) and \( \tau \leftrightarrow x_{n+1} \).
The Stokes system in a cylindrical domain

We obtain the reduced Stokes system (Grubb-Solonnikov approach) with the additional parameter $\tau$:

$$
\begin{align*}
\text{Let } & x = (\bar{x}, x_n+1) \in \Omega, \quad u = (\bar{u}, u_n+1) \text{ etc.} \\
\text{Then we obtain with } & \tau := \xi_n+1 (\Delta + \tau^2) p = 0 \text{ in } \Omega', \\
\partial_\nu \Omega' & p = \gamma_0 \nu \Omega' \cdot \Delta \bar{u} \text{ on } \partial \Omega'
\end{align*}
$$

with a parameter-dependent solution operator $p = G_{\tau}[\gamma_0 \nu \Omega' \cdot \Delta \bar{u}]$. 

Robert Denk (Konstanz)
The Stokes system in a cylindrical domain

We obtain the reduced Stokes system (Grubb-Solonnikov approach) with the additional parameter $\tau$:

Let $x = (\bar{x}, x_{n+1}) \in \Omega$, $u = (\bar{u}, u_{n+1})$ etc.
Then we obtain with $\tau := \xi_{n+1}$

\[
(-\Delta + \tau^2)p = 0 \quad \text{in } \Omega',
\]
\[
\partial_{\nu_{\Omega'}} p = \gamma_0 \nu_{\Omega'} \cdot \Delta \bar{u} \quad \text{on } \partial \Omega'
\]

with a parameter-dependent solution operator $p = G_\tau[\gamma_0 \nu_{\Omega'} \cdot \Delta \bar{u}]$. 
The Stokes system in a cylindrical domain

Inserting this into the first equation, we obtain a nonlocal boundary value problem for $\tilde{u}$ with two parameters:

$$(\lambda - \Delta + \tau^2 + \nabla G_{\tau} \gamma_0 \nu_{\Omega'} \cdot \Delta) \tilde{u} = 0 + \text{b.c.}$$
The Stokes system in a cylindrical domain

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**Aim:** Prove maximal $L^p$-regularity for

$$Au := \mathcal{F}^{-1}_{\tau \to x_{n+1}} \left( -\Delta + \tau^2 + \nabla G_{\tau} \gamma_0 \nu_{\Omega'} \cdot \Delta \right) \mathcal{F}_{x_{n+1} \to \tau} + \text{b.c.}$$
The Stokes system in a cylindrical domain

Inserting this into the first equation, we obtain a nonlocal boundary value problem for \( \bar{u} \) with two parameters:

\[
(\lambda - \Delta + \tau^2 + \nabla G_\tau \gamma_{0\nu_{\Omega'}} \cdot \Delta) \bar{u} = 0 \quad + \text{b.c.}
\]

**Aim:** Prove maximal \( L^p \)-regularity for

\[
Au := \mathcal{F}_{\tau \rightarrow x_{n+1}}^{-1} \left( -\Delta + \tau^2 + \nabla G_\tau \gamma_{0\nu_{\Omega'}} \cdot \Delta \right) \mathcal{F}_{x_{n+1} \rightarrow \tau} \quad + \text{b.c.}
\]

- use Girardi-Weis theorem \( \leadsto \) show \( \mathcal{R} \)-boundedness of the resolvent \( \lambda(\lambda - A_\tau)^{-1} \)
- use Boutet-de Monvel calculus \( \leadsto \) the resolvent is the sum of a PsDO and a singular Green operator

it remains to show that singular Green operators are \( \mathcal{R} \)-bounded!
$\mathcal{R}$-boundedness of singular Green operators

We have to prove the $\mathcal{R}$-boundedness of the operator family

$$[\text{op}_G(k)f](x) = \mathcal{F}^{-1}_{\xi' \to x'} \left[ \int_0^\infty k(x', \xi', \lambda, \tau, x_n, y_n)(\mathcal{F}_{x' \to \xi'}f)(y_n)dy_n \right].$$

**Theorem (Seiler-D. 2011)**

Let $\mathcal{K}$ be a bounded subset in the symbol space of singular Green operators of order $d \leq 0$ (and regularity $\nu \geq 1/2$) on the half-space $\mathbb{R}_n^+$. Let $1 < p < \infty$. Then

$$\{ \text{op}_G(k) : k \in \mathcal{K} \} \subset L(L^p(\mathbb{R}_n^+))$$

is $\mathcal{R}$-bounded.

Ingredients of the proof:

- Girardi-Weis theorem,
- $\mathcal{R}$-boundedness of integral operators in $L^p(\mathbb{R}_+)$,
- interpolation of Sobolev spaces.
Maximal $L^p$-regularity for the Stokes system

Corollary

The Stokes equation in a cylindrical domain $\Omega = \Omega' \times \mathbb{R}$ with Dirichlet boundary conditions has maximal $L^p$-regularity for every $1 < p < \infty$.

(see also (Farwig-Ri 2007))
Corollary

The Stokes equation in a **cylindrical domain** $\Omega = \Omega' \times \mathbb{R}$ with Dirichlet boundary conditions has **maximal $L^p$-regularity** for every $1 < p < \infty$.

(see also (Farwig-Ri 2007))

**Proof:**

- By results of Grubb-Solonnikov, the reduced Stokes system is a parabolic non-local boundary problem in the **Boutet-de Monvel calculus**.

- The resolvent is the sum of a **pseudodifferential operator** and a **singular Green operator**.

- These operators are $\mathcal{R}$-bounded (see above).

- By the theorem of Weis, $\mathcal{R}$-sectoriality is equivalent to **maximal regularity**.
Extensions and remarks

- nonsmooth Boutet-de Monvel calculus (Abels 2005)
- Finite cylinder: $\Omega = \Omega' \times [0, 2\pi]^m$
  - $\mathcal{R}$-boundedness of Fourier series
    (Arendt-Bu 2002, Nau-D. 2011)
- $L^p$ in time and $L^q$ in space
  - Triebel-Lizorkin spaces as trace spaces
    (Hieber-Prüss-D. 2007, Kaip 2011)
- anisotropic symbol structure
  - Newton polygon method
    (Volevich-D. 2008, Saal-Seiler-D. 2008)
Thank you for your attention!